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Weighted inequalities for the two-dimensional Hardy operator*

by

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Abstract. A characterization is obtained for those pairs of weight functions w, v on $(0, \infty)^2$ for which the two-dimensional Hardy operator $I_2 f(x, y) = \int_0^x \int_0^y f(s, t)\, ds\, dt$ is bounded from $L^p(v)$ to $L^q(w)$, $1 < p \leqslant q < \infty$. Related results and some applications are discussed.

§ 1. Introduction. For $n \geqslant 1$, the n-dimensional Hardy operator I_n and its adjoint I_n^* are given by

$$I_n f(x_1, \ldots, x_n) = \int_0^{x_1} \ldots \int_0^{x_n} f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n,$$

$$I_n^* f(x_1, \ldots, x_n) = \int_{x_1}^{\infty} \ldots \int_{x_n}^{\infty} f(t_1, \ldots, t_n)\, dt_1 \ldots dt_n$$

for $x_1, \ldots, x_n > 0$ and suitable functions f. These operators often arise in n-dimensional weighted norm inequalities for classical operators (such as Fourier and double Hilbert transforms) in much the same way that I_1 and I_1^* arise in one-dimensional inequalities (see the survey article by B. Muckenhoupt in [8] and the applications discussed below).

In the one-dimensional case, much is already known concerning weighted inequalities for the Hardy operator. For example, if $1 < p \leqslant q < \infty$, then ([2], [3], [7], [17] and [18])

$$(1.1) \qquad \Big(\int_0^{\infty} (I_1 f(x))^q w(x)\, dx\Big)^{1/q} \leqslant C\Big(\int_0^{\infty} f(x)^p v(x)\, dx\Big)^{1/p} \quad \text{for all } f \geqslant 0$$

if and only if the nonnegative weight functions $w(x)$, $v(x)$ satisfy

$$(1.2) \qquad \sup_{a>0} I_1^* w(a)^{1/q}\, I_1 \sigma(a)^{1/p'} = A < \infty$$

where $\sigma(y) = v(y)^{1-p'}$ and by convention $0 \cdot \infty = 0$. Moreover, if A and C are the least such constants, then $A \leqslant C \leqslant p^{1/q}(p')^{1/p'} A$. We remark that by

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Theorem 1 of [1] condition (1.2) is also necessary and sufficient for the weak type inequality (we write $|E|_w$ for $\int_E w(x)\,dx$)

$$(1.3)\qquad |\{I_1 f > \lambda\}|_w \leqslant C\lambda^{-q}\left(\int f^p v\right)^{q/p} \quad \text{for all } f \geqslant 0 \text{ and } \lambda > 0.$$

More general weak type variants of (1.1) are also considered by K. Andersen and B. Muckenhoupt in [1] and a Lorentz norm analogue is considered by the author in [15].

Unfortunately, weighted inequalities for the higher-dimensional Hardy operators I_n seem more complicated and the difficulties increase with larger n. As a result we concentrate on the two-dimensional Hardy operator I_2, obtaining characterizations of both the strong and weak type inequalities (answering a question of B. Muckenhoupt in [8]) along with applications to some classical operators. We now describe our results. Proofs are given in § 2 and § 3.

In ([8]; p. 72), B. Muckenhoupt observed that the analogue of condition (1.2),

$$(1.4)\qquad \sup_{a,b>0} I_2^* w(a, b)^{1/q} I_2 \sigma(a, b)^{1/p'} = A < \infty$$

(where $\sigma = v^{1-p'}$) is necessary for the weighted inequality

$$(1.5)\qquad \left(\int_0^\infty\int_0^\infty |I_2 f|^q w\right)^{1/q} \leqslant C\left(\int_0^\infty\int_0^\infty |f|^p v\right)^{1/p} \quad \text{for all } f \geqslant 0$$

but pointed out that (1.4) is no longer sufficient for (1.5) (see B. Muckenhoupt [9] and also example 1 in § 4 below).

In order to derive additional necessary conditions for (1.5), we replace f by $f\sigma$ in (1.5) and rewrite this inequality in the natural form

$$(1.5)'\qquad \left(\int_0^\infty\int_0^\infty [I_2(f\sigma)]^q w\right)^{1/q} \leqslant C\left(\int_0^\infty\int_0^\infty f^p \sigma\right)^{1/p} \quad \text{for all } f \geqslant 0,$$

which by duality is equivalent to

$$(1.5)''\qquad \left(\int_0^\infty\int_0^\infty [I_2^*(gw)]^{p'} \sigma\right)^{1/p'} \leqslant C\left(\int_0^\infty\int_0^\infty g^{q'} w\right)^{1/q'} \quad \text{for all } g \geqslant 0.$$

If we now set $f = \chi_{(0,a)\times(0,b)}$ and $g = \chi_{(a,\infty)\times(b,\infty)}$ in (1.5)′ and (1.5)″ respectively and then restrict integration to the corresponding rectangle on the left, we obtain the following necessary conditions for (1.5) with $A = C$:

$$(1.6)\qquad \int_0^a\int_0^b (I_2\sigma)^q w \leqslant A^q [I_2\sigma(a, b)]^{q/p} \quad \text{for all } a, b > 0,$$

$$(1.7)\qquad \int_a^\infty\int_b^\infty (I_2^* w)^{p'} \sigma \leqslant A^{p'} [I_2^* w(a, b)]^{p'/q'} \quad \text{for all } a, b > 0.$$

Our main result is

THEOREM 1. *Suppose* $1 < p \leqslant q < \infty$ *and that* w *and* v *are nonnegative weights on* $(0, \infty)^2$.

(A) *The strong type inequality* (1.5) *holds if and only if* (1.4), (1.6) *and* (1.7) *hold.*

(B) *The weak type inequality*

$$(1.8)\qquad |\{I_2 f > \lambda\}|_w \leqslant C\lambda^{-q}\left(\int_0^\infty\int_0^\infty f^p v\right)^{q/p} \quad \text{for all } f \geqslant 0 \text{ and } \lambda > 0$$

holds if and only if (1.4) *and* (1.7) *hold.*

Examples are given in § 4 to show that (1.4) is not sufficient even for (1.8) and that no two of the conditions (1.4), (1.6) and (1.7) are by themselves sufficient for (1.5). In particular, we see that, unlike the case for I_1, the weak and strong type inequalities for I_2 are not equivalent.

Remark 1. The analogue of this theorem for the adjoint operator I_2^* can easily be obtained as follows. Define $f^*(x, y) = x^{-2} y^{-2} f(x^{-1}, y^{-1})$ for $x, y > 0$, $f \geqslant 0$. Then $I_2^* f(x, y) = I_2(f^*)(x^{-1}, y^{-1})$ and it is easily verified that the weight pair (w_1, v_1) satisfies (1.5), respectively (1.8), with I_2^* in place of I_2, if and only if the pair (w_2, v_2) satisfies (1.5), respectively (1.8), where $w_2 = w_1^*$ and $\sigma_2 = \sigma_1^*$. A similar comment applies to each of the mixed Hardy operators $T_1 f(x, y) = \int_0^x\int_y^\infty f$ and $T_2 f(x, y) = \int_x^\infty\int_0^y f$.

We now indicate applications of Theorem 1 to the Fourier transform, double Hilbert transform and the strong maximal function. The following result of R. Kerman and the author relates a weighted inequality for the Fourier transform to a weighted inequality for I_2 (cf. W. B. Jurkat and G. Sampson [6], H. Heinig [4] and B. Muckenhoupt [10]). Let $\hat{f}$ denote the Fourier transform of f in L^1.

PROPOSITION 1 (R. Kerman and E. Sawyer). *Suppose* $w(x, y)$ *and* $v(x, y)$ *are symmetric about the coordinate axes,* w *is decreasing and* v *increasing in each variable separately on* $(0, \infty)^2$, *and* $\partial^2 w/\partial x\,\partial y \geqslant 0$ *on* $(0, \infty)^2$. *Then*

$$(1.9)\qquad \int_{R^2} |\hat{f}|^2 w \leqslant C \int_{R^2} |f|^2 v \quad \text{for all } f \text{ in } L^1$$

if and only if

$$(1.10)\qquad \int_0^\infty\int_0^\infty (I_2 f)^2 w^* \leqslant C \int_0^\infty\int_0^\infty f^2 v \quad \text{for all } f \geqslant 0$$

where $w^*(x, y) = x^{-2} y^{-2} w(x^{-1}, y^{-1})$.

A proof is sketched in § 3. Combining this Proposition with Theorem 1 (A) yields a set of necessary and sufficient conditions for the weighted

Fourier transform inequality, (1.9), in the case of "sectionally monotone" weights. In particular, this recovers results of [12] for weights of the form $w(x, y) = |x|^{-\alpha}|y|^{-\beta}$. As observed by R. Kerman, such results cannot be obtained from [4], [6] or [10] since the nonincreasing rearrangement of such $w(x, y)$ may be infinite everywhere on $(0, \infty)$.

In [11], B. Muckenhoupt and R. L. Wheeden showed how to obtain two-weight inequalities for the Hilbert transform and the maximal function on $\boldsymbol{R}$ from the weighted inequalities for the 1-dimensional Hardy operators I_1 and I_1^*. In particular, necessary and sufficient conditions were obtained for weights satisfying suitable monotonicity or growth restrictions. Their techniques work just as well for the double Hilbert transform and the strong maximal function on $\boldsymbol{R}^2$, with the 2-dimensional Hardy operators I_2, I_2^*, T_1 and T_2 (see Remark 1 for definitions) in place of I_1 and I_1^*. Theorem 1 and Remark 1 then yield lists of sufficient conditions for the weighted double Hilbert transform and strong maximal function inequalities. The theorems are too long to state here (note that Remark 1 yields 3 conditions corresponding to each of the 4 Hardy operators).

Our final application of Theorem 1 is to the σ-averaging operator $A_\sigma f(x, y) = \left(\int_0^x\int_0^y \sigma\right)^{-1} \int_0^x\int_0^y f\sigma$, defined for f, $\sigma \geqslant 0$. An immediate corollary of Theorem 1(A) is that A_σ is bounded on $L^p(\sigma)$ $(1 < p < \infty)$ if and only if

$$\int_a^\infty\int_b^\infty (I_2\sigma)^{-p}\sigma \leqslant A\, I_2\sigma(a, b)^{1-p} \quad \text{for all } a, b > 0. \tag{1.11}$$

Note that (1.11) is simply (1.4) with $p = q$ and $w = (I_2\sigma)^{-p}\sigma$. Inequality (1.6) is vacuous in this case and (1.7) is a consequence of (1.4). Of course, if σ is a product weight, $\sigma(x, y) = \sigma_1(x)\sigma_2(y)$, then (1.11) is seen to hold by performing the integrations on the left.

We close this introduction with an interesting property of the weight pairs satisfying (1.5) that is not obvious from the characterization in Theorem 1 – namely that (1.5) depends only on the size of $I_2^* w$ and $I_2\sigma$.

PROPOSITION 2. *Suppose $1 < p, q < \infty$ and that the pair of weights (w_2, v_2) satisfies* (1.5). *If $I_2^* w_1 \leqslant I_2^* w_2$ and $I_2\sigma_1 \leqslant I_2\sigma_2$ $(\sigma_i = v_i^{1-p'})$, then (w_1, v_1) also satisfies* (1.5).

§ 2. Proof of Theorem 1. We have already shown that (1.6) and (1.7) are necessary for (1.5). If $I_2\sigma(a, b) < \infty$, then (1.4) follows from (1.5)′ by setting $f = \chi_{(0,a)\times(0,b)}$, restricting integration on the left to $(a, \infty)\times(b, \infty)$, and then dividing both sides by $I_2\sigma(a, b)^{1/p}$. If $I_2\sigma(a, b) = \infty$, then there exists $f \geqslant 0$ such that $I_2(f\sigma)(a, b) = \infty$ and $\int_0^\infty\int_0^\infty f^p\sigma < \infty$, and so (1.5)′ implies $\int_a^\infty\int_b^\infty w = 0$ which yields (1.4). This argument also shows that (1.4) is necessary for the weak type inequality (1.8). Finally, (1.8) is equivalent, by duality, to the inequality in (1.5)″ holding for all $g = \chi_E$, E a measurable subset of $(0, \infty)$ ([16]; Ch. V, § 3). Thus (1.7) is necessary for (1.8) and this completes the proof of the necessity results.

Conversely, suppose that (1.4), (1.6) and (1.7) hold. We show (1.5) by establishing the following inequality:

$$\iint (I_2 f)^q w \leqslant CA\left(\iint f^p v\right)^{1/p}\left(\iint (I_2 f)^q w\right)^{1/q'} + CA^q\left(\iint f^p v\right)^{q/p} \tag{2.1}$$

for all f satisfying

(2.2) $f \geqslant 0$, f bounded, f has compact support contained in $\{I_2\sigma > 0\}$, $\iint f^p v < \infty$.

With this done, we obtain (1.5) as follows. If $K = \operatorname{supp} f$ is compact and contained in $\{I_2\sigma > 0\}$, then we can find finitely many points (ξ_k, η_k), $1 \leqslant k \leqslant m$, such that $I_2\sigma(\xi_k, \eta_k) > 0$ for $1 \leqslant k \leqslant m$ and $K \subset H = \bigcup_{k=1}^m [(\xi_k, \infty)\times(\eta_k, \infty)]$. Then $\{I_2 f > 0\}$ is also contained in H and (1.4) yields

$$|\{I_2 f > 0\}|_w \leqslant |H|_w \leqslant \sum_{k=1}^m \left(\frac{A^{p'}}{I_2\sigma(\xi_k, \eta_k)}\right)^{q/p'} < \infty.$$

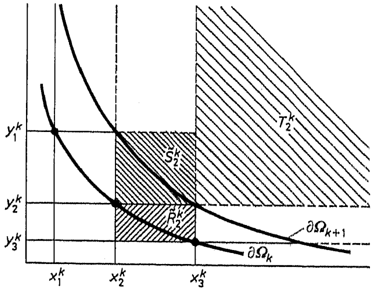
Thus we see that the left side of (2.1) is finite for f satisfying (2.2). For convenience let $a = \left(\iint (I_2 f)^q w\right)^{1/q}$ and $b = A\left(\iint f^p v\right)^{1/p}$. Then $a^q \leqslant C(a^{q-1}b + b^q)$ which implies either $a \leqslant b$ or $a^q \leqslant 2C\, a^{q-1} b$, and so $a \leqslant \max\{1, 2C\}\, b$. This establishes (1.5) for f satisfying (2.2) and a limiting argument yields (1.5) for all $f \geqslant 0$.

It remains to show (2.1) for f satisfying (2.2). Fix such an f and let $\Omega_k = \{I_2 f > 3^k\}$ (the base 3 plays a significant role in (2.11) below). We begin with

$$\iint (I_2 f)^q w \leqslant C_q \sum_k 3^{kq} |\Omega_{k+2} - \Omega_{k+3}|_w. \tag{2.3}$$

To estimate $|\Omega_{k+2} - \Omega_{k+3}|_w$, we introduce rectangles as follows. Fix k for the moment with $\Omega_{k+1} \neq \emptyset$. Choose points (x_j^k, y_j^k), $1 \leqslant j \leqslant N = N_k$, lying on $\partial\Omega_k$ such that (x_j^k, y_{j-1}^k) lies on $\partial\Omega_{k+1}$ for $2 \leqslant j \leqslant N$ and $\Omega_{k+1} \subset \bigcup_{j=1}^N S_j^k$ where S_j^k is the rectangle $(x_j^k, \infty)\times(y_j^k, \infty)$. We also define rectangles $\tilde{S}_j^k = (x_j^k, x_{j+1}^k)\times(y_j^k, y_{j-1}^k)$ for $1 \leqslant j \leqslant N$ and $R_j^k = (0, x_{j+1}^k)\times(0, y_j^k)$,

$\tilde{R}_j^k = (x_j^k, x_{j+1}^k)\times(y_{j+1}^k, y_j^k)$ and $T_j^k = (x_{j+1}^k, \infty)\times(y_j^k, \infty)$ for $1 \leqslant j \leqslant N-1$. Define $y_0^k = x_{N+1}^k = \infty$. See the diagram.



Now choose sets $E_j^k \subset T_j^k$ such that $E_j^k \cap E_i^k = \emptyset$ for $j \neq i$ and such that $\bigcup_j E_j^k = (\Omega_{k+2}-\Omega_{k+3}) \cap (\bigcup_j T_j^k)$. Since $\Omega_{k+2}-\Omega_{k+3} \subset \Omega_{k+1} \subset (\bigcup_j T_j^k) \cup (\bigcup_j \tilde{S}_j^k)$, (2.3) gives

(2.4) $$\iint (I_2 f)^q w \leqslant C \sum_{k,j} 3^{kq} |E_j^k|_w + C \sum_{k,j} 3^{kq} |\tilde{S}_j^k \cap (\Omega_{k+2}-\Omega_{k+3})|_w = \mathrm{I}+\mathrm{II}.$$

To estimate II, let $D_j^k = \tilde{S}_j^k - \Omega_{k+3}$. If (x, y) is in $\tilde{S}_j^k \cap (\Omega_{k+2}-\Omega_{k+3})$, then

$$3^{k+2} < I_2 f(x, y) \leqslant I_2 f(x_{j+1}^k, y_j^k) + I_2 f(x_j^k, y_{j-1}^k) + I_2(\chi_{D_j^k} f)(x, y),$$

which implies $I_2(\chi_{D_j^k} f)(x, y) > 3^k$. Thus

$$|\tilde{S}_j^k \cap (\Omega_{k+2}-\Omega_{k+3})|_w \leqslant 3^{-k} \iint I_2(\chi_{D_j^k} f)\, w = 3^{-k} \iint_{D_j^k} f I_2^* w$$
$$\leqslant 3^{-k}\Big(\iint_{D_j^k} f^p v\Big)^{1/p}\Big(\iint_{S_j^k} (I_2^* w)^{p'} \sigma\Big)^{1/p'} \leqslant 3^{-k} A \Big(\iint_{D_j^k} f^p v\Big)^{1/p} |S_j^k|_w^{1/q'}$$

by hypothesis (1.7). Thus

(2.5) $$\mathrm{II} \leqslant CA \sum_{k,j} 3^{k(q-1)} \Big(\iint_{D_j^k} f^p v\Big)^{1/p} |S_j^k|_w^{1/q'}$$
$$\leqslant CA \Big(\sum_{k,j} \Big(\iint_{D_j^k} f^p v\Big)^{q/p}\Big)^{1/q} \Big(\sum_{k,j} 3^{kq} |S_j^k|_w\Big)^{1/q'}$$
$$\leqslant CA \Big(\iint f^p v\Big)^{1/p} \Big(\iint \Big[\sum_{k,j} 3^{kq} \chi_{S_j^k}\Big] w\Big)^{1/q'}$$

since $p \leqslant q$ and $\sum_{k,j} \chi_{D_j^k} \leqslant \sum_k \chi_{\Omega_k - \Omega_{k+3}} \leqslant 3$. We wish to show that the sum in

square brackets on the right side of (2.5) is dominated by $C(I_2 f)^q$. We first note that

(2.6) $$\sum_j \chi_{S_j^k} \leqslant 3^{-k} \chi_{\Omega_k} I_2 f \quad \text{for all } k,$$

which is an immediate consequence of the following computation for $l \geqslant 1$:

$$I_2 f(x_{j+l}^k, y_j^k) = I_2 f(x_j^k, y_j^k) + \sum_{i=0}^{l-1} [I_2 f(x_{j+i+1}^k, y_j^k) - I_2 f(x_{j+i}^k, y_j^k)]$$
$$\geqslant I_2 f(x_j^k, y_j^k) + \sum_{i=0}^{l-1} [I_2 f(x_{j+i+1}^k, y_{j+i}^k) - I_2 f(x_{j+i}^k, y_{j+i}^k)]$$
$$= 3^k + \sum_{i=0}^{l-1} (3^{k+1} - 3^k) = (2l+1)\, 3^k \geqslant 3^k \sum_i \chi_{S_i^k}(x_{j+l}^k, y_j^k).$$

Thus

$$\sum_{k,j} 3^{kq} \chi_{S_j^k} \leqslant \sum_k 3^{k(q-1)} \chi_{\{I_2 f > 3^k\}} I_2 f \quad \text{by (2.6)} \leqslant C_q (I_2 f)^{q-1} I_2 f = C (I_2 f)^q,$$

and using this in (2.5) we obtain

(2.7) $$\mathrm{II} \leqslant CA \Big(\iint f^p v\Big)^{1/p} \Big(\iint (I_2 f)^q w\Big)^{1/q'}.$$

To estimate term I of (2.4), we will need the following consequence of (1.4) and (1.6):

(2.8) $$\int_0^\infty \int_0^\infty [I_2(\chi_{(0,a)\times(0,b)} \sigma)]^q w \leqslant CA^q I_2 \sigma(a, b)^{q/p} \quad \text{for } a, b > 0.$$

To see (2.8), consider the integral on the left over the rectangles $(0, a)\times(0, b)$, $(a, \infty)\times(b, \infty)$, $(0, a)\times(b, \infty)$ and $(a, \infty)\times(0, b)$ separately. The integrals over the first two rectangles are dominated by $A^q I_2 \sigma(a, b)^{q/p}$ by (1.6) and (1.4), respectively. The integrals over the last two rectangles are dominated by $CA^q I_2 \sigma(a, b)^{q/p}$ upon using (1.4) and appealing to the well-known characterization of the weighted norm inequality for the one-dimensional Hardy operator (see (1.1) and (1.2)). The straightforward details are left to the reader. We write

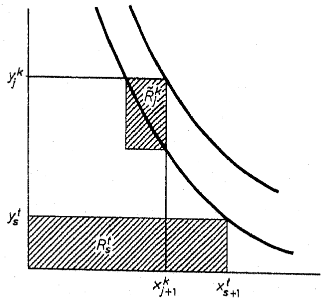
(2.9) $$\mathrm{I} = \sum_{k,j} 3^{kq} |E_j^k|_w = C \sum_{k,j} |E_j^k|_w \Big(\iint_{R_j^k} f\Big)^q$$
$$= C \sum_{k,j} |E_j^k|_w |R_j^k|_\sigma^q \Big(\frac{1}{|R_j^k|_\sigma} \iint_{R_j^k} g\sigma\Big)^q$$

where $g\sigma = f$ or $g = f v^{p'/p}$. Unfortunately, since no two of the R_j^k are disjoint, the map $h \mapsto (|R_j^k|_\sigma^{-1} \iint_{R_j^k} h\sigma)_{(k,j)}$ is not of weak type $(1, q/p)$ with respect to the measure assigning mass $|E_j^k|_w |R_j^k|_\sigma^q$ to the index pair (k, j). Thus we cannot use the argument of ([14]; p. 6 and 7). Instead, we follow the proof of the Marcinkiewicz interpolation theorem and exploit the special geometry of the rectangles R_j^k and their relationship to the function f.

For l an integer, denote by Γ_l the set of pairs of indices (k, j) such that $|E_j^k|_w > 0$ and

$$2^l < \frac{1}{|R_j^k|_\sigma} \iint_{R_j^k} g\sigma, \qquad (k, j) \in \Gamma_l. \tag{2.10}$$

Note that if $|E_j^k|_w > 0$, then $|R_j^k|_\sigma$ is finite by (1.4) and positive since $\iint_{R_j^k} f > 0$ and $\iint f^p v < \infty$. Fix l for the moment and let $\{U_i^l\}_i$ be the maximal rectangles in the collection $\{R_j^k\}_{(k,j)\in\Gamma_l}$. Since $I_2 f$ is bounded, every R_j^k with $(k, j) \in \Gamma_l$ is contained in some U_i^l. While the rectangles $\{U_i^l\}_i$ are not of course disjoint, it is the case that the rectangles $\{\tilde{U}_i^l\}_i$ *are* disjoint (for fixed l).



Here $\tilde{U}_i^l = \tilde{R}_j^k$ if $U_i^l = R_j^k$. Indeed, if $x_{j+1}^k < x_{s+1}^t$ and $y_s^t < y_j^k$ and if, say, $k \geq t$, then $\tilde{R}_j^k$ lies entirely above R_s^t. (If $t \geq k$, then $\tilde{R}_s^t$ lies entirely to the right of R_j^k). To exploit this geometry we will need

$$2^{l-3} < \frac{1}{|U_i^l|_\sigma} \iint_{\tilde{U}_i^l \cap \{g > 2^{l-3}\}} g\sigma, \quad \text{for all } l, i. \tag{2.11}$$

Inequality (2.11) follows from the fact that if $U_i^l = R_j^k$, then

$$\begin{aligned}\iint_{\tilde{U}_i^l} g\sigma = \iint_{\tilde{R}_j^k} f &\geq I_2 f(x_{j+1}^k, y_j^k) - I_2 f(x_j^k, y_j^k) - I_2 f(x_{j+1}^k, y_{j+1}^k) \\ &= 3^{k+1} - 3^k - 3^k = 3^k = \tfrac{1}{3} \iint_{R_j^k} g\sigma > 2^{l-2} |U_i^l|_\sigma \quad \text{by (2.10)}\end{aligned} \tag{2.12}$$

(this accounts for our use of the base 3 rather than 2 in the definition of Ω_k). We now compute that for a fixed l,

$$\begin{aligned}\sum_{(k,j)\in\Gamma_l} |E_j^k|_w |R_j^k|_\sigma^q &\leq \sum_i \sum_{(k,j);R_j^k \subset U_i^l} |E_j^k|_w |R_j^k|_\sigma^q \\ &\leq \sum_i \iint [I_2(\chi_{U_i^l}\sigma)]^q w\end{aligned} \tag{2.13}$$

(since the E_j^k are disjoint and $I_2(\chi_{U_i^l}\sigma) > |R_j^k|_\sigma$ on E_j^k if $R_j^k \subset U_i^l$)

$$\begin{aligned}&\leq CA^q \sum_i |U_i^l|_\sigma^{q/p} && \text{by (2.8)} \\ &\leq CA^q \sum_i \Big(2^{-l} \iint_{\tilde{U}_i^l \cap \{g > 2^{l-3}\}} g\sigma\Big)^{q/p} && \text{by (2.11)} \\ &\leq CA^q \Big(2^{-l} \iint_{\{g > 2^{l-3}\}} g\sigma\Big)^{q/p}\end{aligned}$$

since $q \geq p$ and the $\tilde{U}_i^l$ are disjoint. Combining (2.9) and (2.13) we have

$$\begin{aligned}\mathrm{I} &\leq C \sum_l 2^{lq} \sum_{(k,j)\in\Gamma_l} |E_j^k|_w |R_j^k|_\sigma^q && \text{by (2.9)} \\ &\leq CA^q \sum_l 2^{lq} \Big(2^{-l} \iint_{\{g > 2^{l-3}\}} g\sigma\Big)^{q/p} && \text{by (2.13)} \\ &\leq CA^q \Big(\sum_l 2^{l(p-1)} \iint \chi_{\{g > 2^{l-3}\}} g\sigma\Big)^{q/p} && \text{since } q \geq p \\ &\leq CA^q \Big(\iint g^p \sigma\Big)^{q/p} = CA^q \Big(\iint f^p v\Big)^{q/p}\end{aligned} \tag{2.14}$$

since $\sum_l 2^{l(p-1)} \chi_{\{g > 2^{l-3}\}} \leq C_p g^{p-1}$ for $p > 1$ and since $g^p \sigma = f^p v$. Inequalities (2.4), (2.7) and (2.14) yield (2.1) for f satisfying (2.2), and the proof of part (A) of Theorem 1 is complete.

We now prove that (1.4) and (1.7) imply the weak type inequality (1.8).

Suppose f satisfies (2.2). With notation as above and $\|g\|_{L^{q,\infty}(w)} = \sup_{\lambda>0} \lambda |\{|g|>\lambda\}|_w^{1/q}$, the weak L^q "norm" of g, we have

$$\|I_2 f\|^q_{L^{q,\infty}(w)} \leqslant C_q \sup_k 3^{kq} |\Omega_{k+2}-\Omega_{k+3}|_w$$
$$\leqslant C \sup_k \sum_j 3^{kq} |T_j^k|_w + C \sup_k \sum_j 3^{kq} |\tilde{S}_j^k \cap (\Omega_{k+2}-\Omega_{k+3})|_w$$
$$= \mathrm{III}+\mathrm{IV}.$$

Since $3^{k+1} = \iint_{R_j^k} f \leqslant 3 \iint_{\tilde{R}_j^k} f$ by (2.12), we have

$$\mathrm{III} \leqslant C \sup_k \sum_j \Big(\iint_{\tilde{R}_j^k} f\Big)^q |T_j^k|_w$$
$$\leqslant C \sup_k \sum_j \Big(\iint_{\tilde{R}_j^k} f^p v\Big)^{q/p} |\tilde{R}_j^k|_\sigma^{q/p'} |T_j^k|_w \quad \text{by Hölder's inequality}$$
$$\leqslant C A^q \Big(\iint f^p v\Big)^{q/p} \quad \text{by (1.4)}$$

since $q \geqslant p$ and since, for fixed k, the $\tilde{R}_j^k$ are disjoint. Arguing as for (2.5) above, we obtain

$$\mathrm{IV} \leqslant CA\Big(\iint f^p v\Big)^{1/p} \Big(\sup_k 3^{kq} \sum_j |S_j^k|_w\Big)^{1/q'}.$$

However, inequality (2.6) yields

$$3^{kq} \sum_j |S_j^k|_w \leqslant 3^{k(q-1)} \iint_{\Omega_k} (I_2 f) w$$
$$= 3^{k(q-1)} \int_{3^k}^{\infty} |\{I_2 f > t\}|_w \, dt + 3^{kq} |\Omega_k|_w$$
$$\leqslant 3^{k(q-1)} \int_{3^k}^{\infty} (t^{-1} \|I_2 f\|_{L^{q,\infty}(w)})^q \, dt + 3^{kq} |\Omega_k|_w$$
$$= \frac{q}{q-1} \|I_2 f\|^q_{L^{q,\infty}(w)}$$

and thus $\mathrm{IV} \leqslant CA\big(\iint f^p v\big)^{1/p} \|I_2 f\|^{q-1}_{L^{q,\infty}(w)}$. Altogether then

$$\|I_2 f\|^q_{L^{q,\infty}(w)} \leqslant CA\Big(\iint f^p v\Big)^{1/p} \|I_2 f\|^{q-1}_{L^{q,\infty}(w)} + CA^q \Big(\iint f^p v\Big)^{q/p}$$

and (1.8) is now obtained using the argument following (2.2) above. This completes the proof of Theorem 1.

§ 3. Proofs of Propositions. The key to Proposition 1 is a two-dimensional partial analogue of an inequality of M. Jodeit and A. Torchinsky ([5]; Theorem 4.6). Given $f(x, y)$ measurable on $\boldsymbol{R}^2$, denote by $\tilde{f}(s, t)$ the equimeasurable rearrangement of f on $(0, \infty)^2$ obtained by rearranging first in the "x" variable and then the "y" variable, i.e.

$$\tilde{f}(s, t) = (F_s)^*(t) \quad \text{where} \quad F_s(y) = (f^y)^*(s) \text{ and } f^y(x) = f(x, y).$$

Here g^* denotes the usual nonincreasing rearrangement of g on $(0, \infty)$ (see e.g. [16]; Ch. V, § 3). Set $Ug(x, y) = I_2 g(x^{-1}, y^{-1})$ for $x, y > 0$.

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$$\int_0^S \int_0^T |\hat{f}(\xi, \eta)|^2 \, d\xi \, d\eta \leqslant C \int_0^S \int_0^T [U\tilde{f}(x, y)]^2 \, dx \, dy, \qquad S, T > 0, \tag{3.1}$$

for f in $L^1(\boldsymbol{R}^2)$. Conversely, inequality (3.1) *can be reversed (with a smaller constant C) for f that are symmetric about the coordinate axes and decreasing in each variable separately on* $(0, \infty)^2$.

Proof. The proof of this lemma is a reasonably straightforward adaptation of arguments of W. B. Jurkat and G. Sampson in [6]. We sketch only the main points. Let $D_{S,T}(x, y) = \frac{1}{4ST} \chi_{(-S,S)}(x) \chi_{(-T,T)}(y)$. There are positive constants α, c such that $|(D_{1/S,1/T})^\wedge| \geqslant c$ on $(-\alpha S, \alpha S) \times (-\alpha T, \alpha T)$ and so Plancherel's theorem yields

$$\int_0^{\alpha S} \int_0^{\alpha T} |\hat{f}|^2 \leqslant C \int_{\boldsymbol{R}^2} |(D_{1/S,1/T})^\wedge|^2 |\hat{f}|^2 = C \int_{\boldsymbol{R}^2} |D_{1/S,1/T} * f|^2 \tag{3.2}$$
$$\leqslant C \iiint D_{1/S,1/T}(x-u, y-v) D_{1/S,1/T}(x-s, y-t) \, dx \, dy \times$$
$$\times |f(u, v)| \, |f(s, t)| \, du\, dv\, ds\, dt$$
$$= C \iint L_{S,T}(s-u, t-v) |f(u, v)| \, |f(s, t)| \, du\, dv\, ds\, dt.$$

A result of F. Riesz ([13]), applied one variable at a time, shows that the final integral in (3.2) is dominated by

$$C \iint L_{S,T}(s-u, t-v) f^+(u, v) f^+(s, t) \, du\, dv\, ds\, dt = C \int_{\boldsymbol{R}^2} (D_{1/S,1/T} * f^+)^2$$

where f^+ is the symmetric (with respect to the coordinate axes) rearrangement of $|f|$ on $\boldsymbol{R}^2$ taken first in the "x" variable and then in the "y" variable. This in turn is dominated by

$$C \int_0^\infty \int_0^\infty (D_{1/S,1/T} * \tilde{f})^2 = \int_0^{1/S} \int_0^{1/T} + \int_{1/S}^\infty \int_{1/T}^\infty + \int_0^{1/S} \int_{1/T}^\infty + \int_{1/S}^\infty \int_0^{1/T} = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.$$

Elementary estimates, using the fact that $\tilde{f}$ decreases in each variable separately, show that each of I, II, III and IV is dominated by a constant multiple of $\int_0^S\int_0^T |U\tilde{f}|^2$. For example

$$\mathrm{I} \leqslant \frac{1}{S}\cdot\frac{1}{T}\left[ST\int_0^{1/S}\int_0^{1/T}\tilde{f}\right]^2 \leqslant \int_0^S\int_0^T (U\tilde{f})^2$$

and

$$\mathrm{II} \leqslant \int_{1/S}^{\infty}\int_{1/T}^{\infty}\left(\frac{I_2\tilde{f}(x, y)}{xy}\right)^2 dx\,dy = \int_0^S\int_0^T (U\tilde{f})^2.$$

Inequality (3.1) now follows easily. Conversely, the above inequalities can be reversed for f symmetric about the axes and decreasing in each variable separately on $(0, \infty)^2$. (For (3.2), use a product of Fejér kernels, $K_{S,T}(x, y) = \left(\frac{\sin(x/S)}{x}\right)^2\left(\frac{\sin(y/T)}{y}\right)^2$, in place of $D_{S,T}$.) This completes our sketch of the proof of the lemma.

To obtain the implication $(1.10)\Rightarrow(1.9)$ of Proposition 1, we use integration by parts and the lemma. Suppose $w(x, y)$ is as in Proposition 1. By a limiting argument, we may assume $w(x, y)$ is compactly supported. (If $\theta \in C_c^\infty(\boldsymbol{R})$ satisfies $\theta(0) = 1$, $0 \leqslant \theta(x) \leqslant 1$ and $\theta'(x) \leqslant 0$ for $x \geqslant 0$, then $w_\varepsilon(x, y) = \theta(\varepsilon x)\theta(\varepsilon y)w(x, y)$ has compact support, satisfies the hypotheses of Proposition 1 and increases monotonically to $w(x, y)$ as $\varepsilon \to 0$.) By symmetry, we may assume $\operatorname{supp} f \subset [0, \infty)^2$ and estimate the integral

$$\int_0^\infty\int_0^\infty |\tilde{f}|^2 w = \int_0^\infty\int_0^\infty\left[\int_0^S\int_0^T |\tilde{f}|^2\right]\frac{\partial^2 w}{\partial x\partial y}(S, T)\,dS\,dT$$

$$\leqslant C\int_0^\infty\int_0^\infty\left[\int_0^S\int_0^T (U\tilde{f})^2\right]\frac{\partial^2 w}{\partial x\partial y}(S, T)\,dS\,dT$$

$$= C\int_0^\infty\int_0^\infty (U\tilde{f})^2 w = C\int_0^\infty\int_0^\infty (I_2\tilde{f})^2 w^*$$

$$\leqslant C\int_0^\infty\int_0^\infty \tilde{f}^2 v \leqslant C'\int_0^\infty\int_0^\infty |f|^2 v.$$

The last line follows by (1.10) and because v is increasing in each variable separately.

Conversely, the lemma and (1.9) yield (1.10) for all $f(x, y)$ that decrease in each variable separately on $(0, \infty)^2$, in particular for f of the form $f = v^{-1}I_2^*(gw^*)$, $g \geqslant 0$. But this in turn yields the dual of (1.10) as follows. Let $f = v^{-1}I_2^*(gw^*)$, $g \geqslant 0$. Then

$$\int_0^\infty\int_0^\infty [I_2^*(gw^*)]^2 v^{-1} = \int_0^\infty\int_0^\infty gw^* I_2[v^{-1}I_2^*(gw^*)]$$

$$\leqslant \Big(\int_0^\infty\int_0^\infty g^2 w^*\Big)^{1/2}\Big(\int_0^\infty\int_0^\infty (I_2 f)^2 w^*\Big)^{1/2}$$

$$\leqslant C\,\|g\|_{L^2(w^*)}\Big(\int_0^\infty\int_0^\infty f^2 v\Big)^{1/2}$$

$$= C\,\|g\|_{L^2(w^*)}\Big(\int_0^\infty\int_0^\infty [I_2^*(gw^*)]^2 v^{-1}\Big)^{1/2},$$

which implies $\int_0^\infty\int_0^\infty [I_2^*(gw^*)]^2 v^{-1} \leqslant C\int_0^\infty\int_0^\infty g^2 w^*$ for all $g \geqslant 0$, which is equivalent by duality to (1.10).

We now turn to the proof of Proposition 2. If $1 < q < \infty$, $f \geqslant 0$ is bounded with compact support in $(0, \infty)^2$, and $I_2^* w_1 \leqslant I_2^* w_2$, then integration by parts yields

$$\int_0^\infty\int_0^\infty (I_2 f)^q w_1 = \int_0^\infty\int_0^\infty\left[\frac{\partial^2}{\partial x\partial y}(I_2 f)^q\right]I_2^* w_1$$

$$\leqslant \int_0^\infty\int_0^\infty\left[\frac{\partial^2}{\partial x\partial y}(I_2 f)^q\right]I_2^* w_2 = \int_0^\infty\int_0^\infty (I_2 f)^q w_2$$

since $(\partial^2/\partial x\partial y)(I_2 f)^q \geqslant 0$ for $1 \leqslant q < \infty$ and $f \geqslant 0$. A limiting argument then yields

(3.3) $$\int_0^\infty\int_0^\infty (I_2 f)^q w_1 \leqslant \int_0^\infty\int_0^\infty (I_2 f)^q w_2 \quad \text{for all } f \geqslant 0.$$

Similarly, $I_2\sigma_1 \leqslant I_2\sigma_2$ implies

(3.4) $$\int_0^\infty\int_0^\infty (I_2^* g)^{p'}\sigma_1 \leqslant \int_0^\infty\int_0^\infty (I_2^* g)^{p'}\sigma_2 \quad \text{for all } g \geqslant 0.$$

Combining (3.3) and the hypothesis that (w_2, v_2) satisfies (1.5) we obtain $\|I_2 f\|_{L^q(w_1)} \leqslant C\|f\|_{L^p(v_2)}$ for all $f \geqslant 0$ and so by duality

(3.5) $$\|I_2^* g\|_{L^{p'}(v_2^{1-p'})} \leqslant C\|g\|_{L^{q'}(w_1^{1-q'})} \quad \text{for all } g \geqslant 0.$$

Here we have used the fact that $L^{r'}(u^{1-r'})$ is the dual of $L^r(u)$ under the pairing $\langle f, g\rangle = \iint fg$. From (3.4) and (3.5) we conclude that

$$\|I_2^* g\|_{L^{p'}(\sigma_1)} \leqslant C\|g\|_{L^{q'}(w_1^{1-q'})} \quad \text{for all } g \geqslant 0$$

and duality now yields (1.5) for the pair (w_1, v_1). This completes the proof of Proposition 2.

§ 4. **Counterexamples.** We give three examples here. The first is a pair of weights (w, v) that satisfies Muckenhoupt's condition (1.4) with $p = q = 2$ but not the corresponding weak type inequality (1.8). The second example shows that (1.4) and (1.7) are not sufficient for the strong type inequality (1.5). Duality then shows that (1.4) and (1.6) are not sufficient for (1.5), and the third example shows that (1.6) and (1.7) are insufficient.

Example 1. Let N be a large positive integer and set

$$v(x, y) = \begin{cases} (N+1-x-y)^3 & \text{if } x+y \leqslant N,\\ \infty & \text{if } x+y > N,\end{cases}$$

$$w(x, y) = \begin{cases} 1 & \text{if } N \leqslant x+y \leqslant N+1,\\ 0 & \text{otherwise.}\end{cases}$$

Then with $p = 2$ so that $\sigma = v^{-1}$, we have

$$2I_2\sigma(x, y) = \frac{1}{N+1-x-y} - \frac{1}{N+1-x} - \frac{1}{N+1-y} + \frac{1}{N+1} \leqslant \frac{2}{N+1-x-y},$$

$$I_2^* w(x, y) \leqslant \sqrt{2}(N+1-x-y)$$

for $x+y \leqslant N$ and it follows easily that (1.4) holds with $p = q = 2$. Let $f = (N+1-x-y)\sigma$. Then for $x+y = N$ and $N/4 \leqslant x, y \leqslant 3N/4$ we have $I_2 f(x, y) \geqslant \alpha\log N$ where α is a positive constant independent of N large. Thus $|\{I_2 f > \alpha\log N\}|_w \geqslant N/2$ while $\iint f^2 v = N\log N$ and so (1.8) with $p = q = 2$ fails for sufficiently large N.

Example 2. Let

$$v(x, y) = \begin{cases} 1 & \text{for } xy \leqslant 1,\\ \infty & \text{for } xy > 1.\end{cases}$$

Then with $\sigma = v^{-1}$ we have

(4.1) $$I_2\sigma(x, y) = \begin{cases} xy & \text{for } xy \leqslant 1,\\ 1+\log xy & \text{for } xy \geqslant 1.\end{cases}$$

Now let w be such that $I_2^* w I_2\sigma \equiv 1$, i.e.

(4.2) $$w(x, y) = \begin{cases} x^{-2}y^{-2} & \text{for } xy < 1,\\ 2x^{-1}y^{-1}(1+\log xy)^{-3} & \text{for } xy > 1.\end{cases}$$

For N large set $f_N = \sigma\chi_{[0,N]^2}$. Then

$$\iint f_N^2 v = 1 + 2\log N,$$

$$\iint (I_2 f_N)^2 w \geqslant 2(\log N)\log(1+\log N)$$

and so (1.5) fails (for $p = q = 2$) for large N. However, (1.4) holds by construction and (1.7) is easily verified using (4.1) and (4.2). Indeed,

$$\int_x^\infty\int_y^\infty (I_2^* w)^2\sigma = \int_x^\infty\int_y^\infty (I_2\sigma)^{-2}\sigma = \int_x^\infty\int_y^\infty (st)^{-2}\sigma = \int_x^\infty\int_y^\infty w\sigma \quad \text{by (4.1) and (4.2)}$$
$$\leqslant \int_x^\infty\int_y^\infty w = I_2^* w(x, y).$$

Example 3. Let w, σ be such that $\int_0^\infty\int_0^\infty w = \int_0^\infty\int_0^\infty \sigma = \infty$ and $\operatorname{supp} w \subset \subset (1, 2)\times(1, 2)$ while $\operatorname{supp}\sigma \subset (0, 1)\times(0, 1)$. Then (1.6) and (1.7) hold trivially but (1.4) fails with $a = b = 1$.

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Generalizations of Calderón–Zygmund operators

by

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Dedicated to Professor Sigeru Mizohata on the occasion of his sixtieth birthday

Abstract. Calderón–Zygmund operators are introduced by R. Coifman and Y. Meyer to treat systematically the classical Calderón–Zygmund singular integrals, commutators of Calderón, and some classes of pseudo-differential operators. In this note we generalize the notion of Calderón–Zygmund operators and apply it to the study of, for example, weighted norm inequalities for certain classes of pseudo-differential operators, treated by Coifman and Meyer, and recently G. Bourdaud. We also refer to a recent work of David and Journé on the L^2-boundedness criterion for operators of the Calderón–Zygmund type.

1. Introduction. Calderón–Zygmund operators are introduced by R. Coifman and Y. Meyer [5] to treat systematically the classical Calderón–Zygmund singular integrals, the commutators of Calderón and some classes of pseudo-differential operators, etc., and are further developed by Journé [10]. They are used in many places, [3], [6], [7], [16], [17], etc. In this note we introduce two classes of operators which are generalizations of Calderón–Zygmund operators, and apply them to some classes of pseudo-differential operators considered in [1], [5] and [13].

A Calderón–Zygmund operator defined by Coifman and Meyer is as follows: Let T be a bounded operator from the class $\mathscr{S}(\boldsymbol{R}^n)$ of Schwartz functions to its dual $\mathscr{S}'(\boldsymbol{R}^n)$, satisfying the following two conditions.

(A-1) There exists $C > 0$ such that for any $f \in C_0^\infty(\boldsymbol{R}^n)$

$$\|Tf\|_{L^2(\boldsymbol{R}^n)} \leqslant C\|f\|_{L^2(\boldsymbol{R}^n)};$$

(A-2) There exist a continuous function $K(x, y)$ defined on $\Omega = \boldsymbol{R}^n \times \boldsymbol{R}^n \setminus \{(x, y);\ x = y\}$ and $C_K > 0$ such that

(1.1) for all $(x, y) \in \Omega$

$$|K(x, y)| \leqslant C_K |x-y|^{-n};$$

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