A $B_0$-algebra without generalized topological divisors of zero

by

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Abstract. We construct a commutative algebra possessing the property announced in the title. This solves in the negative a problem posed in [3] and [6].

1. Introduction. All algebras considered in this paper are commutative algebras with unit elements. A topological algebra is a topological linear space together with a jointly continuous associative multiplication making it an algebra over $C$. Thus if $A$ is a topological algebra and $\Phi$ is a basis of neighbourhoods of the origin in $A$, then for each $U$ in $\Phi$ there is a neighbourhood $V \in \Phi$ such that

$$V^2 \subseteq U.$$  

(1)

A locally convex algebra is a topological algebra which is a locally convex space. The topology of such an algebra $A$ can be introduced by means of a family $(\|x\|_i)_{i \in \mathbb{N}}$, $x \in A$, of seminorms such that for each index $x$ there is an index $i$ with

$$\|x\|_i \leq \|x\|_j$$

for all $x, y \in A$. Relations (2) follow clearly from relations (1). We can also assume that for each finite subset $\{x_1, \ldots, x_n\} \subseteq A$ there is an index $i \in \mathbb{N}$ such that

$$\|x_i\|_i \leq \|x\|_i$$

for all elements $x$ in $A$ and $i = 1, \ldots, n$.

A $B_0$-algebra is a locally convex algebra which is a $B_0$-space, i.e. a completely metrizable locally convex space. Its topology can be introduced by means of a sequence $(\|x\|_i)$ of seminorms satisfying

$$\|x\|_1 \leq \|x\|_2 \leq \ldots$$

(4)

* This paper was written during the second author's stay at the Instituto de Matemáticas, Universidad Nacional Autónoma de México, in the summer of 1984.
for all $x$ in $A$, which corresponds to relations (3), and

$$
\|xy\| \leq \|x\| \|y\|, \quad i = 1, 2, \ldots,
$$

for all elements $x$ and $y$ in $A$, which corresponds to relations (2).

A locally convex algebra (in particular a $B_0$-algebra) $A$ is said to be

**locally multiplicatively convex** (shortly $m$-convex) if relations (2) or (5) can be replaced by

$$
\|xy\| \leq \|x\| \|y\|,
$$

for all elements $x$ and $y$ in $A$ and all indices $i$ in $\mathbb{N}$.

In particular, all Banach algebras are $m$-convex algebras.

One of the basic concepts in the theory of Banach algebras is the concept of,
a **topological divisor of zero**. This is a nonzero element $x$ of the algebra in
question such that there exists a sequence of elements $x_n$, $\|x_n\| = 1$, with

$$
limit_{n \to \infty} x_n x = 0.
$$

This concept is due to G. E. Shilov, who proved that a Banach algebra either possesses topological divisors of zero or is isomorphic to the field of complex numbers. An analogous theorem, however, fails for general locally convex algebras, even for $m$-convex $B_0$-algebras: in the algebra $E$ of all entire functions, provided with the pointwise algebra operations and the topology of uniform convergence on compacts, the relations $\lim x_n x = 0$ and $x \neq 0$ imply $\lim x_n = 0$ (cf. [4] and [6]). In view of this example the second author introduced in [3] a more general concept of generalized topological divisors of zero (for the definition cf. Section 2) and proved that any topological algebra containing the field of rational functions always possesses such divisors. He proved later in [4] (for complex algebras) and in [5] (for real algebras) that an $m$-convex algebra either possesses generalized topological divisors of zero or is isomorphic to the field of complex numbers (in the case of complex algebras), or to one of the three finite-dimensional division algebras over $\mathbb{R}$ (reals, complex numbers, or quaternions). In [3] and [6] a general conjecture is posed that an arbitrary topological algebra $A$ either possesses generalized topological divisors of zero or is isomorphic to the field of complex numbers (or to one of the three standard finite-dimensional division algebras in the real case).

The purpose of the present note is to disprove this conjecture. We are going to construct a commutative $B_0$-algebra possessing no generalized topological divisors of zero. We shall construct our example in the form of a matrix algebra. This is an algebra of power series in one variable, with complex coefficients, whose seminorms are given by means of a matrix $d^n$, $n = 1, 2, \ldots$, $i = 0, 1, 2, \ldots$, of nonnegative real numbers satisfying the following conditions:

$$
da^n \leq d_i^{n+1}
$$

for $n = 1, 2, \ldots$, and

$$
da_j^0 \leq d_i^{n+1}
$$

for $i, j = 0, 1, 2, \ldots$, and $n = 1, 2, \ldots$

For a power series $x = \sum_{i=0}^{\infty} c_i z^i$ we put

$$
\|x\|_n = \sum_{i=0}^{\infty} d^n |c_i|,
$$

and the matrix algebra associated with the matrix $(d^n)$ is the algebra of all power series for which the seminorms (10) are finite. One can easily see that this algebra is a $B_0$-space, i.e. it is complete. Relation (9) follows immediately from relation (8), while relation (4) follows from (8). Relation (7) means that for each seminorm (10) we have $\|e\|_n = 1$, where $e$ is the unit element of

the algebra in question. For more details on the above the reader is referred to [1]–[7].

2. Generalized topological divisors of zero. Let $A$ be a topological algebra

and $\Phi$ a basis of neighbourhoods of its origin. $A$ is said to possess

generalized topological divisors of zero if for some $U \in \Phi$ we have

$$
0 \notin (A \setminus U)^2,
$$

where the bar denotes the closure. In other words, generalized topological divisors of zero are sets $S_1, S_2 \subset A$ such that $0 \notin S_1 \cup S_2$, but $0 \in S_1 \cap S_2$. If one of the sets $S_i$ consists of a single point $x$, then $x$ is a topological divisor of zero. If the sets $S_1$ and $S_2$ both consist of single points, then they are divisors of zero. If $A$ is a locally convex algebra with topology given by means of a family $(\|x\|_e)$, $x \in A$, of seminorms satisfying (2) and (3), then $A$ possesses generalized topological divisors of zero if and only if there is an index $e$ in $\mathbb{N}$ such that

$$
\inf \{ \|x\|_e : \|x\|_e = \|y\|_e = 1 \} = 0
$$

for each $e \in \mathbb{N}$. Thus there are no generalized topological divisors of zero if and only if for each index $e$ in $\mathbb{N}$ there is an index $f \in \mathbb{N}$ and a positive constant $C_f$ such that the relations $\|x\|_e = \|y\|_e = 1$ imply $\|xy\| \geq C_f$, or, equivalently,

$$
\|xy\|_f \geq C_f \|x\|_e \|y\|_e
$$

for all elements $x$ and $y$ in $A$. Since for a $B_0$-algebra any subsequence of the sequence (4) gives the same topology, we can state by (11) that a $B_0$-algebra has no generalized topological divisors of zero if and only if its topology is
given by means of a sequence of seminorms satisfying relations (4), (5) and
\[ \|x\|_i \leq C_i \|x\|_{i+1}, \]
for i = 1, 2, ..., and all elements x and y in A. In the next section we shall construct a matrix algebra A whose seminorms (10) satisfy relations (12) with all C_i equal to 2.

3. Construction of an example. Let p be a monic polynomial in the variable t with complex coefficients. So \( p(t) = t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0. \)
We can also write \( p(t) = (t + \lambda_1)(t + \lambda_2) \cdots (t + \lambda_n); \) it is more convenient to use here the roots of polynomials equipped with the sign "-". Denote by \( \beta \) the polynomial with roots equal to \(-|\lambda_i|\), where \(-|\lambda_i|\) are the roots of the polynomial \( p \). The coefficients \( a_i \) of \( p \) and \( \beta \) are expressed by means of \( \lambda_i \) as
\[ a_i^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n = k} |\lambda_{i_1}| \lambda_{i_2} \cdots \lambda_{i_n}. \]
k = 1, 2, ..., n - 1, and similarly
\[ \beta_i^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n = k} |\lambda_{i_1}| \lambda_{i_2} \cdots \lambda_{i_n}. \]
where \( n \) is the degree of \( p \) and \( \beta \). We also set \( a_i^{(n)}(\lambda_1, \ldots, \lambda_n) = 1 \) if \( a_i^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) = 0 \) for all natural \( n \).

Our construction is based upon the following

Lemma 1. For each sequence \( A_0, A_1, \ldots \) of positive real numbers there is a sequence \( B_0, B_1, \ldots \) of positive real numbers such that for each natural \( n \) and all complex numbers \( \lambda_1, \ldots, \lambda_n \) we have
\[ A_0 B_0^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) + A_1 B_1^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) + \cdots + A_{n-1} B_{n-1}^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) + A_n B_n^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n) \]
\[ \leq B_0 |a_1^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n)| + B_1 |a_2^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n)| + \cdots + B_n |a_n^{(n)}(\lambda_1, \lambda_2, \ldots, \lambda_n)| + B_{n+1}, \]
where the functions \( a_i^{(n)} \) and \( B_i^{(n)} \) are given by (13) and (14) respectively.

Proof. We shall define the numbers \( B_i \) in an inductive way, starting with \( n = 0 \) and setting \( B_0 = A_0 + 1. \) Suppose that we have already defined the numbers \( B_0, B_1, \ldots, B_{n-1} \) in such a way that for each \( k, 0 \leq k \leq n, \) we have
\[ \sup_{\lambda_1, \ldots, \lambda_k} \left( A_0 B_0^{(k)}(\lambda_1, \ldots, \lambda_k) + \cdots + A_k B_k^{(k)}(\lambda_1, \ldots, \lambda_k) - B_0 |a_1^{(k)}(\lambda_1, \lambda_2, \ldots, \lambda_k)| - \cdots - B_k |a_k^{(k)}(\lambda_1, \lambda_2, \ldots, \lambda_k)| \right) \leq B_k - 1. \]

If we show that the left-hand supremum in (16) is finite and equals \( C \) for \( k = n, \) then setting \( B_n = C + 1 \) we obtain (16) for \( k = n. \) So by induction we have (16) for all \( n, \) which, in turn, implies (15) for all \( n \) and we are done.

Denote by \( \Phi_s(\lambda_1, \ldots, \lambda_n) \) the expression under the supremum sign on the left side of (16). Assume that for a certain choice of \( \lambda_1^{(p)}, \ldots, \lambda_n^{(p)}, s = 1, 2, \ldots, \) we have
\[ \lim_{s \to \infty} \Phi_s(\lambda_1^{(p)}, \ldots, \lambda_n^{(p)}) = \infty. \]
We have to show that this leads to a contradiction. Since the role of variables \( \lambda_1^{(p)}, \ldots, \lambda_n^{(p)} \) in (17) is symmetric, we can assume without loss of generality that for each \( s \) we have
\[ \lambda_1^{(p)} \leq \cdots \leq \lambda_n^{(p)}. \]
Since \( \Phi_s \) is a continuous function in its variables, it is bounded on each bounded subset of \( C^n. \) Therefore we must have \( \lim_{s \to \infty} |\lambda_1^{(p)}| = \infty. \) Denote by \( k \) the smallest integer for which we have \( \lim_{s \to \infty} |\lambda_{k+1}^{(p)}| = \infty. \) Thus we have \( |\lambda_p^{(p)}| = \infty \) for \( k < p \leq n \) and \( |\lambda_p^{(p)}| \leq M \) for some positive \( M \) if \( p \leq k. \) Note that \( 0 \leq k < n, \) so that it can happen that \( \lim_{s \to \infty} |\lambda_{k+1}^{(p)}| = \infty \) for all \( p = 1, 2, \ldots, n. \)

Consider now the functions \( \beta_i^{(n)}(\lambda_1, \ldots, \lambda_n). \) If \( p \leq k, \) we can write them as
\[ \beta_i^{(n)}(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < i_2 < \cdots < i_k} |\lambda_{i_1}| \lambda_{i_2} \cdots \lambda_{i_k}. \]
where the second sum less than \( n - k \) of the indices \( j_1, \ldots, j_{k-1} \) are greater than \( k. \) This implies, in particular, that
\[ \lim_{s \to \infty} |\lambda_1^{(p)}| \cdots |\lambda_n^{(p)}| = 0 \]
for all such \( (n-p) \)-tuples \( (j_1, \ldots, j_{n-p}). \) Assuming the notation \( o(1) \) for an expression depending upon \( s \) which tends to zero as \( s \to \infty, \) and taking into account the fact that for \( p > k \) in formula (14) (with \( p \) instead of \( k \)) only such \( (n-p) \)-tuples occur, we can rewrite (18) as
\[ \beta_i^{(n)}(\lambda_1^{(p)}, \ldots, \lambda_n^{(p)}) = |\lambda_1^{(p)}| \cdots |\lambda_{k-1}^{(p)}| |\lambda_k^{(p)}| \cdots |\lambda_n^{(p)}| + o(1) \]
for \( 0 \leq p \leq k, \)
\[ |\lambda_1^{(p)}| \cdots |\lambda_{k-1}^{(p)}| o(1) \]
for \( k < p \leq n. \)

In exactly the same way we obtain
\[ a_i^{(n)}(\lambda_1^{(p)}, \ldots, \lambda_n^{(p)}) = |\lambda_1^{(p)}| \cdots |\lambda_{k-1}^{(p)}| |\lambda_k^{(p)}| \cdots |\lambda_n^{(p)}| + o(1) \]
for \( 0 \leq p \leq k, \)
\[ |\lambda_1^{(p)}| \cdots |\lambda_{k-1}^{(p)}| o(1) \]
for \( k < p \leq n. \)
Denote by $\|p\|_s$ the seminorm defined on the algebra of all polynomials $p = a_n t^n + \ldots + a_1 t + a_0$ by the formula

$$\|p\|_s = \sum_{k} |a_k|.$$  

With this notation we have the following.

**Lemma 3.** The sequence $b$ of Lemma 2 can be chosen so that it has in addition the following property: for all polynomials $p$ and $q$ in one variable

$$\|p\|_b < 2\|q\|_b.$$  

**Proof.** Observe first that if for some $b$ relations (22) and (23) are satisfied, then they are also satisfied if we replace $b$ by any coarselywise larger sequence. The same applies to relation (24). Thus it is sufficient to find a sequence $b$ with $b_n = 1$ which satisfies relation (24). For, if $b' = (\beta_0, \beta_1, \ldots)$ satisfies (22) and (23) and $b'' = (\beta_0, \beta_n = 1$, satisfies (24), then the sequence $b$ with $b_n > \max \{b'_n, b''_n\}$ satisfies (22), (23) and (24), and $b_n = 1$.

One can easily see that it is sufficient to construct a sequence $b$ for which relation (24) is satisfied for monic polynomials. Let $p(t)$ and $q(t)$ be such polynomials. We can write

$$p(t) = (t + \lambda_1)(t + \lambda_2)\ldots(t + \lambda_n),$$

$$q(t) = (t + \lambda_{n+1})(t + \lambda_{n+2})\ldots(t + \lambda_{2n}).$$

We have

$$\|p\|_b = |\lambda_1 + \ldots + \lambda_n| + \sum_{k=1}^{n} |\lambda_1 + \ldots + \lambda_k - 1| + \ldots + |\lambda_1 + \ldots + \lambda_m - 1| + \ldots + |\lambda_1 + \ldots + \lambda_{k+1} - 1| + \ldots + a_k,$$

$$\|q\|_b = |\lambda_{n+1} + \ldots + \lambda_n| + \sum_{k=1}^{n} |\lambda_{n+1} + \ldots + \lambda_k - 1| + \ldots + |\lambda_{n+1} + \ldots + \lambda_{k+1} - 1| + \ldots + a_{k+1},$$

where $1 < j < k$ and $k + 1 < j < n$. Thus

$$\|p\|_b < 2\|q\|_b.$$  

**Remark.** If for a given polynomial $p$ we define $\beta$ as the polynomial whose roots are equal to $-|\lambda|$, where $\lambda$ are the roots of $p$, and whose leading coefficient equals $|\lambda|$, where $a_0$ is the leading coefficient of the polynomial $p$, then formula (15) is valid also for arbitrary polynomials.

For the sake of completeness we repeat here the proof of the following simple lemma proved in ([8], Lemma 3).

**Lemma 2.** Let $a = (a_n), n = 0, 1, \ldots$, be a sequence of positive real numbers with $a_0 = 1$. There exists a sequence $b = (b_n), n = 0, 1, \ldots$, of positive real numbers with $b_0 = 1$ such that

$$a_{i+j} \leq b_i b_j$$  

for all nonnegative integers $i$ and $j$. In particular,

$$a_i \leq b_i$$  

for all $i$.  

**Proof.** Put $b_0 = 1$ and suppose that the numbers $b_1, \ldots, b_{n-1}$ are already constructed. Then put

$$b_n = \max \{a_n a_{n-1}/b_1, a_{n-2}/b_2, \ldots, a_{2n-1-1}/b_{2n-1}, a_{n-1}^2 \}.$$  

Let $s = (s_n), n = 0, 1, \ldots$, be a sequence of nonnegative real numbers.
Theorem. There exists a $B_0$-algebra $A$ for which relations (12) hold true
with $C_1 = 2$, and which consequently has no generalized topological divisors of
zero.

Proof. We shall construct a matrix $a_{ij}^n$, $i = 0, 1, 2, \ldots, n = 1, 2, \ldots,$
satisfying relations (7), (8), and (9) such that for the seminorms (10) formulas
(12) hold true with $C_1 = 2$. To this end take as the first row $a_1^n$ an arbitrary
sequence of positive entries with $a_0 = 1$. According to Lemma 3, for the
sequence $a$ there is a sequence $b$ with $b_0 = 1$ which together with $a$ satisfies
relations (22), (23) and (24). Put $a_{11}^{(2)} = b$. Then, inductively, if we are given
$(a_{ij}^{(n-1)}$, we take it as $a$ in Lemma 3 and call $(a_{ij}^{(n)})$ the resulting $b$. We have
$a_{ij}^{(n)} = 1$ for all $n$, so that the resulting matrix satisfies relation (7). Relations
(22) and (23) now imply relations (8) and (9). Relations (24) imply relations
(12), with $C_1 = 2$, for all polynomials. Approximating elements of the
constructed matrix algebra by polynomials we obtain, by continuity, relations (12) for arbitrary elements $x$ and $y$ of the algebra in question. The
conclusion follows.

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Received August 13, 1984

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