

Almost everywhere convergence of some summability methods for Laguerre series

by

JOLANTA DŁUGOSZ (Wrocław)

Abstract. We consider various types of convergence for Laguerre expansions. In the main theorem we formulate a general sufficient condition on a summability kernel K to imply that the corresponding summability method applied to the Laguerre expansion of a function f in $L^p(\mathbf{R}_+)$ gives almost everywhere convergence if $1 \leq p \leq \infty$ and the L^p norm convergence if $1 \leq p < \infty$. The theorem works for the classical Abel summability method, and for Riesz and Riemann methods with large exponents.

Introduction. Let $\mathcal{L}_n^\alpha(x) = (n!/\Gamma(n+\alpha+1))^{1/2} e^{-x/2} x^{\alpha/2} L_n^\alpha(x)$, where L_n^α , $n = 0, 1, \dots$, are the Laguerre polynomials on \mathbf{R}_+ . For a function f in $L^p(\mathbf{R}_+)$, $1 \leq p \leq \infty$, write its formal Laguerre expansion

$$(0.1) \quad f \sim \sum_n c_n \mathcal{L}_n^\alpha,$$

where

$$c_n = (f, \mathcal{L}_n^\alpha) = \int_0^\infty f(x) \mathcal{L}_n^\alpha(x) dx.$$

The aim of this paper is to prove theorems on the mean convergence of (0.1) almost everywhere and in the L^p norm. For integral values of the parameter α we obtain a general sufficient condition on a summability kernel K to imply that for a function f in $L^p(\mathbf{R}_+)$

$$(0.2) \quad f(x) = \lim_{t \rightarrow 0} \sum_n K(tn) c_n \mathcal{L}_n^\alpha(x)$$

almost everywhere if $1 \leq p \leq \infty$, and in the L^p norm if $1 \leq p < \infty$. In particular, this condition is satisfied by $K(\lambda) = e^{-\lambda}$, $K(\lambda) = (1-\lambda)_+^N$, $N > 9$, $K(\lambda) = (\sin \lambda/\lambda)^\beta$, β large. Thus, our theorem includes the cases of the classical Abel, Riesz and Riemann means.

The problem of the mean convergence of Laguerre expansions has been considered by a number of authors. In 1965 Askey and Wainger [1] proved

that if $4/3 < p < 4$ and $\alpha \geq 0$ then the partial sums of the Laguerre expansion of a function $f \in L^p(\mathbf{R}_+)$ converge to f in the L^p norm. They also showed that this is not true if $1 \leq p \leq 4/3$ or $p \geq 4$. In [10] and [11] Muckenhoupt extended their results to all $\alpha > -1$ and weighted L^p .

Abel summability (i.e. for $K(\lambda) = e^{-\lambda}$) has been treated by Muckenhoupt in [9] where instead of \mathcal{L}_n^α he considered L_n^α and $f \in L^p(\mathbf{R}_+, e^{-x} x^\alpha)$. In particular, he obtained the almost everywhere convergence of Abel means for all $\alpha > -1$ and $1 \leq p \leq \infty$. This is the only result concerning almost everywhere convergence for Laguerre expansions known to the author.

The norm convergence of the first Cesàro means (i.e. $K(\lambda) = (1-\lambda)_+$) has been proved by Poiani [12] for weighted L^p spaces. Her results include the case of the ordinary L^p norm convergence. General Riesz means (i.e. $K(\lambda) = (1-\lambda)_+^\delta$, $\delta \geq 0$) for Laguerre expansions with the parameter $\alpha \geq 0$ have been considered by Markett [7]. He has proved the L^p norm convergence of (0.2) for $\delta > \frac{1}{2}$ and $1 \leq p < \infty$.

The above-mentioned authors have used classical methods. Recently Hulanicki and Jenkins [5], [6] have developed a technique which enables one to obtain mean convergence theorems for eigenfunction expansions of some differential operators on \mathbf{R}^n from theorems concerning homogeneous groups and spectral expansions of Rockland operators. In particular, using this technique they obtained results parallel to ours for Hermite expansions. Their results do not give theorems for Laguerre expansions, nevertheless considering this case we follow their ideas.

Our approach is based on the observation that the Laguerre functions with integral parameter α appear in eigenexpansions of the sublaplacian acting on the spaces of functions on the Heisenberg group considered by Geller [4]. We pass to a quotient group to obtain a discrete spectrum. A theorem of Hulanicki and Jenkins (quoted in Section 2) enables us to formulate problems concerning the mean summability of Laguerre expansions in terms of the convergence of an appropriate family of operators on this quotient group. As usual to prove theorems on almost everywhere convergence we investigate the corresponding maximal operator. Results obtained for the Heisenberg group are interpreted for Laguerre expansions.

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1. Let H_m be the $(2m+1)$ -dimensional Heisenberg group. We shall write elements of H_m as pairs (z, u) where $z = (z_1, \dots, z_m) \in \mathbf{C}^m$ and $u \in \mathbf{R}$, the multiplication law being

$$(1.1) \quad (z, u)(z', u') = (z+z', u+u'+\operatorname{Im} \sum_{j=1}^m \bar{z}_j z'_j).$$

Let $\Gamma = \{(0, u) : u \in \mathbf{Z}\}$. We identify H_m/Γ as a set with $\mathbf{R}^{2m} \times$

$\times T(T = [0, 1))$, the Lebesgue measure being the Haar measure on H_m/Γ . We write \dot{x} for the element of H_m/Γ corresponding to x in H_m .

In the sequel we present some considerations involving spaces of homogeneous type. We recommend [2] as a reference.

For $t > 0$ a dilation on H_m is defined by

$$(1.2) \quad \delta_t(z, u) = (tz, t^2 u).$$

Let $|\cdot|$ be any symmetric dilation-homogeneous norm on H_m . Define a quasi-distance d on H_m/Γ by

$$d(\dot{x}, \dot{y}) = \inf_{\gamma \in \Gamma} |x^{-1} \gamma y|.$$

Set

$$B_s(\dot{x}) = \{\dot{y} \in H_m/\Gamma : d(\dot{y}, \dot{x}) < s\}.$$

LEMMA 1. H_m/Γ with the quasi-distance d is a space of homogeneous type.

Proof. According to [2] it is sufficient to show that there exists a constant C such that for any $\dot{x} \in H_m/\Gamma$ and $s > 0$

$$(1.3) \quad |B_s(\dot{x})| \leq C |B_{s/2}(\dot{x})|,$$

where $|B_s(\dot{x})|$ is the Lebesgue measure of $B_s(\dot{x})$. Since the quasi-distance d and the Lebesgue measure on H_m/Γ are left-invariant, we have only to verify (1.3) for $\dot{x} = \dot{0}$ (0 being the neutral element of H_m). Since all dilation-homogeneous norms on H_m are equivalent, we may assume that

$$|(z, u)| = \max \{|\operatorname{Re} z_1|, |\operatorname{Im} z_1|, \dots, |\operatorname{Re} z_m|, |\operatorname{Im} z_m|, \sqrt{|u|}\}.$$

If the quasi-distance d on H_m/Γ is defined by means of this norm we have

$$(1.4) \quad |B_s(\dot{0})| = \begin{cases} (2s)^{2m} 2s^2 & \text{if } s \leq 1/\sqrt{2}, \\ (2s)^{2m} & \text{if } s > 1/\sqrt{2}. \end{cases}$$

Now (1.3) follows easily. Thus the lemma is proved.

By [2], Chapter III we obtain

COROLLARY 1. The Hardy-Littlewood maximal function on H_m/Γ ,

$$(m^* f)(\dot{x}) = \sup_{s>0} \frac{1}{|B_s(\dot{x})|} \int_{B_s(\dot{x})} |f(\dot{y})| d\dot{y},$$

is of weak type (1,1).

Now define a symmetric dilation-homogeneous norm on H_m by

$$|x| = |(z, u)| = |\operatorname{Re} z_1| + |\operatorname{Im} z_1| + \dots + |\operatorname{Re} z_m| + |\operatorname{Im} z_m| + \sqrt{|u|}.$$

Let $|\dot{x}|_{\sim}$ be the corresponding quotient norm on H_m/Γ , i.e.

$$|\dot{x}|_{\sim} = \inf_{\gamma \in \Gamma} |x\gamma|.$$

For a function k on H_m let

$$(1.5) \quad \varrho_t k(x) = t^{-Q} k(\delta_{t^{-1}} x) = k_t(x),$$

where $Q = 2m+2$ is the homogeneous dimension of H_m . Set

$$(1.6) \quad \dot{k}_t(\dot{x}) = \sum_{\gamma \in \Gamma} k_t(x\gamma).$$

Define a function ω on H_m by

$$(1.7) \quad \omega(x) = 1 + |x|.$$

LEMMA 2. Let k be a function on H_m such that

$$(1.8) \quad \sup_{x \in H_m} |k(x) \omega^l(x)| < \infty$$

where $l > Q+2 = 2m+4$. Let \dot{k}_t be defined by (1.5) and (1.6). Then there is a constant C such that for all $t \in (0, 1]$ and all f in $L^1(H_m/\Gamma)$ we have

$$(1.9) \quad |(f * \dot{k}_t)(\dot{x})| \leq C(m^* f)(\dot{x}) \quad \text{for all } \dot{x} \in H_m/\Gamma.$$

Proof. By (1.8) we have $|\dot{k}_t(\dot{x})| \leq C_0 \dot{h}_t(\dot{x})$, where $h(x) = \omega^{-l}(x)$, $l = Q+2+\varepsilon$, $\varepsilon > 0$. Thus it is sufficient to prove the lemma for the function \dot{h}_t . To do this we use the following estimate:

$$(1.10) \quad \sum_{\gamma \in \Gamma} \frac{1}{(1+t^{-1}|x\gamma|)^l} \leq C_1 t^l \left(\frac{1}{(t+|\dot{x}|_{\sim})^l} + \frac{1}{(t+|\dot{x}|_{\sim})^{l-2}} \right),$$

where the constant C_1 is independent of $t \in (0, 1)$ and $x \in H_m$.

The first summand on the right-hand side of (1.10) appears because we have $|x\gamma| \geq |\dot{x}|_{\sim}$ for every $\gamma \in \Gamma$. Now to see (1.10) write $x = (z, u)$, $\gamma = (0, n)$, $n \in \mathbb{Z}$. We may assume that $u \in [0, 1)$. We have $|x\gamma| = a + \sqrt{|u+n|}$, where $a = |\operatorname{Re} z_1| + |\operatorname{Im} z_1| + \dots + |\operatorname{Re} z_m| + |\operatorname{Im} z_m|$. Using this notation we obtain

$$\begin{aligned} \sum_{n \in \mathbb{Z} \setminus \{0, -1, -2\}} \frac{1}{(1+t^{-1}(a+\sqrt{|u+n|}))^l} &\leq 2 \int_0^\infty \frac{dv}{(1+t^{-1}(a+\sqrt{u+v}))^l} \\ &\leq \frac{4}{l-2} \cdot \frac{t^l}{(t+a+\sqrt{u})^{l-2}} \leq \frac{4}{l-2} \cdot \frac{t^l}{(t+|\dot{x}|_{\sim})^{l-2}}. \end{aligned}$$

Thus the estimate (1.10) is established. As an immediate consequence of it we have

$$(1.11) \quad \dot{h}_t(\dot{x}) \leq \begin{cases} C_2 t^{-Q} & \text{if } |\dot{x}|_{\sim} \leq t, \\ C_2 (2^j t)^{-Q} 2^{-j\varepsilon} & \text{if } 2^{j-1} t \leq |\dot{x}|_{\sim} \leq 2^j t, j = 1, 2, \dots, \end{cases}$$

where the constant C_2 is independent of j .

Now we prove the lemma for \dot{h}_t by a well-known technique. We include the proof here for completeness sake. Denote $B_t = B_t(0)$. We have

$$\begin{aligned} |(f * \dot{h}_t)(\dot{x})| &\leq \int_{|\dot{y}|_{\sim} \leq t} |f(\dot{x}\dot{y}^{-1})| \dot{h}_t(\dot{y}) d\dot{y} + \sum_{j=1}^\infty \int_{2^{j-1}t \leq |\dot{y}|_{\sim} \leq 2^j t} |f(\dot{x}\dot{y}^{-1})| \dot{h}_t(\dot{y}) d\dot{y} \\ &\leq C_2 \left[t^{-Q} \int_{B_t} |f(\dot{x}\dot{y}^{-1})| d\dot{y} + \sum_{j=1}^\infty (2^j t)^{-Q} 2^{-j\varepsilon} \int_{B_{2^j t}} |f(\dot{x}\dot{y}^{-1})| d\dot{y} \right] \\ &\leq C m^* f(\dot{x}). \end{aligned}$$

The second inequality holds by (1.11), and the third by the estimate $|B_t| \leq c t^Q$ (cf. (1.4)).

PROPOSITION 1. Let k be a function as in Lemma 2 and let

$$(1.12) \quad \int_{H_m} k(x) dx = 1.$$

For a function f in $L^p(H_m/\Gamma)$ we have

$$(1.13) \quad \lim_{t \rightarrow 0} (f * \dot{k}_t)(\dot{x}) = f(\dot{x})$$

almost everywhere on H_m/Γ if $1 \leq p \leq \infty$, and in the L^p norm if $1 \leq p < \infty$.

Proof. Since \dot{k}_t is an approximate identity, convergence (1.13) is proved for f continuous with compact support as usual. These functions form a dense subset of $L^p(H_m/\Gamma)$ if $1 \leq p < \infty$. Hence to complete the proof for the case of norm convergence it is sufficient to observe that the operators of convolution with \dot{k}_t acting on $L^p(H_m/\Gamma)$, $1 \leq p < \infty$, have norms commonly bounded by $\|k\|_{L^1(H_m)}$.

As regards the almost everywhere convergence of (1.13), it is sufficient to prove it for $p = 1$ and $p = \infty$. To do this for $p = 1$ consider the operator $Mf(\dot{x}) = \sup_{t \in (0, 1]} |(f * \dot{k}_t)(\dot{x})|$. Since the operator $f \rightarrow m^* f$ on H_m/Γ is of weak type (1,1) (Corollary 1) by Lemma 2 the same is true for M . Now the almost everywhere convergence of (1.13) for any $f \in L^1(H_m/\Gamma)$ follows if we use a well-known theorem ([14], p. 60, Theorem 3.12).

For f in $L^\infty(H_m/\Gamma)$ we show that $\lim_{t \rightarrow 0} (f * \dot{k}_t)(\dot{x}) = f(\dot{x})$ for almost all \dot{x} in every ball B_s and so for almost all $\dot{x} \in H_m/\Gamma$. To do this we write f as a sum $f = f_1 + f_2$, where $f_1 = f$ on B_{2As} (A is a constant such that $|\dot{y}\dot{x}|_{\sim} \leq A(|\dot{y}|_{\sim} + |\dot{x}|_{\sim})$ for all $\dot{x}, \dot{y} \in H_m/\Gamma$), and $f_1 = 0$ elsewhere. Since $f_1 \in L^1(H_m/\Gamma)$ we have

$\lim_{t \rightarrow 0} (f_1 * k_t)(x) = f_1(x) = f(x)$ almost everywhere on B_s . For f_2 and $x \in B_s$ we have

$$|(f_2 * k_t)(x)| = \left| \int_{H_m/\Gamma} f_2(x\dot{y}^{-1}) k_t(\dot{y}) d\dot{y} \right| \leq \int_{|\dot{y}| \sim \geq s} |k_t(\dot{y})| d\dot{y} \|f_2\|_\infty \rightarrow 0$$

as $t \rightarrow 0$. Thus Proposition 1 is proved.

2. Let L_n^α denote the Laguerre polynomial of type α and degree n on \mathbf{R}_+ :

$$L_n^\alpha(w) = \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{(-w)^j}{j!},$$

and let \mathcal{L}_n^α be the corresponding Laguerre function:

$$\mathcal{L}_n^\alpha(w) = \left(\frac{n!}{\Gamma(n+\alpha+1)} \right)^{1/2} e^{-w/2} w^{\alpha/2} L_n^\alpha(w).$$

For $\mathbf{n} = (n_1, \dots, n_m)$, $\alpha = (\alpha_1, \dots, \alpha_m) \in N^m$ (where $N = \{0, 1, \dots\}$), and $\mathbf{w} = (w_1, \dots, w_m) \in \mathbf{R}_+^m$ we use the notation $\mathcal{L}_{\mathbf{n}}^\alpha(\mathbf{w}) = \mathcal{L}_{n_1}^{\alpha_1}(w_1) \dots \mathcal{L}_{n_m}^{\alpha_m}(w_m)$.

On H_m we use coordinates $(x_1, \dots, x_m, y_1, \dots, y_m, u)$ where $z_j = x_j + iy_j$, $j = 1, \dots, m$. Let X_j, Y_j denote the element of the Lie algebra of H_m corresponding to the one-parameter subgroup $(0, \dots, 0, t, 0, \dots, 0)$, t in the x_j or y_j position respectively. Let L be the homogeneous sublaplacian on H_m , i.e. $L = -\sum_{j=1}^m (X_j^2 + Y_j^2)$.

On H_m/Γ consider the functions of the form

$$(2.1) \quad f(z_1, \dots, z_m, u) = \exp(2\pi i u) \exp\left(-i \sum_{j=1}^m \alpha_j \theta_j\right) f_0(r_1, \dots, r_m),$$

where $z_j = r_j e^{i\theta_j}$, $j = 1, \dots, m$, α_j , $j = 1, \dots, m$, are nonnegative integers, and f_0 is a function defined on \mathbf{R}_+^m . We denote $L_{e,\alpha}^p(H_m/\Gamma)$ the space of functions in $L^p(H_m/\Gamma)$ of the form (2.1).

PROPOSITION 2. The operator L maps functions of the form (2.1) into functions of the same form, has discrete spectrum on $L_{e,\alpha}^2(H_m/\Gamma)$, and the normalized eigenfunctions of L on this space are

$$\varphi_n^\alpha(z, u) = 2^{m/2} \exp(2\pi i u) \exp\left(-i \sum_{j=1}^m \alpha_j \theta_j\right) \mathcal{L}_n^\alpha(2\pi \mathbf{r}^2), \quad \mathbf{n} \in N^m,$$

where $\mathbf{r}^2 = (r_1^2, \dots, r_m^2)$, the corresponding eigenvalues of L being $8\pi(|\mathbf{n}| + m/2)$, where $|\mathbf{n}| = n_1 + \dots + n_m$.

Proof. It is routine to verify this writing L in polar coordinates

$$\begin{aligned} x_j &= r_j \cos \theta_j, \\ y_j &= r_j \sin \theta_j, \end{aligned}$$

$j = 1, \dots, m$, and utilizing the second order differential equation satisfied by $L_n^\alpha(w)$ (see [3]):

$$w \frac{d^2}{dw^2} L_n^\alpha(w) + (\alpha + 1 - w) \frac{d}{dw} L_n^\alpha(w) + n L_n^\alpha(w) = 0.$$

Let $\{p_t\}_{t>0}$ be the convolution semigroup on H_m whose infinitesimal generator is L and let \mathcal{A} be the Banach *-subalgebra of $L^1(H_m)$ generated by $\{p_t\}_{t>0}$. For a function $k \in \mathcal{A}$ let \hat{k} be its Gelfand transform. The following theorem enables us to use results of Section 1 to obtain theorems on mean summability for Laguerre expansions.

THEOREM (Hulanicki and Jenkins [5]). Consider the condition: $K \in C^N(\mathbf{R}_+)$ and

$$(K \cdot a) \quad \sup_{\lambda \geq 0} |K^{(j)}(\lambda)(1+\lambda)^{(a+N)S+1}| < \infty, \quad j = 0, 1, \dots, N,$$

where S is the smallest integer such that

$$\|\varphi\|_{L^\infty(H_m)} \leq C \|(I+L)^S \varphi\|_{L^2(H_m)}.$$

Then:

- (i) if $N > Q/2 + 1$ and $a = 0$, there is a $k \in \mathcal{A}$ such that $\hat{k} = K$,
- (ii) if $N > Q/2 + l + 1$, where $l \geq 0$, and $a = 4$, there is a $k \in \mathcal{A}$ such that $\hat{k} = K$ and $\sup_{x \in H_m} |k(x) \omega^l(x)| < \infty$.

Now we are ready to prove our main theorem.

THEOREM. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in N^m$. Let $K \in C^N(\mathbf{R}_+)$, $K(0) = 1$.

- (a) If $N > m + 2$ and K satisfies $(K \cdot 0)$ then for a function $g \in L^p(\mathbf{R}_+^m)$, $1 \leq p < \infty$,

$$(2.2) \quad \lim_{t \rightarrow 0} \left\| g - \sum_{\mathbf{n} \in N^m} K(t(|\mathbf{n}| + m/2)) (g, \mathcal{L}_n^\alpha) \mathcal{L}_n^\alpha \right\|_{L^p(\mathbf{R}_+^m)} = 0.$$

- (b) If $N > 3m + 6$ and K satisfies $(K \cdot 4)$ then for a function $g \in L^p(\mathbf{R}_+^m)$, $1 \leq p \leq \infty$,

$$(2.3) \quad g(\mathbf{w}) = \lim_{t \rightarrow 0} \sum_{\mathbf{n} \in N^m} K(t(|\mathbf{n}| + m/2)) (g, \mathcal{L}_n^\alpha) \mathcal{L}_n^\alpha(\mathbf{w})$$

for almost all $\mathbf{w} \in \mathbf{R}_+^m$.

In particular, if the function K is of the form (A), (B) or (C) below (cf. [5]) then it satisfies the assumptions of the theorem.

$$(A) \quad K(\lambda) = e^{-\lambda},$$

$$(B) \quad K(\lambda) = \begin{cases} (1-\lambda)^N & \text{if } 0 \leq \lambda \leq 1, \\ 0 & \text{if } \lambda > 1, \end{cases}$$

$$(C) \quad K(\lambda) = (\varphi(\lambda)/\lambda)^p, \text{ where } \varphi \in C^N(\mathbf{R}_+), \varphi'(0) = 1, \sup_{\lambda \geq 0} |\varphi^{(j)}(\lambda)| < \infty \text{ for } j$$

$= Q, 1, \dots, N, \beta > NS+1$ for the case (a), $\beta > (4+N)S+1$ for the case (b).

Thus we obtain almost everywhere convergence and appropriate L^p norm convergence of Abel, Riesz and Riemann means of Laguerre expansions respectively.

Proof of the Theorem. To simplify the notation we present the proof for the case $m=1$, the changes for $m>1$ being obvious.

We first prove part (b). If K satisfies the assumptions of part (b) of the theorem then the function k corresponding to it by the theorem of Hulanicki and Jenkins satisfies the assumptions of Proposition 1. We also have

$$f * k_{\sqrt{t}} = \sum_{n \in \mathbb{N}} K(8\pi t(n+1/2))(f, \varphi_n^\alpha) \varphi_n^\alpha$$

for $f \in L_{e,\alpha}^p(H_1/\Gamma)$, $1 \leq p \leq \infty$, (cf. Proposition 2). So by Proposition 1

$$f(x) = \lim_{t \rightarrow 0} \sum_{n \in \mathbb{N}} K(8\pi t(n+1/2))(f, \varphi_n^\alpha) \varphi_n^\alpha(x)$$

for almost all $x \in H_1/\Gamma$. Thus, since f and φ_n^α are of the form (2.1), we obtain

$$(2.4) \quad f_0(r) = \lim_{t \rightarrow 0} \sum_{n \in \mathbb{N}} K(t(n+1/2))(f, \varphi_n^\alpha) 2^{1/2} \mathcal{L}_n^\alpha(2\pi r^2)$$

for almost all $r \in \mathbb{R}_+$, where the function f_0 corresponds to f by the formula (2.1). But

$$(f, \varphi_n^\alpha) = 2^{1/2} \int_{\mathbb{C}} f_0(|z|) \mathcal{L}_n^\alpha(2\pi|z|^2) dz = 2^{-1/2} \int_0^\infty f_0(\sqrt{w/2\pi}) \mathcal{L}_n^\alpha(w) dw.$$

Set

$$(2.5) \quad g(w) = f_0(\sqrt{w/2\pi}).$$

Putting in (2.4) $r = \sqrt{w/2\pi}$ we obtain

$$(2.6) \quad g(w) = \lim_{t \rightarrow 0} \sum_{n \in \mathbb{N}} K(t(n+1/2))(g, \mathcal{L}_n^\alpha) \mathcal{L}_n^\alpha(w)$$

for almost all $w \in \mathbb{R}_+$.

As $\|f\|_{L_{e,\alpha}^p(H_1/\Gamma)} = C_p \|g\|_{L^p(\mathbb{R}_+)}$, $1 \leq p \leq \infty$, the correspondence $f \rightarrow f_0 \rightarrow g$ defined by (2.1) and (2.5) maps the space $L_{e,\alpha}^p(H_1/\Gamma)$ onto the space $L^p(\mathbb{R}_+)$. Thus part (b) follows.

Now observe that to prove norm convergence in Proposition 1 it is sufficient to assume that $k \in L^1(H_m)$ and k satisfies (1.12). So to prove (a) we use part (i) of the theorem of Hulanicki and Jenkins, and repeat previous considerations where almost everywhere convergence is replaced by L^p norm convergence.

3. It would be nice to replace $K(t(|\mathbf{n}|+m/2))$ by $K(t|\mathbf{n}|)$ in (2.2) and (2.3). We can obviously do this in the case of Abel means. To do this for Riesz means denote $\lambda_n = |\mathbf{n}|+m/2$, $\mu_n = |\mathbf{n}|$, $g_0(\mathbf{w}) = (g, \mathcal{L}_0^\alpha) \mathcal{L}_0^\alpha(\mathbf{w}) - g(\mathbf{w})$, $g_n(\mathbf{w}) = (g, \mathcal{L}_n^\alpha) \mathcal{L}_n^\alpha(\mathbf{w})$ for $\mathbf{n} \neq 0$, $\mathbf{n} \in \mathbb{N}^m$. Let $x = 1/t$. With this notation we can write the formula (2.3) for Riesz means as follows:

$$(3.1) \quad \lim_{x \rightarrow \infty} x^{-N} \sum_{\lambda_n \leq x} (x - \lambda_n)^N g_n(\mathbf{w}) = 0.$$

Changing the variable $y = x - m/2$ we see that (3.1) is equivalent to

$$(3.2) \quad \lim_{y \rightarrow \infty} y^{-N} \sum_{\mu_n \leq y} (y - \mu_n)^N g_n(\mathbf{w}) = 0.$$

Thus we obtain the following

COROLLARY 2. Let $\alpha_1, \dots, \alpha_m$ be nonnegative integers. For $g \in L^p(\mathbb{R}_+^m)$ we have

$$(3.3) \quad \lim_{y \rightarrow \infty} \sum_{|\mathbf{n}| \leq y} (1 - |\mathbf{n}|/y)^N (g, \mathcal{L}_n^\alpha) \mathcal{L}_n^\alpha(\mathbf{w}) = g(\mathbf{w})$$

almost everywhere if $1 \leq p \leq \infty$ and $N > 3m+6$, and in the L^p norm if $1 \leq p < \infty$ and $N > m+2$.

Remark. It seems that if in the proof of almost everywhere and appropriate L^p norm convergence of Riesz means for Laguerre expansions we use the results of Mauceri [8] in place of those of Hulanicki and Jenkins [5] then the exponent in (3.3) can be taken smaller.

To replace $K(t(n+m/2))$ by $K(tn)$ in the one-dimensional case we need the following lemma.

LEMMA 3. Let K be a function as in the Theorem, part (a), and for an $a < 1$ let $|s_n| = |b_1 + \dots + b_n| = O(n^a)$ as $n \rightarrow \infty$. We have

$$(3.4) \quad \lim_{t \rightarrow 0} \sum_{n=1}^{\infty} K(t(n+m/2)) b_n = s$$

if and only if

$$(3.5) \quad \lim_{t \rightarrow 0} \sum_{n=1}^{\infty} K(tn) b_n = s.$$

Proof. To prove the lemma it is sufficient to show that

$$(3.6) \quad \lim_{t \rightarrow 0} \sum_{n=1}^{\infty} [K(t(n+m/2)) - K(tn)] b_n = 0.$$

Using Abel transformation we deduce that (3.6) is equivalent to

$$(3.7) \quad \lim_{t \rightarrow 0} \sum_{n=1}^{\infty} [K(t(n+m/2)) - K(t(n+1+m/2)) - K(tn) + K(t(n+1))] s_n = 0.$$

The series in (3.7) can be estimated by

$$(3.8) \quad t^2 \sum_{n=1}^{\infty} |K''(t(n + \Theta_n'))| |s_n|$$

where $\Theta_n' \in (0, \frac{1}{2}m + 1)$. Using $(K \cdot 0)$ and denoting $M = NS + 1$ we can estimate (3.8) by

$$Ct^2 \sum_{n=1}^{\infty} \frac{n^a}{(1 + t(n + \Theta_n'))^M} \leq C_1 t^2 \int_1^{\infty} \frac{x^a dx}{(1 + tx)^M} \leq C_1 t^{1-a} \int_0^{\infty} \frac{z^a dz}{(1 + z)^M} \rightarrow 0$$

as $t \rightarrow 0$ and Lemma 3 follows.

Using the form of the n th partial sum for Laguerre expansion given in [1], p. 703, we can estimate it pointwise by $C(w)n^{3/4}$ for almost all $w \in \mathbf{R}_+$ for $g \in L^p(\mathbf{R}_+)$, $1 \leq p \leq \infty$. If $1 \leq p < \infty$ its L^p norms are estimated by $Cn^{1/2}$ (see [7]). So the assumptions of Lemma 3 are satisfied and we obtain the following

COROLLARY 3. Let $K \in C^N(\mathbf{R}_+)$, $K(0) = 1$ and let $\alpha \in \mathbf{N}$.

(a) If $N > 3$ and K satisfies $(K \cdot 0)$ then for $g \in L^p(\mathbf{R}_+)$, $1 \leq p < \infty$,

$$(3.9) \quad \lim_{t \rightarrow 0} \|g - \sum_{n=0}^{\infty} K(tn)(g, \mathcal{L}_n^{\alpha}) \mathcal{L}_n^{\alpha}\|_{L^p(\mathbf{R}_+)} = 0.$$

(b) If $N > 9$ and K satisfies $(K \cdot 4)$ then for $g \in L^p(\mathbf{R}_+)$, $1 \leq p \leq \infty$,

$$(3.10) \quad g(w) = \lim_{t \rightarrow 0} \sum_{n=0}^{\infty} K(tn)(g, \mathcal{L}_n^{\alpha}) \mathcal{L}_n^{\alpha}(w)$$

for almost all $w \in \mathbf{R}_+$.

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
ul. Kopernika 18, 51-617 Wrocław, Poland

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