$$\leqslant 4\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{n}\sum_{i=1}^{m}\sum_{k=1}^{n}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2\left[1-\frac{A_{n-k}^{\beta}}{A_n^{\beta}}\right]^2 i a_{ik}^2$$

$$\leqslant 4\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} i a_{ik}^2 \sum_{m=i}^{\infty}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2\sum_{n=k}^{\infty}\frac{1}{n}\left[1-\frac{A_{n-k}^{\beta}}{A_n^{\beta}}\right]^2 < \infty,$$

the last inequality is thanks to (4.8), (4.9) and (1.2). Hence B. Levi's theorem yields (8.9).

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Received October 26, 1983
Shortened version January 16, 1984 (1930)

STUDIA MATHEMATICA, T. LXXXI (1985)

Fourier series and Hilbert transforms with values in UMD Banach spaces

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Abstract. Let B be a Banach space with the unconditional martingale differences property and let T be the circle group. It is shown that if in addition B has an unconditional basis then the Fourier series of $f\in L^p_B(T)$, $p>1$, converges to f a.e.

§ 1. Introduction. The Banach spaces B for which the Hilbert transform $H\colon L^p\to L^p$ admits a bounded B-valued extension to L^p_B, $1<p<\infty$, were recently characterized by a condition called *ζ-convexity* (see [4] and [2]). The class of all such spaces is also denoted as UMD, due to the fact that the unconditionality of martingale differences holds for B-valued random variables if and only if B is ζ-convex.

It is natural to ask to which extent the most important estimates of harmonic analysis carry over to the B-valued setting, $B\in$ UMD. Since the rotation method still applies, the singular integral operators falling under the scope of this method have B-valued extensions which are bounded in L^p_B. A different class of singular integral operators is considered in [3] but the proof requires that the space $B\in$ UMD has an unconditional basis. With the same restriction, we aim to extend here the pointwise convergence theorems for Fourier series ([5] and [10]) to the B-valued setting. Thus, it is shown in Section 3 that the Fourier series of $f\in L^p_B(T)$, $p>1$, converges to $f(x)$ a.e., and in Section 4, that the lacunar sequences of partial sum operators converge to $f(x)$ a.e. if $f\in H^1_B(T)$. These are exactly the same results which hold for the scalar case. Finally, Section 5 contains an interesting stability property of UMD spaces.

§ 2. Notation and basic lemmas. Throughout the paper, B will denote a Banach space in the class UMD. The UMD-constant for B will be the least C such that the inequality

$$\int\|\tilde{H}f(x)\|_B^2\,dx\leqslant C^2\int\|f(x)\|_B^2\,dx$$

holds for every $f \in L^2_B(\mathbf{R})$, where $\tilde{H}$ is the B-valued extension of the Hilbert transform. We shall also assume that B has an unconditional basis, so that the elements of B can be identified with sequences $(b_j)_{j\in N}$; then, a B-valued function is a sequence of functions: $f(x) = (f_j(x))_{j\in N}$, and $\tilde{H}f(x) = (Hf_j(x))_{j\in N}$.

We shall denote by Mf the Hardy–Littlewood maximal function of $f \in L^1_{\rm loc}(\mathbf{R}^n)$ and, more generally, for $1 \leqslant r < \infty$, we shall write

$$M_r f(x) = \sup_{x\in Q} \left\{ \frac{1}{|Q|} \int_Q |f(y)|^r \, dy \right\}^{1/r}.$$

For all $1 < p < \infty$, and for $f = (f_j)_{j\in N} \in L^p_B(\mathbf{R}^n)$, the following inequality holds:

$$\int \|(Mf_j(x))_{j\in N}\|_B^p \, dx \leqslant C_p^p \int \|(f_j(x))_{j\in N}\|_B^p \, dx \tag{2.1}$$

where C_p depends only on p and on the UMD-constant for B. When $B = l^q$, $1 < q < \infty$, this was proved by Fefferman and Stein [8]; the more general B-valued case is due to Bourgain [3]. It follows from the results for vector valued singular integrals (see [1], [13]) that a weak type inequality holds in the limiting case $p = 1$ of (2.1). However, we are rather interested in the following slight improvement of (2.1):

2.2. Lemma. *Given p with $1 < p < \infty$, there exists $r > 1$ depending only on p and on the* UMD-*constant for B such that the operator*

$$(f_j(x))_{j\in N} \to (M_r f_j(x))_{j\in N}$$

is bounded in $L^p_B(\mathbf{R}^n)$.

Proof. Given $f = (f_j)_{j\in N} \in L^p_B$, we define

$$F(x) = (F_j(x))_{j\in N} = \sum_{k=0}^{\infty} [2C_p]^{-k} (M^k f_j(x))_{j\in N}$$

where C_p is the constant in (2.1) and M^k denotes the kth iteration of the operator M (with $M^0 = $ Identity). It is obvious that the series in the right-hand side converges in L^p_B, and

$$\int \|F(x)\|_B^p \, dx \leqslant 2^p \int \|f(x)\|_B^p \, dx.$$

On the other hand, $Mf_j(x) \leqslant 2C_p F_j(x)$ for every j, which means that each F_j is an A_1 weight and therefore satisfies a reverse Hölder inequality of order $r > 1$ depending only on C_p (see [7]). Thus, $M_r F_j(x) \leqslant C'_p F_j(x)$ for all $j \in N$, and this completes the proof.

The next tool that we shall need is the sharp maximal function of Fefferman–Stein, which can also be defined for B-valued functions $f \in L^1_{{\rm loc},B}(\mathbf{R}^n)$ in the obvious way:

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q \|f(x) - f_Q\|_B \, dx$$

where f_Q stands for the mean value of f over the cube Q. The basic result for our purposes is only concerned with scalar valued functions $f(x)$. It is the following inequality which reflects in a very nice way the duality between H^1 and BMO:

2.3. Lemma. *There exists an absolute constant $C > 0$ such that*

$$\left|\int f(x) g(x) \, dx\right| \leqslant C \int f^{\#}(x) P^* g(x) \, dx$$

for all Schwartz functions g such that $\hat{g}(0) = 0$, and all $f \in L^p$, $p > 1$, where $P^* g(x) = \sup\{|P_t * g(y)|:\ t > 0,\ y \in \mathbf{R}^n,\ |x-y| \leqslant t\}$ is the nontangential maximal Poisson integral of g.

See [15], Theorem 4.5, for a proof of this result.

On the other hand, for Banach space valued functions, the usual estimates for the sharp maximal operator of a (smooth enough) singular integral hold, namely:

2.4. Lemma. *Given a Banach space $Y \in$ UMD, and denoting by $\tilde{H}$ the Hilbert transform on Y-valued functions, for every r, $1 < r < \infty$, we have the estimate*

$$(\tilde{H}f)^{\#}(x) \leqslant C_r M_r(\|f\|_Y)(x) \qquad (f \in L^2_Y(\mathbf{R}))$$

where C_r depends only on r and on the UMD-*constant for Y.*

The proof is exactly as in the scalar case (see [9]). We remark that it is not necessary for Y to have an unconditional basis.

Finally, the space $H^1_B(\mathbf{R}) \subset L^1_B(\mathbf{R})$, which will appear in § 4, is defined in terms of B-atoms in the usual way, where a *B-atom* is a function $a \in L^\infty_B$ supported in a bounded interval $I \subset \mathbf{R}$ and satisfying

$$\|a(x)\|_B \leqslant |I|^{-1}, \qquad \int a(x) \, dx = 0.$$

Actually, this definition makes sense for arbitrary Banach spaces, but for the class UMD we get something more:

2.5. Lemma. *$f \in H^1_B(\mathbf{R})$ if and only if $f \in L^1_B(\mathbf{R})$ and $\tilde{H}f \in L^1_B(\mathbf{R})$. Moreover:*

$$\|f\|_{H^1_B(\mathbf{R})} \sim \|f\|_{L^1_B(\mathbf{R})} + \|\tilde{H}f\|_{L^1_B(\mathbf{R})}.$$

This was first pointed out by J. Garcia-Cuerva (oral communication). The proof follows [6] with some minor modifications. An analogous result holds for $\mathbf{R}^n$, with the Hilbert transform replaced by the Riesz transforms.

§ 3. Pointwise convergence of B-valued Fourier series. Let B be as indicated in § 2. We denote by S_m the mth partial sum operator of Fourier series for complex valued functions, and by $\tilde{S}_m$ its extension to functions $f = (f_j) \in L^1_B(\boldsymbol{T})$ (where $\boldsymbol{T} \simeq [0, 1)$ is the 1-dimensional torus), which is given by

$$\tilde{S}_m f(x) = \sum_{-m}^{m} e^{2\pi i k x} \hat{f}(k) = (S_m f_j(x))_{j \in \boldsymbol{N}}.$$

We shall also consider the maximal partial sum operator: $S^* \varphi(x) = \sup_m |S_m \varphi(x)|$ for complex valued functions φ. Our main result is

3.1. THEOREM. *For all $f = (f_j)_{j \in \boldsymbol{N}} \in L^p_B(\boldsymbol{T})$, $1 < p < \infty$, we have*

$$\int \|(S^* f_j(x))_{j \in \boldsymbol{N}}\|_B^p \, dx \leqslant C_p^p \int \|(f_j(x))_{j \in \boldsymbol{N}}\|_B^p \, dx \tag{3.2}$$

and

$$\lim_{m \to \infty} \|\tilde{S}_m f(x) - f(x)\|_B = 0 \quad (\text{a.e. } x \in \boldsymbol{T}).$$

Observe that $\sup_m \|\tilde{S}_m f(x)\|_B \leqslant \|(S^* f_j(x))_{j \in \boldsymbol{N}}\|_B$, so that (3.2) implies the boundedness in $L^p_B(\boldsymbol{T})$ of the maximal partial sum operator for B-valued functions, and from this, the a.e. pointwise convergence follows by a standard argument. The inequalities (3.2) were obtained for the special case $B = l^q$ in [12].

Proof of (3.2). It turns out to be formally easier to deal with partial sum operators for the Fourier transform in $\boldsymbol{R}$ (though the problem of a.e. convergence is equivalent in both settings). Thus, for scalar functions $\varphi \in L^p(\boldsymbol{R})$, we define

$$S_R \varphi(x) = \int_{-R}^{R} \hat{\varphi}(\xi) e^{2\pi i x \xi} \, d\xi = T_R \varphi(x) - T_{-R} \varphi(x)$$

where

$$T_R \varphi(x) = e^{2\pi i R x} H(e^{-2\pi i R \cdot} \varphi)(x).$$

Thus, instead of S^*, we may as well consider

$$T^* \varphi(x) = \sup_R |T_R \varphi(x)| = \sup_R \frac{1}{\pi} \left| \text{p.v.} \int e^{-2\pi i R y} \varphi(y) \frac{dy}{x-y} \right|$$

which is (the continuous analogue of) Carleson's maximal operator. Now, our first step consists in stablishing the inequality

$$(T^* \varphi)^{\#}(x) \leqslant C_r M_r \varphi(x) \quad (1 < r < \infty). \tag{3.3}$$

This was already stated in [13], but, for the sake of completeness, we shall give a detailed proof, which is really an adaptation of the Fefferman–Stein argument in [9]. Since both sides of (3.3) are translation invariant, it suffices to consider the point $x = 0$. For every $t > 0$, let $\varphi_t = \varphi \chi_{[-2t, 2t]}$, and $\varphi^t = \varphi - \varphi_t$. Then

$$\begin{aligned}(T^* \varphi)^{\#}(0) &\leqslant 4 \sup_{t>0} \frac{1}{2t} \int_{-t}^{t} |T^* \varphi(x) - T^* \varphi^t(0)| \, dx \\ &\leqslant 4 \sup_{t>0} \frac{1}{2t} \left\{ \int_{-t}^{t} |T^* \varphi_t(x)| \, dx + \int_{-t}^{t} |T^* \varphi^t(x) - T^* \varphi^t(0)| \, dx \right\} \\ &= 4 \sup_{t>0} (I_t + I^t).\end{aligned}$$

Now, if $|x| < t$, we have

$$\begin{aligned}|T^* \varphi^t(x) - T^* \varphi^t(0)| &\leqslant \sup_{R>0} \frac{1}{\pi} \left| \text{p.v.} \int e^{-2\pi i R y} \varphi^t(y) \left(\frac{1}{y} + \frac{1}{x-y} \right) dy \right| \\ &\leqslant \frac{1}{\pi} \int_{|y|>2t} |\varphi(y)| \, 2|x| \, y^{-2} \, dy \leqslant \frac{2|x|}{\pi t} M\varphi(0)\end{aligned}$$

so that

$$\sup_{t>0} |I^t| \leqslant \pi M\varphi(0) t^{-2} \int_{-t}^{t} |x| \, dx = \pi M\varphi(0).$$

Thus, we have correctly estimated the second term. For the first term, we must use the Carleson–Hunt theorem ([5] and [10]) which asserts that

$$\|T^* \varphi\|_r \leqslant C_r \|\varphi\|_r \quad (\varphi \in L^r;\ 1 < r < \infty).$$

Then, for arbitrary $r > 1$,

$$\begin{aligned}|I_t| &\leqslant \frac{1}{2t} \|T^* \varphi_t\|_r (2t)^{1-1/r} \\ &\leqslant C_r \left\{ \frac{1}{2t} \int_{-2t}^{2t} |\varphi(x)|^r \, dx \right\}^{1/r} \leqslant C_r M_r \varphi(0)\end{aligned}$$

and the proof of (3.3) is complete.

To conclude the theorem, we shall use Lemmas 2.2 and 2.3. First of all, we point out that every UMD-space is reflexive, and therefore, the dual B^* of B has an unconditional basis dual of the one fixed in B. Thus, we can also view the elements of B^* as sequences, and if $b = (b_j) \in B$ and $b^* = (b_j^*) \in B^*$, the duality is given by

$$\langle b, b^* \rangle = \sum_j b_j b_j^*.$$

The duality between L_B^p and $L_{B^*}^{p'}$, $1 < p < \infty$, can be expressed in the same way. Now, we shall prove (3.2) for $f = (f_j)_{j\in N} \in L_B^p(\boldsymbol{R})$ and with T^* instead of S^*. Let $g = (g_j)_{j\in N}$ be an arbitrary element of the unit ball of $L_{B^*}^{p'}(\boldsymbol{R})$ such that $g \in \mathscr{S}(\boldsymbol{R})$ and $\hat{g}(0) = 0$ (such g are dense in the unit ball of $L_{B^*}^{p'}$), so that

$$\begin{aligned}\|(T^* f_j)_{j\in N}\|_{L_B^p} &= \sup_g \sum_j \int T^* f_j(x)\, g_j(x)\,dx \\ &\leqslant C \sup_g \sum_j \int (T^* f_j)^{\#}(x)\, P^* g_j(x)\,dx \\ &\leqslant C_r \sup_g \int \sum_j M_r f_j(x)\, P^* g_j(x)\,dx\end{aligned}$$

where $r > 1$ can be chosen arbitrarily. We take it so that Lemma 2.2 holds for our given p, and observe that $P^* g_j(x) \leqslant C M g_j(x)$ and that we are allowed to use (2.1) for B^*-valued functions, because the dual of a UMD-space is again UMD. This gives us finally

$$\begin{aligned}\|(T^* f_j)_{j\in N}\|_{L_B^p} &\leqslant C_r \|(M_r f_j)_{j\in N}\|_{L_B^p} \sup_g \|(M g_j)_{j\in N}\|_{L_{B^*}^{p'}} \\ &\leqslant C_{r,p} \|(f_j)_{j\in N}\|_{L_B^p}.\end{aligned}$$

§ 4. Lacunary convergence in H_B^1. We denote by S_R the partial sum operators for the Fourier transform defined in § 3, and by $\tilde{S}_R$ their B-valued extensions:

$$\tilde{S}_R f(x) = \int_{|\xi| \leqslant R} \hat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi = \int_{-\infty}^{\infty} \frac{\operatorname{sen} 2\pi R y}{\pi y} f(x-y)\,dy$$

(the last expression makes sense for every $f \in L_B^p(\boldsymbol{R})$, $1 \leqslant p < \infty$).

4.1. THEOREM. *For every $f \in H_B^1(\boldsymbol{R})$, we have*

$$\Big|\{x\colon \sup_{k\in Z} \|\tilde{S}_{2^k} f(x)\|_B > \lambda\}\Big| \leqslant C \int (\|f(x)\|_B + \|\tilde{H} f(x)\|_B)\,dx \tag{4.2}$$

and as a consequence,

$$\lim_{k\to+\infty} \|\tilde{S}_{2^k} f(x) - f(x)\|_B = 0 \quad (\text{a.e. } x \in \boldsymbol{R}).$$

As before, only the maximal inequality needs to be proved, and for this, we shall use two auxiliary results. First of all, we recall that every UMD-space is B-convex, so that, in particular, B has (Rademacher) cotype $q < \infty$. For this q, we have

4.3. LEMMA. *Let ψ be a Schwartz function in $\boldsymbol{R}$ such that $\hat{\psi}(0) = 0$, and write $\psi_k(x) = 2^k \psi(2^k x)$ for each $k \in \boldsymbol{Z}$. Then, for every B-atom $a(x)$ we have*

$$\int \Big(\sum_{k=-\infty}^{\infty} \|\psi_k * a(x)\|_B^q\Big)^{1/q} dx \leqslant C.$$

Proof. Let $\{r_k(t)\}_{-\infty}^{\infty}$ be Rademacher functions, $t \in [0, 1]$, and set

$$L_t(x) = \sum_k r_k(t)\, \psi_k(x) \quad (x \neq 0).$$

By the definition of cotype, the left-hand side of the inequality to be proved is majorized by the cotype constant times

$$\int_R \int_0^1 \Big\|\sum_k r_k(t)\, \psi_k * a(x)\Big\|_B\, dt\, dx = \int_0^1 \int_R \|L_t * a(x)\|_B\, dx\, dt$$

and the lemma will be proved if we show that

$$\|L_t * a\|_{L_B^1} \leqslant C \quad (0 \leqslant t \leqslant 1)$$

(C being also independent of the B-atom $a(x)$). But the kernel L_t satisfies the standard conditions for singular integrals:

$$\begin{aligned}|\hat{L}_t(\xi)| &\leqslant C && (\xi \in \boldsymbol{R}),\\ |L_t(x)| &\leqslant C|x|^{-1} && (x \in \boldsymbol{R},\ x \neq 0),\\ |L_t(x-y) - L_t(x)| &\leqslant C|y|\,|x|^{-2} && (|x| > 2|y|)\end{aligned}$$

(see [14]), and therefore, the main theorem in [3] shows that convolution with L_t defines a bounded operator in $L_B^p(\boldsymbol{R})$, $1 < p < \infty$. (This can also be seen by the method used in Theorem 3.1, since $(L_t * \varphi)^{\#}(x) \leqslant C_r M_r \varphi(x)$ for all $r > 1$.) Thus, if I is the supporting interval of $a(x)$ and its center is c, we get

$$\begin{aligned}\|L_t * a\|_{L_B^1} &\leqslant C\|a\|_{L_B^2} \|\chi_{2I}\|_2 + \int_{x\notin 2I} \Big|\int_{y\in I} (L_t(x-y) - L_t(x-c))\, a(y)\,dy\Big|\,dx \\ &\leqslant C\sqrt{2} + C|I|^{-1} \int_{y\in I} \int_{x\notin 2I} |y-c|\,|x-c|^{-2}\,dx\,dy \leqslant \text{Const}\end{aligned}$$

and this ends the proof.

4.4. LEMMA. *For arbitrary numbers $R_j > 0$, and functions $f_j \in L_B^1(\boldsymbol{R})$, we have the inequality*

$$\Big\|\Big(\sum_j \|\tilde{S}_{R_j} f_j\|_B^s\Big)^{1/s}\Big\|_{WL^1} \leqslant C_s \Big\|\Big(\sum_j \|f_j\|_B^s\Big)^{1/s}\Big\|_1$$

where $1<s<\infty$ *and* $\|\cdot\|_{WL^1}$ *stands for the weak-L^1 "norm" of a scalar function.*

Proof. By the formula expressing S_R in terms of the Hilbert transform (see the proof of (3.2)), it is equivalent to prove the same inequality with $\tilde{H}f_j$ instead of $\tilde{S}_{R_j}f_j$. But $(\tilde{H}f_j)_{j\in N}$ is the Hilbert transform of the function $(f_j)_{j\in N}\in L^1_{l^s(B)}$, and the lemma is therefore a consequence of two well-known facts:

1st. The Hilbert transform acting on UMD-valued functions satisfies a weak type (1, 1) inequality.

2nd. $l^s(B)$ is a UMD-space whenever B is, and $1<s<\infty$.

Proof of (4.2). Take Schwartz functions φ and ψ such that

$$\hat{\varphi}(\xi)=\begin{cases}1 & \text{when } |\xi|\leqslant 1/3,\\ 0 & \text{when } |\xi|\geqslant 2/3,\end{cases}$$

and

$$\hat{\varphi}(\xi)+\hat{\psi}(\xi)=1 \quad \text{when} \quad |\xi|\leqslant 1.$$

Then, for arbitrary functions $f\in L^1_B(\boldsymbol{R})$ we can write

$$\tilde{S}_{2^k}f(x)=\varphi_k*f(x)+\tilde{S}_{2^k}(\psi_k*f)(x),$$

where φ_k, ψ_k are defined as in Lemma 4.3. Therefore,

$$\sup_k\|\tilde{S}_{2^k}f(x)\|_B\leqslant CM(\|f\|_B)(x)+\big(\sum_k\|\tilde{S}_{2^k}(\psi_k*f)(x)\|_B^q\big)^{1/q}.$$

The WL^1-norm of the first term is majorized by $C\|f\|_{L^1_B}$. For the second term, we use Lemma 4.4 (with $s=q$) and then decompose $f\in H^1_B$ into B-atoms: $f(x)=\sum_j\lambda_j a_j(x)$, with $\sum_j|\lambda_j|\leqslant C\|f\|_{H^1_B}$. Thus

$$\begin{aligned}\Big\|\big(\sum_k\|\tilde{S}_{2^k}(\psi_k*f)\|_B^q\big)^{1/q}\Big\|_{WL^1} &\leqslant C_q\int\big(\sum_k\|\psi_k*f(x)\|_B^q\big)^{1/q}dx\\ &\leqslant C_q\sum_j|\lambda_j|\int\big(\sum_k\|\psi_k*a_j(x)\|_B^q\big)^{1/q}dx\\ &\leqslant (\text{by Lemma 4.3})\leqslant C\|f\|_{H^1_B}. \ \blacksquare\end{aligned}$$

It is evident that Theorem 4.1 has an analogous formulation in the periodic setting. We state it as a corollary.

4.5. Corollary. *Let $f\in L^1_B(\boldsymbol{T})$ be a function whose Fourier series is of analytic type, i.e. $\hat{f}(k)=0$ for all $k<0$. Then*

$$\lim_{N\to\infty}\sum_{k=0}^{2^N}e^{2\pi ikx}\hat{f}(k)=f(x)\quad (\text{a.e. } x\in T).$$

The arguments that we have used are very close to those of classical Littlewood–Paley theory. Without going into the details, let us merely state what one can obtain in the vector valued setting as a substitute for the standard results for scalar functions:

$$\|f\|_p\sim\Big\|\big(\sum_{I\in\Delta}|S_If|^2\big)^{1/2}\Big\|_p\qquad(1<p<\infty)$$

where Δ is the family of dyadic intervals in $\boldsymbol{R}$ and $(S_If)\hat{}=\hat{f}\chi_I$. Since S_I can be easily written in terms of the Hilbert transform, we remark that the partial sum operators $\tilde{S}_I$ make sense and are uniformly bounded in $L^p_B(\boldsymbol{R})$, $1<p<\infty$.

4.6. Theorem. *Let B be of (Rademacher) type $p>1$ and cotype $q<\infty$. Then, for every $f\in L^r_B(\boldsymbol{R})$, $1<r<\infty$,*

$$c_r\Big\|\big(\sum_{I\in\Delta}\|\tilde{S}_If\|_B^q\big)^{1/q}\Big\|_r\leqslant\|f\|_{L^r_B}\leqslant C_r\Big\|\big(\sum_{I\in\Delta}\|\tilde{S}_If\|_B^p\big)^{1/p}\Big\|_r$$

and both inequalities are best possible in the sense that the first one implies that B has cotype q and the second one implies that B has type p.

A similar result holds for $\boldsymbol{R}^n$.

§ 5. A stability property of the class UMD. The method of proof of Theorem 3.1 can be used to find a procedure to obtain new UMD spaces. We have already mentioned the fact that, if $B\in$ UMD, then also $l^s(B)\in$ UMD for $1<s<\infty$. When B has an unconditional basis (or is a lattice), and Y is an arbitrary Banach space, one can also define the space

$$B(Y)=\{(y_j)_{j\in N}:\ (\|y_j\|_Y)_{j\in N}\in B\}$$

with the natural norm. When Y is a Hilbert space, it follows from Grothendieck's fundamental inequality (see [11]) that $B(Y)\in$ UMD. The same turns out to be true for an arbitrary $Y\in$ UMD and, more generally, we can state the following

5.1. Theorem. *Let $(Y_j)_{j\in N}$ be a sequence of UMD Banach spaces with uniformly bounded UMD-constants, and let B be, as in § 2, another UMD-space with an unconditional basis. Define the space*

$$Y=B\big(\bigoplus_{j=1}^{\infty}Y_j\big)$$

which consists of all sequences $y=(y_j)_{j\in N}$, with $y_j\in Y_j$, such that $(\|y_j\|_{Y_j})_{j\in N}\in B$, the norm in Y being

$$\|y\|_Y=\big\|(\|y_j\|_{Y_j})_{j\in N}\big\|_B.$$

Then $Y\in$ UMD.

Such a statement was conjectured to the author by J. Bourgain during the Colloque Laurent Schwartz, 1983, and an independent, somewhat different, proof has been simultaneously found by Bourgain himself.

Proof. A function $F \in L^2_Y(\boldsymbol{R})$ is a sequence $F = (F_j)_{j\in \boldsymbol{N}}$ with $F_j \in L^2_{Y_j}(\boldsymbol{R})$ and

$$\|F\|_{L^2_Y} = \{\int \|(\|F_j(x)\|_{Y_j})_{j\in \boldsymbol{N}}\|_B^2\}^{1/2}.$$

We shall use the same notation, $\tilde{H}$, to indicate the Y_j-valued Hilbert transform for all $j \in \boldsymbol{N}$. We shall also define

$$\tilde{H}F(x) = (\tilde{H}F_j(x))_{j\in \boldsymbol{N}}$$

and try to prove that this is a bounded operator in L^2_Y. Let f_j be the nonnegative L^2 functions $f_j(x) = \|\tilde{H}F_j(x)\|_{Y_j}$, $j \in \boldsymbol{N}$. Then,

$$\|\tilde{H}F\|_{L^2_Y} = \|(f_j)_{j\in \boldsymbol{N}}\|_{L^2_B} = \sup_g \sum_j \int f_j(x)\, g_j(x)\, dx$$

where the "sup" is taken over all $g = (g_j)_{j\in \boldsymbol{N}} \in L^2_{B^*}$ of unit norm, such that $g_j \in \mathscr{S}(\boldsymbol{R})$, $\hat{g}_j(0) = 0$. We apply Lemma 2.4 to get

$$f_j^{\#}(x) \leqslant 2(\tilde{H}F_j)^{\#}(x) \leqslant C_r M_r(\|F_j\|_{Y_j})(x)$$

for every $r > 1$, C_r being independent of j due to the uniform boundedness of the UMD-constants for Y_j. Then, Lemma 2.3 gives

$$\|\tilde{H}F\|_{L^2_Y} \leqslant C_r \sup_g \int \sum_j M_r(\|F_j\|_{Y_j})(x)\, M g_j(x)\, dx$$

$$\leqslant C'_r \{\int \|(M_r(\|F_j\|_{Y_j})(x))_{j\in \boldsymbol{N}}\|_B^2\, dx\}^{1/2}.$$

Finally, we take r close enough to 1 so that Lemma 2.2 applies, and it allows us to drop the operator M_r in the last expression by suitably enlarging the constant C'_r. This ends the proof.

As a final remark, let me mention that the results of Sections 3 and 4 are probably true for arbitrary $B \in \mathrm{UMD}$. The restriction which consists in assuming the existence of unconditional basis in B is due to the method of proof, and a different approach should be found in order to get rid of such a restriction in our results as well as in those of [3].

Acknowledgements. All this work was inspired by paper [3] by J. Bourgain, to whom I express my thanks for sending his preprints and for the useful conversations from which, among other things, the question considered in Section 5 was suggested. The work was finished during a stay at the Institut Mittag-Leffler, and I wish to thank L. Carleson and P. W. Jones for the opportunity to work here, and Karin Lindberg for typing the manuscript. I am also grateful to M. Kaneko for pointing out an error in the proof of Theorem 4.1.

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Received January 21, 1984
Corrected February 27, 1984 (1950)