\[ \leq 4 \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \sum_{l=1}^{n-k} \left( \frac{A_{l-k}}{A_k} \right)^2 \left( \frac{A_{l-k+1}}{A_k} \right)^2 < \infty, \]

the last inequality is thanks to (4.8), (4.9) and (1.2). Hence B. Levi's theorem yields (8.9).

References


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Fourier series and Hilbert transforms with values in UMD Banach spaces

by

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Abstract. Let \( B \) be a Banach space with the unconditional martingale differences property and let \( T \) be the circle group. It is shown that if in addition \( B \) has an unconditional basis then the Fourier series of \( f \in \ell_p^2 \), \( p > 1 \), converges to \( f \) a.e.

§ 1. Introduction. The Banach spaces \( B \) for which the Hilbert transform \( H: L^p \to L^q \) admits a bounded \( B \)-valued extension to \( L^q \), \( 1 < p < \infty \), were recently characterized by a condition called \( \zeta \)-convexity (see [4] and [22]). The class of all such spaces is also denoted as UMD, due to the fact that the unconditionality of martingale differences holds for \( B \)-valued random variables if and only if \( B \) is \( \zeta \)-convex.

It is natural to ask to which extent the most important estimates of harmonic analysis carry over to the \( B \)-valued setting. \( B \in UMD \). Since the rotation method still applies, the singular integral operators falling under the scope of this method have \( B \)-valued extensions which are bounded in \( L^q \). A different class of singular integral operators is considered in [3] but the proof requires that the space \( B \in UMD \) has an unconditional basis. With the same restriction, we aim to extend here the pointwise convergence theorems for Fourier series ([5] and [10]) to the \( B \)-valued setting. Thus, it is shown in Section 3 that the Fourier series of \( f \in \ell_2^2 \), \( p > 1 \), converges to \( f \) a.e. and in Section 4, that the lacunary sequences of partial sum operators converge to \( f \) a.e. if \( f \in H_2^2 \). These are exactly the same results which hold for the scalar case. Finally, Section 5 contains an interesting stability property of UMD spaces.

§ 2. Notation and basic lemmas. Throughout the paper, \( B \) will denote a Banach space in the class UMD. The UMD-constant for \( B \) will be the least \( C \) such that the inequality

\[ \|Hf(x)\|_B \leq C \|f(x)\|_B \]
holds for every \( f \in L^1_{\text{loc}}(\mathbb{R}) \), where \( \hat{H} \) is the \( B \)-valued extension of the Hilbert transform. We shall also assume that \( B \) has an unconditional basis, so that the elements of \( B \) can be identified with sequences \((b_j)_{j=0}^\infty\); then, a \( B \)-valued function is a sequence of functions: \( \hat{f}(x) = (\hat{f}_j(x))_{j=0}^\infty \), and \( \hat{H}f(x) = (\hat{H}f_j(x))_{j=0}^\infty \).

We shall denote by \( Mf \) the Hardy–Littlewood maximal function of \( f \in L^1_{\text{loc}}(\mathbb{R}) \) and, more generally, for \( 1 < r < \infty \), we shall write

\[
M_rf(x) = \sup_{x \in \Omega} \left( \frac{1}{|\Omega|} \int_{|y-x| < r} |f(y)|^r \, dy \right)^{1/r}.
\]

For all \( 1 < p < \infty \), and for \( f = (f_j)_{j=0}^\infty \in L^p_B(\mathbb{R}) \), the following inequality holds:

\[
|||Mf_j|||_B \leq C_{p,r} |||f_j|||_B \quad \text{for all } j.
\]

where \( C_{p,r} \) depends only on \( p \) and on the UMD-constant for \( B \). When \( B = \mathbb{R} \), \( 1 < q < \infty \), this was proved by Fefferman and Stein [8]; the more general \( B \)-valued case is due to Bourgain [2]. It follows from the results for vector-valued singular integrals (see [1], [13]) that a weak type inequality holds in the limiting case \( p = 1 \) of (2.1). However, we are rather interested in the following slight improvement of (2.1):

2.2. Lemma. Given \( p \) with \( 1 < p < \infty \), there exists \( r > 1 \) depending only on \( p \) and on the UMD-constant for \( B \) such that the operator

\[
(Mf_j)_{j=0}^\infty \rightarrow (M_rf_j)_{j=0}^\infty
\]

is bounded in \( L^p_B(\mathbb{R}) \). Proof. Given \( f = (f_j)_{j=0}^\infty \in L^p_B \), we define

\[
F(x) = (f_j(x))_{j=0}^\infty = \sum_{k=0}^\infty [2^{kr}]^{-1} (M^k f_j(x))_{j=0}^\infty
\]

where \( C_r \) is constant in (2.1) and \( M^k \) denotes the \( k \)-th iteration of the operator \( M \) with \( M^0 = \text{Identity} \). It is obvious that the series in the right-hand side converges in \( L^p_B \), and

\[
|||F|||_B \leq 2 \int f(x) \, dx = 2 \int f(x) \, dx.
\]

On the other hand, \( M^k f_j(x) \leq 2 C_r F_j(x) \) for every \( j \), which means that each \( F_j \) is an \( A_1 \) weight and therefore satisfies a reverse H"older inequality of order \( r > 1 \) depending only on \( C_r \) (see [7]). Thus, \( M, M^k f_j(x) \leq C_r F_j(x) \) for all \( j \in \mathbb{N} \), and this completes the proof.

The next tool that we shall need is the sharp maximal function of Fefferman-Stein, which can also be defined for \( B \)-valued functions \( f \in L^1_{\text{loc}}(\mathbb{R}) \) in the obvious way:

\[
f^*(x) = \sup_{x \in \Omega} \left( \frac{1}{|\Omega|} \int_{|y-x| < r} |f(y)|^r \, dy \right)^{1/r}.
\]

where \( f_0 \) stands for the mean value of \( f \) over the cube \( Q \). The basic result for our purposes is only concerned with scalar valued functions \( f(x) \). It is the following inequality which reflects in a very nice way the duality between \( H^1 \) and \( BMO \):

2.3. Lemma. There exists an absolute constant \( C > 0 \) such that

\[
||f(g)\, dx||_B \leq C ||f^*(g)\, dx||_B
\]

for all Schwartz functions \( g \) such that \( \hat{g}(0) = 0 \), and all \( f \in L^1, p > 1 \), where \( P \)-\( g \) is the non-tangential maximal Poisson integral of \( g \).

See [15], Theorem 4.5, for a proof of this result.

On the other hand, for Banach space valued functions, the usual estimates for the sharp maximal operator of a (smooth enough) singular integral hold, namely:

2.4. Lemma. Given a Banach space \( Y \in \text{UMD} \), and denoting by \( \hat{H} \) the Hilbert transform on \( Y \)-valued functions, for every \( r, 1 < r < \infty \), we have the estimate

\[
||H_f(x)\, dx||_B \leq C_{r,y} ||f||_Y \quad (f \in L^r_B(Y))
\]

where \( C_{r,y} \) depends only on \( r \) and on the UMD-constant for \( Y \).

The proof is exactly as in the scalar case (see [9]). We remark that it is not necessary for \( Y \) to have an unconditional basis.

Finally, the space \( H^1_B(\mathbb{R}) \subset L^1_B(\mathbb{R}) \), which will appear in \$ 4 \$, is defined in terms of \( B \)-atoms in the usual way, where a \( B \)-atom is a function \( a \in L^2_B \) supported in a bounded interval \( I \subset \mathbb{R} \) and satisfying

\[
||a||_B \leq ||I||^{-1}, \quad a(x) \, dx = 0.
\]

Actually, this definition makes sense for arbitrary Banach spaces, but for the class \( \text{UMD} \) we get something more:

2.5. Lemma. \( f \in H^1_B(\mathbb{R}) \) if and only if \( \hat{f} \in L^1_B(\mathbb{R}) \) and \( \hat{H}f \in L^1_B(\mathbb{R}) \). Moreover:

\[
||f||_{H^1_B(\mathbb{R})} \sim ||f||_{L^1_B(\mathbb{R})} + ||\hat{H}f||_{L^1_B(\mathbb{R})}.
\]

This was first pointed out by J. Garcia-Cuerva (oral communication).

The proof follows [6] with some minor modifications. An analogous result holds for \( \mathbb{R}^n \), with the Hilbert transform replaced by the Riesz transforms.
§ 3. Pointwise convergence of \( B \)-valued Fourier series. Let \( B \) be as indicated in § 2. We denote by \( S_n \) the \( n \)-th partial sum operator of Fourier series for complex valued functions, and by \( S_n \) its extension to functions \( f = (f_j) \in L_{\infty}(T) \) (where \( T = [0, 1] \) is the 1-dimensional torus), which is given by

\[
S_n f(x) = \sum_{k=-n}^{n} e^{2\pi i k x} f(k) = (S_n f_j)(x).
\]

We shall also consider the maximal partial sum operator: \( S^* \psi(x) = \sup |S_n \psi(x)| \) for complex valued functions \( \psi \). Our main result is

3.1. Theorem. For all \( f = (f_j) \in L_{\infty}(T) \), \( 1 < p < \infty \), we have

\[
\mu \left( \left\{ \| |S^* f_j(x)| \|_p \right\} dx \leq C \mu \left( \left\{ \| |f_j(x)| \|_p \right\} dx \right.
\]

and

\[
\lim_{n \to \infty} \| |S_n f(x)| - |f(x)| \|_p = 0 \quad (a.e. \ x \in T)\]

Observe that \( \sup |S_n f_j(x)| \|_p \leq \| |S^* f_j(x)| \|_p \|_p \), so that (3.2) implies the boundedness in \( L_{\infty}(T) \) of the maximal partial sum operator for \( B \)-valued functions, and from this, the a.e. pointwise convergence follows by a standard argument. The inequalities (3.2) were obtained for the special case \( B = l^1 \) in [12].

Proof of (3.2). It turns out to be formally easier to deal with partial sum operators for the Fourier transform in \( \mathbb{R} \) (though the problem of a.e. convergence is equivalent in both settings). Thus, for scalar functions \( \phi \in \mathcal{L}(\mathbb{R}) \), we define

\[
S_{\phi} \psi(x) = \int_{\mathbb{R}} \hat{\phi}(\xi) e^{2\pi i \xi x} \, d\xi = T_{\phi}(\psi)(x) - T_{-\psi}(\psi)(x)
\]

where

\[
T_{\phi} \psi(x) = e^{2\pi i x} H(e^{-2\pi i x} \psi(x))
\]

Thus, instead of \( S^* \), we may as well consider

\[
T^* \psi(x) = \sup_{x \in \mathbb{R}} \left\| T_{\phi} \psi(x) \right\|_{L^p} \leq \frac{1}{\pi} \int_{\mathbb{R}} \left| e^{-2\pi i x} \psi(y) \right| \frac{dy}{x - y}
\]

which is (the continuous analogue of) Carleson's maximal operator. Now, our first step consists in establishing the inequality

\[
(T^* \psi)^{(r)}(x) \leq C, \quad \| M \psi (x) \|_{L^r} \leq C \| M \psi (x) \|_{L^r}
\]

and the proof of (3.2) is complete.

To conclude the theorem, we shall use Lemmas 2.2 and 2.3. First of all, we point out that every UMD-space is reflexive, and therefore, the dual \( B^* \) of \( B \) has an unconditional basis dual of the one fixed in \( B \). Thus, we can also view the elements of \( B^* \) as sequences, and if \( b = (b_j) \in B \) and \( b^* = (b_j^*) \in B^* \),

argument in [9]. Since both sides of (3.3) are translation invariant, it suffices to consider the point \( x = 0 \). For every \( t > 0 \), let \( \phi_t = \phi_{x^{-1} \cdot t, t} \), and \( \phi' = \phi - \phi_t \). Then

\[
(T^* \psi)^{(r)}(0) \leq 4 \sup_{t > 0} \frac{1}{2t} \int_{-t}^{t} |T^* \psi(x) - T^* \psi'(0)| \, dx
\]

\[
\leq 4 \sup_{t > 0} \frac{1}{2t} \left\{ \int_{-t}^{t} |T^* \phi_t(x)| \, dx + \int_{-t}^{t} |T^* \phi'(x) - T^* \phi'(0)| \, dx \right\}
\]

\[
= 4 \sup_{t > 0} (I_t + F_t).
\]

Now, if \( |x| < t \), we have

\[
|T^* \phi'(x) - T^* \phi'(0)| \leq \sup_{t > 0} \frac{1}{2t} \left\{ \int_{|y| > t} e^{-2\pi R^2} \psi(y) \left( \frac{1}{y} + \frac{1}{x - y} \right) dy \right\}
\]

\[
\leq \frac{1}{\pi} \int_{|y| > t} |\psi(y)| \frac{|x|}{y} \, dy \leq \frac{2|x|}{\pi t} M \psi(0)
\]

so that

\[
\sup_{t > 0} |F_t| \leq \pi M \psi(0) t^{-2} \left\| \frac{x}{|x|} \right\|_{L^r} = \pi M \psi(0).
\]

Thus, we have correctly estimated the second term. For the first term, we must use the Carleson–Hunt theorem ([15] and [10]) which asserts that

\[
\| T^* \psi \|_{L^p} \leq C \| \psi \|_{L^p} \quad (\psi \in \mathcal{L}; \ 1 < r < \infty).
\]

Then, for arbitrary \( r > 1 \),

\[
|I| \leq \frac{1}{2^r} \| T^* \psi \|_{L^r} (2t)^{-1 + \frac{1}{r}}
\]

\[
\leq C, \quad \frac{1}{2^r} \left\{ \int_{-t}^{t} |\psi(x)| \, dx \right\}^{\frac{1}{r}} \leq C, \ M \psi(0)
\]

and the proof of (3.2) is complete.
the duality is given by
\[
\langle b, b^* \rangle = \sum_j b_j b_j^* .
\]

The duality between \( L_p \) and \( L_p^r \), \( 1 < p < \infty \), can be expressed in the same way. Now, we shall prove (3.2) for \( f = (f_j)_{j \in \mathbb{N}} \in L_p^r(\mathbb{R}) \) and with \( T^* \) instead of \( S^* \). Let \( g = (g_j)_{j \in \mathbb{N}} \) be an arbitrary element of the unit ball of \( L_p^r(\mathbb{R}) \) such that \( g \in \mathcal{V}(\mathbb{R}) \) and \( \hat{g}(0) = 0 \) (such \( g \) are dense in the unit ball of \( L_p^r \)), so that
\[
\| (T^* f_j) \|_{L^r_p} = \sup_s \sum_j \| (T^* f_j)^s (x) \|_{L^p} dx 
\leq C \sup_s \sum_j \| (T^* f_j)^s (x) \|_{L^p} g_j(x) dx 
\leq C \sup_s \sum_j \| (M f_j)^s (x) \|_{L^p} g_j(x) dx
\]
where \( r > 1 \) can be chosen arbitrarily. We take it so that Lemma 2.2 holds for our given \( p \), and observe that \( \| (T^* f_j) \|_{L^r_p} \leq CM^p g_j(x) \) and that we are allowed to use (2.1) for \( B^* \)-valued functions, because the dual of a UMD-space is again UMD. This gives us finally
\[
\| (T^* f_j) \|_{L^r_p} \leq C \sup_s \| (M f_j)^s \|_{L^p} \| g_j \|_{L^r_p} 
\leq C \sup_s \| (f_j)^s \|_{L^p} \| g_j \|_{L^r_p}.
\]

4. Lacunary convergence in \( H_p^r \). We denote by \( S_{\mathbb{R}} \) the partial sum operators for the Fourier transform defined in §3, and by \( S_{\mathbb{R}}^r \) their \( B \)-valued extensions:
\[
S_{\mathbb{R}}^r f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{-2\pi i x \xi} d\xi = \int_{-\infty}^{\infty} \text{sech} 2\pi R y f(x-y) dy
\]
the last expression makes sense for every \( f \in L_p^r(\mathbb{R}), 1 \leq p < \infty \).

4.1. Theorem. For every \( f \in H_p^r(\mathbb{R}) \), we have
\[
\| [x, \text{sup}_{n \in \mathbb{Z}} \| S_{\mathbb{R}}^r f(x) \|_p > \lambda] \|_p \leq C \| (f(x))_p \|_p + \| \mathcal{A} f(x) \|_p dx
\]
and as a consequence,
\[
\lim_{k \to \infty} \| S_{\mathbb{R}}^r f(x) - f(x) \|_p = 0 \quad (a.e. \ x \in \mathbb{R}).
\]
As before, only the maximal inequality needs to be proved, and for this, we shall use two auxiliary results. First of all, we recall that every UMD-space is \( B \)-convex, so that, in particular, \( B \) has (Rademacher) cotype \( q < \infty \). For this \( q \), we have

4.3. Lemma. Let \( \psi \) be a Schwartz function in \( \mathbb{R} \) such that \( \hat{\psi}(0) = 0 \), and write \( \psi_k(x) = 2^k \psi(2^k x) \) for each \( k \in \mathbb{Z} \). Then, for every \( B \)-atom \( a(x) \) we have
\[
\int \sum_k \| \psi_k \ast a(x) \|_{L^q} dx \leq C.
\]
Proof. Let \( \{ r_k(t) \}_{t \in [0,1]} \) be Rademacher functions, \( r_k(t) \), and set
\[
L_k(x) = \sum_k r_k(t) \psi_k(x) \quad (x \neq 0).
\]
By the definition of cotype, the left-hand side of the inequality to be proved is majorized by the cotype constant times
\[
\int \| \sum_k \int r_k(t) \psi_k \ast a(x) dx \|_{L^q} dt dx = \int \| L_k \ast a(x) \|_{L^q} dx dt
\]
and the lemma will be proved if we show that
\[
\| L_k \ast a(x) \|_{L^q} \leq C \quad (0 \leq t \leq 1)
\]
(\( C \) being also independent of the \( B \)-atom \( a(x) \)). But the kernel \( L_k \) satisfies the standard conditions for singular integrals:
\[
\begin{align*}
|L_k(\xi)| & \leq C & (\xi \in \mathbb{R}), \\
|L_k(x)| & \leq C \| x \|^{-1} & (x \in \mathbb{R}, x \neq 0), \\
|L_k(x-y) - L_k(x) - L_k(x-y)| & \leq C \| y \|^{-1} & (|x| > 2 \| y \|)
\end{align*}
\]
(see [14]), and therefore, the main theorem in [3] shows that convolution with \( L_k \) defines a bounded operator in \( L_p^r(\mathbb{R}), 1 \leq p < \infty \). (This can also be seen by the method used in Theorem 3.1, since \( (\mathcal{M} \ast \phi) \) \( \psi \) \( \leq C \mathcal{M} \phi \) \( \psi \) for all \( \psi \).) Thus, if \( I \) is the supporting interval of \( a(x) \) and its center is \( c \), we get
\[
\| L_k \ast a \|_{L^q} \leq C \| a \|_{L^q} \| L_k \|_{L^q} + \int \| L_k(x-y) - L_k(x-c) \| a(y) dy dx 
\leq C \sqrt{2 + C} \| I \|^{-1} \int \| y-c \| \| x-c \|^{-2} dx dy \leq \text{constant}
\]
and this ends the proof.

4.4. Lemma. For arbitrary numbers \( R_j > 0 \), and functions \( f_j \in L_p^r(\mathbb{R}) \), we have the inequality
\[
\| \sum_j \| S_{\mathbb{R}} f_j(x) \|_{L^q} \|_{L^p} \leq C \| \sum_j \| f_j(x) \|_{L^q} \|_{L^p} \|
\]
where $1 < s < \infty$ and $\| \cdot \|_{L^q}$ stands for the weak-$L^q$ “norm” of a scalar function.

Proof. By the formula expressing $S_n$ in terms of the Hilbert transform (see the proof of (3.2)), it is equivalent to prove the same inequality with $H_f$ instead of $S_n f$. But $(H_f)_{t \in T}$ is the Hilbert transform of the function $(f_{t})_{t \in T} \in L^1_{\mathbb{R}}$, and the lemma is therefore a consequence of two well-known facts:

1. The Hilbert transform acting on UMD-valued functions satisfies a weak-type $(1,1)$ inequality.

2. $\mathcal{F}(B)$ is a UMD-space whenever $B$ is, and $1 < s < \infty$.

Proof of (4.2). Take Schwartz functions $\varphi$ and $\psi$ such that

\[ \varphi(\xi) = \begin{cases} 1 & \text{when } |\xi| \leq 1/3, \\ 0 & \text{when } |\xi| \geq 2/3, \\ \end{cases} \]

and

\[ \varphi(\xi) + \psi(\xi) = 1 \quad \text{when } |\xi| \leq 1. \]

Then, for arbitrary functions $f \in \text{L}^p_{\mathbb{R}}(\mathbb{R})$ we can write

\[ S_{2^k} f(x) = \varphi_{2^k} f(x) + S_{2^k} (\psi_{2^k} * f)(x), \]

where $\varphi_{2^k}, \psi_{2^k}$ are defined as in Lemma 4.3. Therefore,

\[ \sup |S_{2^k} f(x)| \leq CM \left( \sum_{k} |S_{2^k} f(x)| \right)^{1/2}. \]

The $W^{1,1}$-norm of the first term is majorized by $C \| f \|_{L^{1/2}}$. For the second term, we use Lemma 4.4 (with $s = q$) and then decompose $f \in H^1_{\mathbb{R}}$ into B-atoms: $f(x) = \sum_{j} \lambda_j \varphi_j(x)$, with $\sum |\lambda_j| \leq C \| f \|_{L^{1/2}}$. Thus

\[ \left( \sum_{k} \| S_{2^k} (\psi_{2^k} * f)(x) \| \right)^{1/2} \leq C \left( \sum_{k} \| \psi_{2^k} * f(x) \| \right)^{1/2} \leq C \sum_{j} |\lambda_j| \left( \sum_{k} \| \psi_{2^k} \varphi_{2^k} \| \right)^{1/2} dx \]

\[ \leq C \sum_{j} |\lambda_j| \left( \sum_{k} \| \psi_{2^k} \varphi_{2^k} \| \right)^{1/2} dx \]

\[ \leq (\text{by Lemma 4.3}) \leq C \| f \|_{L^{1/2}}. \]

It is evident that Theorem 4.1 has an analogous formulation in the periodic setting. We state it as a corollary.

4.5. Corollary. Let $f \in \text{L}^1_{\mathbb{R}}(\mathbb{T})$ be a function whose Fourier series is of analytic type, i.e. $f(k) = 0$ for all $k < 0$. Then

\[ \lim_{N \to \infty} \sum_{k=0}^{N} e^{ikx} f(k) = f(x) \quad (a.e. \ x \in \mathbb{T}). \]

The arguments that we have used are very close to those of classical Littlewood-Paley theory. Without going into the details, let us merely state what one can obtain in the vector valued setting as a substitute for the standard results for scalar functions:

\[ \| f \|_{L^p} \sim \left( \sum_{k} \| S_{k} f(x) \|_{L^p}^{2} \right)^{1/2} \quad (1 < p < \infty) \]

where $\Delta$ is the family of dyadic intervals in $\mathbb{R}$ and $(S_{k} f) \sim f_{k}$. Since $S_{k}$ can be easily written in terms of the Hilbert transform, we remark that the partial sums of operators $S_{k}$ make sense and are uniformly bounded in $L^p_{\mathbb{R}}$, $1 < p < \infty$.

4.6. Theorem. Let $B$ be of (Rademacher) type $p > 1$ and cotype $q < \infty$. Then, for every $f \in \text{L}^p_{\mathbb{R}}(\mathbb{R})$, $1 < r < \infty$,

\[ C_{r} \left( \sum_{i} \| S_{i} f(x) \|_{L^q}^{2} \right)^{1/2} \leq \| f \|_{L^p_{\mathbb{R}}} \leq C_{r} \left( \sum_{i} \| S_{i} f(x) \|_{L^q}^{2} \right)^{1/2}, \]

and both inequalities are best possible in the sense that the first one implies that $B$ has cotype $q$ and the second one implies that $B$ has type $p$. A similar result holds for $\mathbb{F}^{*}$.

§ 5. A stability property of the class UMD. The method of proof of Theorem 3.1 can be used to find a procedure to obtain new UMD spaces. We have already mentioned the fact that, if $B \in \text{UMD}$, then also $\mathcal{F}(B) \in \text{UMD}$ for $1 < s < \infty$. We know that $B$ has an unconditional basis (or is a lattice), and $\mathbb{Y}$ is an arbitrary Banach space, one can also define the space

\[ B(Y) = \{(y_{n})_{n \in \mathbb{N}} : (y_{n} \varpi_{n})_{n \in \mathbb{N}} \in B \} \]

with the natural norm. When $\mathbb{Y}$ is a Hilbert space, it follows from Grothendieck's fundamental inequality (see [11]) that $B(Y) \in \text{UMD}$. The same turns out to be true for an arbitrary $B \in \text{UMD}$ and, more generally, we can state the following

5.1. Theorem. Let $(Y_{n})_{n \in \mathbb{N}}$ be a sequence of UMD Banach spaces with uniformly bounded UMD-constants, and let $B$ be, as in § 2, another UMD-space with an unconditional basis. Define the space

\[ B = B \left( \bigoplus_{n=1}^{\infty} Y_{n} \right) \]

which consists of all sequences $y = (y_{n})_{n \in \mathbb{N}}$ with $y_{n} \in Y_{n}$, such that $\{(y_{n})_{n \in \mathbb{N}} \in B \}$, the norm in $Y$ being

\[ \| y \|_{B} = \| \{(y_{n})_{n \in \mathbb{N}} \} \|_{B}. \]

Then $\mathbb{Y} \in \text{UMD}$. 
Such a statement was conjectured to the author by J. Bourgain during the Colloque Laurent Schwartz, 1983, and an independent, somewhat different, proof has been simultaneously found by Bourgain himself.

Proof. A function \( F \) is a sequence \( F = (F_j)_{j \in \mathbb{N}} \) with \( F_j \in L^2_\beta (\mathbb{R}) \) and

\[
\|F\|_{L^2_\beta}^2 = \left\{ \sum_j (\|F_j(x)\|_{L^2_\beta})^2 \right\}^{1/2}.
\]

We shall use the same notation, \( \tilde{H} \), to indicate the \( Y_j \)-valued Hilbert transform for all \( j \in \mathbb{N} \). We shall also define

\[
\tilde{H} F(x) = \langle \tilde{H} F_j(x) \rangle_{j \in \mathbb{N}}
\]

and try to prove that this is a bounded operator in \( L^2_\beta \). Let \( f_j \) be the nonnegative \( L^2 \) functions \( f_j (x) = \|F_j(x)\|_{L^2_\beta} \), \( f_j \in L^2_\mu \), \( f_j (0) = 0 \). We apply Lemma 2.4 to get

\[
\|\tilde{H} f_j\|_{L^2_\mu}^2 = \sum_j f_j^*(x) = \|F_j f_j\|_{L^2_\beta} \leq C, \text{ for } r > 1, C,
\]

where the "sup" is taken over all \( g = (g_j)_{j \in \mathbb{N}} \in L^p_\beta \) of unit norm, such that \( g_j \in \mathcal{C}(\mathbb{R}) \), \( g_j (0) = 0 \). We apply Lemma 2.4 to get

\[
\|\tilde{H} f_j\|_{L^2_\mu}^2 \leq C, \sup_j \left\{ \sum_j M_j (\|F_j f_j\|_{L^2_\beta}) (x) M_j g_j (x) \right\} \leq C, \text{ for } r > 1, C.
\]

Finally, we take \( r \) close enough to 1 so that Lemma 2.2 applies, and it allows us to drop the operator \( M_j \) in the last expression by suitably enlarging the constant \( C \). This ends the proof.

As a final remark, let me mention that the results of Sections 3 and 4 are probably true for arbitrary \( B \subseteq UMD, \) the restriction which consists in assuming the existence of unconditional basis in \( B \) is due to the method of proof, and a different approach should be found in order to get rid of such a restriction in our results as well as in those of [33].

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