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A B_0-algebra without generalized topological divisors of zero

by

HUGO ARIZMENDI PEIMBERT (México D. F.) and WIESŁAW ŻELAZKO (Warszawa)*

Abstract. We construct a commutative algebra possessing the property announced in the title. This solves in the negative a problem posed in [3] and [6].

1. Introduction. All algebras considered in this paper are commutative algebras with unit elements. A *topological algebra* is a topological linear space together with a jointly continuous associative multiplication making of it an algebra over C. Thus if A is a topological algebra and Φ is a basis of neighbourhoods of the origin in A, then for each U in Φ there is a neighbourhood $V \in \Phi$ such that

(1) $$V^2 \subset U.$$

A *locally convex algebra* is a topological algebra which is a locally convex space. The topology of such an algebra A can be introduced by means of a family $(\|x\|_\alpha)$, $\alpha \in \mathfrak{A}$, of seminorms such that for each index α there is an index β with

(2) $$\|xy\|_\alpha \leqslant \|x\|_\beta \|y\|_\beta$$

for all $x, y \in A$. Relations (2) follow clearly from relations (1). We can also assume that for each finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{A}$ there is an index $\beta \in \mathfrak{A}$ such that

(3) $$\|x\|_{\alpha_i} \leqslant \|x\|_\beta$$

for all elements x in A and $i = 1, \ldots, n$.

A *B_0-algebra* is a locally convex algebra which is a B_0-space, i.e. a completely metrizable locally convex space. Its topology can be introduced by means of a sequence $(\|x\|_i)$ of seminorms satisfying

(4) $$\|x\|_1 \leqslant \|x\|_2 \leqslant \ldots$$

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for all x in A, which corresponds to relations (3), and

(5) $$\|xy\|_i \leqslant \|x\|_{i+1}\|y\|_{i+1},$$

$i = 1, 2, \ldots$, for all elements x and y in A, which corresponds to relations (2).

A locally convex algebra (in particular a B_0-algebra) A is said to be *locally multiplicatively convex* (shortly *m-convex*) if relations (2) or (5) can be replaced by

(6) $$\|xy\|_\alpha \leqslant \|x\|_\alpha \|y\|_\alpha$$

for all elements x and y in A and all indices α in $\mathfrak{A}$.

In particular, all Banach algebras are m-convex algebras.

One of basic concepts in the theory of Banach algebras is the concept of a *topological divisor of zero*. This is a nonzero element x of the algebra in question such that there exists a sequence of elements x_n, $\|x_n\| = 1$, with $\lim x_n x = 0$. This concept is due to G. E. Shilov, who proved that a Banach algebra either possesses topological divisors of zero or is isomorphic to the field of complex numbers. An analogous theorem, however, fails for general locally convex algebras, even for m-convex B_0-algebras: in the algebra E of all entire functions, provided with the pointwise algebra operations and the topology of uniform convergence on compacts, the relations $\lim x_n x = 0$ and $x \neq 0$ imply $\lim x_n = 0$ (cf. [4] and [6]). In view of this example the second author introduced in [3] a more general concept of generalized topological divisors of zero (for the definition cf. Section 2) and proved that any topological algebra containing the field of rational functions always possesses such divisors. He proved later in [4] (for complex algebras) and in [5] (for real algebras) that an m-convex algebra either possesses generalized topological divisors of zero or is isomorphic to the field of complex numbers (in the case of complex algebras), or to one of the three finite-dimensional division algebras over $\boldsymbol{R}$ (reals, complex numbers, or quaternions). In [3] and [6] a general conjecture is posed that an arbitrary topological algebra A either possesses generalized topological divisors of zero or is isomorphic to the field of complex numbers (or to one of the three standard finite-dimensional division algebras in the real case).

The purpose of the present note is to disprove this conjecture. We are going to construct a commutative B_0-algebra possessing no generalized topological divisors of zero. We shall construct our example in the form of a matrix algebra. This is an algebra of power series in one variable, with complex coefficients, whose seminorms are given by means of a matrix $a_i^{(n)}$, $n = 1, 2, \ldots$, $i = 0, 1, 2, \ldots$, of nonnegative real numbers satisfying the following conditions:

(7) $$a_0^{(n)} = 1$$

for $n = 1, 2, \ldots$,

(8) $$a_k^{(n)} \leqslant a_k^{(n+1)}$$

for $n = 1, 2, \ldots$, $k = 0, 1, 2, \ldots$, and

(9) $$a_{i+j}^{(n)} \leqslant a_i^{(n+1)} a_j^{(n+1)}$$

for $i, j = 0, 1, 2, \ldots$, and $n = 1, 2, \ldots$

For a power series $x = \sum_{i=0}^{\infty} \xi_i t^i$ we put

(10) $$\|x\|_n = \sum_{i=0}^{\infty} a_i^{(n)} |\xi_i|,$$

and the *matrix algebra associated with the matrix* $(a_i^{(n)})$ is the algebra of all power series for which the seminorms (10) are finite. One can easily see that this algebra is a B_0-space, i.e. it is complete. Relation (5) follows immediately from relation (9), while relation (4) follows from (8). Relation (7) means that for each seminorm (10) we have $\|e\|_n = 1$, where e is the unit element of the algebra in question. For more details on the above the reader is referred to [1]–[7].

2. Generalized topological divisors of zero. Let A be a topological algebra and Φ a basis of neighbourhoods of its origin. A is said to possess *generalized topological divisors of zero* if for some $U \in \Phi$ we have

$$0 \in \overline{(A \setminus U)^2},$$

where the bar denotes the closure. In other words, generalized topological divisors of zero are sets $S_1, S_2 \subset A$ such that $0 \notin \bar{S}_1 \cup \bar{S}_2$, but $0 \in \overline{S_1 S_2}$. If one of the sets S_i consists of a single point x, then x is a topological divisor of zero. If the sets S_1 and S_2 both consist of single points, then they are divisors of zero. If A is a locally convex algebra with topology given by means of a family $(\|x\|_\alpha)$, $\alpha \in \mathfrak{A}$, of seminorms satisfying (2) and (3), then A possesses generalized topological divisors of zero if and only if there is an index α in $\mathfrak{A}$ such that

$$\inf\{\|xy\|_\beta\colon \|x\|_\alpha = \|y\|_\alpha = 1\} = 0$$

for each $\beta \in \mathfrak{A}$. Thus there are no generalized topological divisors of zero if and only if for each index α in $\mathfrak{A}$ there is an index $\beta \in \mathfrak{A}$ and a positive constant C_β such that the relations $\|x\|_\alpha = \|y\|_\alpha = 1$ imply $\|xy\|_\beta \geqslant C_\beta$, or, equivalently,

(11) $$\|xy\|_\beta \geqslant C_\beta \|x\|_\alpha \|y\|_\alpha$$

for all elements x and y in A. Since for a B_0-algebra any subsequence of the sequence (4) gives the same topology, we can state by (11) that a B_0-algebra has no generalized topological divisors of zero if and only if its topology is given by means of a sequence of seminorms satisfying relations (4), (5) and

(12) $$\|x\|_i \|y\|_i \leqslant C_i \|xy\|_{i+1},$$

for $i = 1, 2, \ldots$, and all elements x and y in A. In the next section we shall construct a matrix algebra A whose seminorms (10) satisfy relations (12) with all C_i equal to 2.

3. Construction of an example. Let p be a monic polynomial in the variable t with complex coefficients. So $p(t) = t^n + \alpha_{n-1} t^{n-1} + \ldots + \alpha_1 t + \alpha_0$. We can also write $p(t) = (t+\lambda_1)(t+\lambda_2)\ldots(t+\lambda_n)$; it is more convenient to use here the roots of polynomials equipped with the sign "$-$". Denote by $\tilde{p}$ the polynomial with roots equal to $-|\lambda_i|$, where $-\lambda_i$ are the roots of the polynomial p. The coefficients α_i of p and $\tilde{\alpha}_i$ of $\tilde{p}$ can be expressed by means of λ_i as

(13) $$\alpha_k^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i_1 < i_2 < \ldots < i_{n-k}} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_{n-k}},$$

$k = 1, 2, \ldots, n-1$, and similarly

(14) $$\tilde{\alpha}_k^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i_1 < i_2 < \ldots < i_{n-k}} |\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_{n-k}}|,$$

where n is the degree of p and $\tilde{p}$. We also set $\alpha_n^{(n)} \equiv 1 \equiv \tilde{\alpha}_n^{(n)}$ for all natural n.

Our construction is based upon the following

LEMMA 1. *For each sequence $A_0, A_1, \ldots$ of positive real numbers there is a sequence $B_0, B_1, \ldots$ of positive real numbers such that for each natural n and all complex numbers $\lambda_1, \ldots, \lambda_n$ we have*

(15) $$A_0 \tilde{\alpha}_0^{(n)}(\lambda_1, \ldots, \lambda_n) + A_1 \tilde{\alpha}_1^{(n)}(\lambda_1, \ldots, \lambda_n) + \ldots + A_{n-1} \tilde{\alpha}_{n-1}^{(n)}(\lambda_1, \ldots, \lambda_n) + A_n$$
$$\leqslant B_0 |\alpha_0^{(n)}(\lambda_1, \ldots, \lambda_n)| + B_1 |\alpha_1^{(n)}(\lambda_1, \ldots, \lambda_n)| + \ldots + B_{n-1} |\alpha_{n-1}^{(n)}(\lambda_1, \ldots, \lambda_n)| + B_n,$$

where the functions $\alpha_i^{(n)}$ and $\tilde{\alpha}_i^{(n)}$ are given by (13) *and* (14) *respectively.*

Proof. We shall define the numbers B_n in an inductive way, starting with $n = 0$ and setting $B_0 = A_0 + 1$. Suppose that we have already defined the numbers $B_0, B_1, \ldots, B_{n-1}$ in such a way that for each k, $0 \leqslant k < n$, we have

(16) $$\sup_{\lambda_1, \ldots, \lambda_n \in C} \{A_0 \tilde{\alpha}_0^{(k)}(\lambda_1, \ldots, \lambda_k) + \ldots + A_k \tilde{\alpha}_k^{(k)}(\lambda_1, \ldots, \lambda_k) -$$
$$- B_0 |\alpha_0^{(k)}(\lambda_1, \ldots, \lambda_k)| - \ldots - B_{k-1} |\alpha_{k-1}^{(k)}(\lambda_1, \ldots, \lambda_k)|\} \leqslant B_k - 1.$$

If we show that the left-hand supremum in (16) is finite and equals C for $k = n$, then setting $B_n = C+1$ we obtain (16) for $k = n$. So by induction we have (16) for all n, which, in turn, implies (15) for all n and we are done.

Denote by $\Phi_k(\lambda_1, \ldots, \lambda_k)$ the expression under the supremum sign on the left side of (16). Assume that for a certain choice of $\lambda_1^{(s)}, \ldots, \lambda_n^{(s)}$, $s = 1, 2, \ldots$, we have

(17) $$\lim_s \Phi_n(\lambda_1^{(s)}, \ldots, \lambda_n^{(s)}) = \infty.$$

We have to show that this leads to a contradiction. Since the role of variables $\lambda_1^{(s)}, \ldots, \lambda_n^{(s)}$ in (17) is symmetric, we can assume without loss of generality that for each s we have

$$|\lambda_1^{(s)}| \leqslant \ldots \leqslant |\lambda_n^{(s)}|.$$

Since Φ_n is a continuous function in its variables, it is bounded on each bounded subset of C^n. Therefore we must have $\lim_s |\lambda_n^{(s)}| = \infty$. Denote by k the smallest integer for which we have $\lim_s |\lambda_{k+1}^{(s)}| = \infty$. Thus we have $\lim_s |\lambda_p^{(s)}| = \infty$ for $k < p \leqslant n$ and $|\lambda_p^{(s)}| \leqslant M$ for some positive M if $p \leqslant k$. Note that $0 \leqslant k < n$, so that it can happen that $\lim_s |\lambda_p^{(s)}| = \infty$ for all $p = 1, 2, \ldots, n$.

Consider now the functions $\tilde{\alpha}_p^{(n)}(\lambda_1, \ldots, \lambda_n)$. If $p \leqslant k$, we can write them as

(18) $$\tilde{\alpha}_p^{(n)}(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < \ldots < i_{k-p} \leqslant k} |\lambda_{i_1} \ldots \lambda_{i_{k-p}} \lambda_{k+1} \ldots \lambda_n|$$
$$+ \sum_{j_1 < \ldots < j_{n-p}} |\lambda_{j_1} \ldots \lambda_{j_{n-p}}|,$$

where in the second sum less than $n-k$ of the indices $j_1, \ldots, j_{n-p}$ are greater than k. This implies, in particular, that

$$\lim_s |\lambda_{j_1}^{(s)} \ldots \lambda_{j_{n-p}}^{(s)}| / |\lambda_{k+1}^{(s)} \ldots \lambda_n^{(s)}| = 0$$

for all such $(n-p)$-tuples $(j_1, \ldots, j_{n-p})$. Assuming the notation $o(1)$ for an expression depending upon s which tends to zero as $s \to \infty$, and taking into account the fact that for $p > k$ in formula (14) (with p instead of k) only such $(n-p)$-tuples occur, we can rewrite (18) as

(19) $$\tilde{\alpha}_p^{(n)}(\lambda_1^{(s)}, \ldots, \lambda_n^{(s)}) = \begin{cases} |\lambda_{k+1}^{(s)} \ldots \lambda_n^{(s)}| (\tilde{\alpha}_p^{(k)}(\lambda_1^{(s)}, \ldots, \lambda_k^{(s)}) + o(1)) & \text{for } 0 \leqslant p \leqslant k, \\ |\lambda_{k+1}^{(s)} \ldots \lambda_n^{(s)}| o(1) & \text{for } k < p \leqslant n. \end{cases}$$

In exactly the same way we obtain

$$|\alpha_p^{(n)}(\lambda_1^{(s)}, \ldots, \lambda_n^{(s)})| = \begin{cases} |\lambda_{k+1}^{(s)} \ldots \lambda_n^{(s)}| \, |\alpha_p^{(k)}(\lambda_1^{(s)}, \ldots, \lambda_k^{(s)}) + o(1)| & \text{for } 0 \leqslant p \leqslant k, \\ |\lambda_{k+1}^{(s)} \ldots \lambda_n^{(s)}| \, |o(1)| & \text{for } k < p \leqslant n. \end{cases}$$

Observing now that $|\alpha+o(1)| = |\alpha|+o(1)$, we can rewrite the above as

(20)
$$|\alpha_p^{(n)}(\lambda_1^{(s)}, \ldots, \lambda_n^{(s)})| = \begin{cases} |\lambda_{k+1}^{(s)}\ldots\lambda_n^{(s)}|\,(|\alpha_p^{(k)}(\lambda_1^{(s)}, \ldots, \lambda_k^{(s)})|+o(1)) & \text{for } 0 \leqslant p \leqslant k, \\ |\lambda_{k+1}^{(s)}\ldots\lambda_n^{(s)}|\,|o(1)| & \text{for } k < p \leqslant n. \end{cases}$$

Taking into account the fact that $\alpha_k^{(k)} \equiv 1$, we have by (19) and (20)

(21)
$$\begin{aligned} \Phi_n(\lambda_1^{(s)}, \ldots, \lambda_n^{(s)}) &= |\lambda_{k+1}^{(s)}\ldots\lambda_n^{(s)}|\,(A_0\tilde{\alpha}_0^{(k)}(\lambda_1^{(s)}, \ldots, \lambda_k^{(s)})+\ldots+A_k\tilde{\alpha}_k^{(k)}- \\ &\quad -B_0|\alpha_0^{(k)}(\lambda_1^{(s)}, \ldots, \lambda_k^{(s)})|-\ldots-B_{k-1}|\alpha_{k-1}^{(k)}(\lambda_1^{(s)}, \ldots, \lambda_k^{(s)})|-B_k+o(1)) \\ &= |\lambda_{k+1}^{(s)}\ldots\lambda_n^{(s)}|\,(\Phi_k(\lambda_1^{(s)}, \ldots, \lambda_k^{(s)})-B_k+o(1)). \end{aligned}$$

Since $k < n$, we have by (16)

$$\Phi_k(\lambda_1^{(s)}, \ldots, \lambda_k^{(s)})-B_k \leqslant -1$$

for each choice of $\lambda_1^{(s)}, \ldots, \lambda_k^{(s)}$. For large s we also have $|o(1)| < 1$, and since $\lim_s |\lambda_{k+1}^{(s)}\ldots\lambda_n^{(s)}| = \infty$, we obtain by (21)

$$\lim_s \Phi_n(\lambda_1^{(s)}, \ldots, \lambda_n^{(s)}) = -\infty,$$

which contradicts formula (17). Since $B_n = \sup\Phi_n+1 \geqslant A_n+1$, all B_n are positive. The conclusion follows.

Remark. If for a given polynomial p we define $\tilde{p}$ as the polynomial whose roots are equal to $-|\lambda_i|$, where $-\lambda_i$ are the roots of p, and whose leading coefficient equals $|\alpha_n|$, where α_n is the leading coefficient of the polynomial p, then formula (15) is valid also for arbitrary polynomials.

For the sake of completeness we repeat here the proof of the following simple lemma proved in ([8], Lemma 3).

LEMMA 2. *Let $a = (a_n)$, $n = 0, 1, \ldots$, be a sequence of positive real numbers with $a_0 = 1$. There exists a sequence $b = (b_n)$, $n = 0, 1, \ldots$, of positive real numbers with $b_0 = 1$ such that*

(22)
$$a_{i+j} \leqslant b_i b_j$$

for all nonnegative integers i and j. In particular,

(23)
$$a_i \leqslant b_i$$

for all i.

Proof. Put $b_0 = 1$ and suppose that the numbers $b_1, \ldots, b_{n-1}$ are already constructed. Then put

$$b_n = \max\{a_n, a_{n+1}/b_1, a_{n+2}/b_2, \ldots, a_{2n-1}/b_{n-1}, a_{2n}^{1/2}\}.$$

Let $s = (s_n)$, $n = 0, 1, \ldots$, be a sequence of nonnegative real numbers.

Denote by $\|p\|_s$ the seminorm defined on the algebra of all polynomials $p = \alpha_n t^n+\alpha_{n-1}t^{n-1}+\ldots+\alpha_1 t+\alpha_0$ by the formula

$$\|p\|_s = \sum_i s_i|\alpha_i|.$$

With this notation we have the following

LEMMA 3. *The sequence b of Lemma 2 can be chosen so that it has in addition the following property: for all polynomials p and q in one variable*

(24)
$$\|p\|_a\|q\|_a \leqslant 2\|pq\|_b.$$

Proof. Observe first that if for some b relations (22) and (23) are satisfied, then they are also satisfied if we replace b by any coordinatewise larger sequence. The same applies to relation (24). Thus it is sufficient to find a sequence b with $b_0 = 1$ which satisfies relation (24). For, if $b' = (b'_n)$, $b'_0 = 1$, satisfies (22) and (23) and $b'' = (b''_n)$, $b''_0 = 1$, satisfies (24), then the sequence b with $b_i = \max\{b'_i, b''_i\}$ satisfies (22), (23) and (24), and $b_0 = 1$.

One can easily see that it is sufficient to construct a sequence b for which relation (24) is satisfied for monic polynomials. Let $p(t)$ and $q(t)$ be such polynomials. We can write

$$p(t) = (t+\lambda_1)(t+\lambda_2)\ldots(t+\lambda_k),$$
$$q(t) = (t+\lambda_{k+1})(t+\lambda_{k+2})\ldots(t+\lambda_n).$$

We have

$$\|p\|_a = |\lambda_1\ldots\lambda_k|+a_1\Big|\sum_{j_s<j_{s+1}}\lambda_{j_1}\ldots\lambda_{j_{k-1}}\Big|+\ldots+a_{k-1}\Big|\sum_{i=1}^{k}\lambda_i\Big|+a_k,$$

$$\|q\|_a = |\lambda_{k+1}\ldots\lambda_n|+a_1\Big|\sum_{j'_s<j'_{s+1}}\lambda_{j'_1}\ldots\lambda_{j'_{n-k-1}}\Big|+\ldots+a_{n-k},$$

where $1 \leqslant j_s \leqslant k$ and $k+1 \leqslant j'_s \leqslant n$. Thus

(25)
$$\begin{aligned} \|p\|_a\|q\|_a &\leqslant |\lambda_1\ldots\lambda_n|+2a_1\sum_{i_s<i_{s+1}}|\lambda_{i_1}\ldots\lambda_{i_{n-1}}|+\ldots \\ &\quad \ldots+\sum_{i=0}^{s}a_{s-i}a_i\sum_{i_s<i_{s+1}}|\lambda_{i_1}\ldots\lambda_{i_{n-s}}|+\ldots+\sum_{i=0}^{n}a_{n-i}a_i \\ &= A_0\tilde{\alpha}_0^{(n)}(\lambda_1, \ldots, \lambda_n)+A_1\tilde{\alpha}_1^{(n)}(\lambda_1, \ldots, \lambda_n)+\ldots+A_n, \end{aligned}$$

where $A_s = \sum_{i=0}^{s}a_{s-i}a_i$, $0 \leqslant s \leqslant n$, and $\tilde{\alpha}_p^{(n)}$ are the functions defined by formula (14). On the other hand, the coefficients of pq are given by formula (13). By Lemma 1, to the sequence (A_i) corresponds a sequence (B_i) such that relation (15) is satisfied. We now put $b_i = B_i/2$, $i = 0, 1, 2, \ldots$ Since $A_0 = a_0^2$

$=1$ we have $B_0 = A_0 + 1 = 2$ and so $b_0 = 1$. Relations (15) and (25) now imply (24). The lemma is proved.

We can now prove our main result.

THEOREM. *There exists a B_0-algebra A for which relations* (12) *hold true with $C_i = 2$, and which consequently has no generalized topological divisors of zero.*

Proof. We shall construct a matrix $a_i^{(n)}$, $i = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$, satisfying relations (7), (8), and (9) such that for the seminorms (10) formulas (12) hold true with $C_i = 2$. To this end take as the first row $a_i^{(1)}$ an arbitrary sequence a of positive entries with $a_0 = 1$. According to Lemma 3, for the sequence a there is a sequence b with $b_0 = 1$ which together with a satisfies relations (22), (23) and (24). Put $(a_i^{(2)}) = b$. Then, inductively, if we are given $(a_i^{(n-1)})$, we take it as a in Lemma 3 and call $(a_i^{(n)})$ the resulting b. We have $a_0^{(n)} = 1$ for all n, so that the resulting matrix satisfies relation (7). Relations (22) and (23) now imply relations (8) and (9). Relations (24) imply relations (12), with $C_i = 2$, for all polynomials. Approximating elements of the constructed matrix algebra by polynomials we obtain, by continuity, relations (12) for arbitrary elements x and y of the algebra in question. The conclusion follows.

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INSTITUTO DE MATEMATICAS, UNIVERSIDAD NACIONAL AUTONOMA DE MÉXICO
and
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

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