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On the $(C, \alpha \geqslant 0, \beta \geqslant 0)$-summability of double orthogonal series

by

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Abstract. The classical coefficient test for the $(C, \alpha > 0)$-summability of single orthogonal series is due to Men'shov (1926) and Kaczmarz (1927) for $\alpha = 1$, and to Zygmund (1927) for every $\alpha > 0$. The present author has already extended the Men'shov–Kaczmarz theorem for double orthogonal series in [6] giving a coefficient test for the $(C, 1, 1)$-summability. The main purpose of this paper is to supply the corresponding extension of the Zygmund theorem for double orthogonal series. This yields a coefficient test for the $(C, \alpha > 0, \beta > 0)$-summability. In addition, the problem of $(C, \alpha > 0, \beta = 0)$-summability of double orthogonal series is also dealt with.

1. Introduction. Let $(X, \mathscr{F}, \mu)$ be an arbitrary positive measure space and $\{\varphi_{ik}(x):\ i, k = 0, 1, \ldots\}$ an orthonormal system on X. We consider the double orthogonal series

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik} \varphi_{ik}(x), \tag{1.1}$$

where $\{a_{ik}:\ i, k = 0, 1, \ldots\}$ is a sequence of real numbers for which

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

By the well-known Riesz–Fischer theorem, there exists a function $f(x) \in L^2 = L^2(X, \mathscr{F}, \mu)$ such that the rectangular partial sums

$$s_{mn}(x) = \sum_{i=0}^{m} \sum_{k=0}^{n} a_{ik} \varphi_{ik}(x) \qquad (m, n = 0, 1, \ldots)$$

of series (1.1) converge to $f(x)$ in the L^2-metric:

$$\int [s_{mn}(x) - f(x)]^2 \, d\mu(x) \to 0 \quad \text{as} \quad \min(m, n) \to \infty.$$

Here and in the sequel, the integrals are taken over the entire space X.

Let α and β be real numbers, $\alpha > -1$ and $\beta > -1$. We remind that the (C, α, β)-means of series (1.1) are defined as follows:

$$\sigma_{mn}^{\alpha\beta}(x) = \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{\alpha-1} A_{n-k}^{\beta-1} s_{ik}(x)$$
$$= \frac{1}{A_m^\alpha A_n^\beta} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{\alpha} A_{n-k}^{\beta} a_{ik} \varphi_{ik}(x) \qquad (m, n = 0, 1, \ldots),$$

where

$$A_m^\alpha = \binom{m+\alpha}{m} = \begin{cases} (\alpha+1)(\alpha+2)\ldots(\alpha+m)/m! & \text{for } m = 1, 2, \ldots, \\ 1 & \text{for } m = 0 \end{cases}$$

(see, e.g. [10], p. 77).

The case $\alpha = \beta = 0$ gives back the rectangular partial sums $s_{mn}(x) = \sigma_{mn}^{00}(x)$. The case $\alpha = \beta = 1$ provides the first arithmetic means with respect to m and n:

$$\sigma_{mn}^{11}(x) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} s_{ik}(x)$$
$$= \sum_{i=0}^{m} \sum_{k=0}^{n} \left(1 - \frac{i}{m+1}\right)\left(1 - \frac{k}{n+1}\right) a_{ik} \varphi_{ik}(x).$$

Furthermore, the case $\alpha = 1$ and $\beta = 0$ provides the first arithmetic means with respect to m:

$$\sigma_{mn}^{10}(x) = \frac{1}{m+1} \sum_{i=0}^{m} s_{in}(x) = \sum_{i=0}^{m} \sum_{k=0}^{n} \left(1 - \frac{i}{m+1}\right) a_{ik} \varphi_{ik}(x),$$

while the case $\alpha = 0$ and $\beta = 1$ provides the first arithmetic means with respect to n.

Before stating the preliminary results, we make the following convention. Given a double sequence $\{f_{mn}(x)\colon m, n = 0, 1, \ldots\}$ of functions in L^2 and a double sequence $\{\lambda_{mn}\}$ of positive numbers, we write

$$f_{mn}(x) = o_x\{\lambda_{mn}\} \text{ a.e.} \quad \text{as} \quad \min(m, n) \to \infty$$
$$(\text{or } \max(m, n) \to \infty, \text{ or } m \to \infty, \text{ or } n \to \infty)$$

if

$$f_{mn}(x)/\lambda_{mn} \to 0 \text{ a.e.} \quad \text{as} \quad \min(m, n) \to \infty$$
$$(\text{or } \max(m, n) \to \infty, \text{ or } m \to \infty, \text{ or } n \to \infty)$$

and, in addition, there exists a function $F(x) \in L^2$ such that

$$\sup_{m,n \geq 0} |f_{mn}(x)|/\lambda_{mn} \leqslant F(x) \quad \text{a.e.}$$

2. Preliminary results. The extension of the famous Rademacher–Men'shov theorem proved by a number of authors (see, e.g. [1], [5], etc.) reads as follows.

THEOREM A. *If*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 [\log(i+2)]^2 [\log(k+2)]^2 < \infty, \tag{2.1}$$

then

$$s_{mn}(x) - f(x) = o_x\{1\} \quad \text{a.e. as} \quad \min(m, n) \to \infty.$$

In this paper the logarithms are to the base 2.

This theorem is exact in the sense that $\log(t+2)$ cannot be replaced in it by any sequence $\varrho(t)$ tending to ∞ slower as $t \to \infty$ (cf. [8]).

The convergence behavior improves when considering $\sigma_{mn}^{11}(x)$ instead of $s_{mn}(x)$. The following extension of the Men'shov–Kaczmarz theorem was proved in [6].

THEOREM B. *If*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 [\log\log(i+4)]^2 [\log\log(k+4)]^2 < \infty, \tag{2.2}$$

then

$$\sigma_{mn}^{11}(x) - f(x) = o_x\{1\} \quad \text{a.e. as} \quad \min(m, n) \to \infty.$$

It was pointed out by Fedulov [3] that Theorem B is the best possible in the same sense as Theorem A is.

The coefficient test that ensures the a.e. convergence of $\sigma_{mn}^{10}(x)$ lies between (2.1) and (2.2). (See again [6].)

THEOREM C. *If*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 [\log\log(i+4)]^2 [\log(k+2)]^2 < \infty, \tag{2.3}$$

then

$$\sigma_{mn}^{10}(x) - f(x) = o_x\{1\} \quad \text{a.e. as} \quad \min(m, n) \to \infty.$$

3. Main results. Our first goal is to prove that condition (2.2) is also sufficient for the a.e. (C, α, β)-summability of series (1.1) for all $\alpha > 0$, $\beta > 0$. Besides, we show that condition (2.3) ensures the a.e. $(C, \alpha, 0)$-summability for every $\alpha > 0$.

THEOREM 1. *If $\alpha > 0$ and condition (2.3) is satisfied, then*

$$\sigma_{mn}^{\alpha 0}(x) - f(x) = o_x\{1\} \quad \text{a.e. as} \quad \min(m, n) \to \infty. \tag{3.1}$$

THEOREM 2. *If $\alpha > 0$, $\beta > 0$ and condition* (2.2) *is satisfied, then*

$$\sigma_{mn}^{\alpha\beta}(x) - f(x) = o_x\{1\} \quad \text{a.e. as} \quad \min(m, n) \to \infty. \tag{3.2}$$

These two theorems can be considered the extension of the well-known result of Zygmund [9] from single orthogonal series to double ones.

The following two theorems play a key role in the proofs of Theorems 1 and 2.

THEOREM 3. *If $\alpha > \frac{1}{2}$ and the condition*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2 [\log(k+2)]^2 < \infty \tag{3.3}$$

is satisfied, then

$$\delta_{Mn}^{\alpha}(x) = \left\{\frac{1}{M+1}\sum_{m=0}^{M} [\sigma_{mn}^{\alpha-1,0}(x) - \sigma_{mn}^{\alpha 0}(x)]^2\right\}^{1/2} = o_x\{1\} \quad \text{a.e. as} \quad M \to \infty,$$

uniformly in n.

THEOREM 4. *If $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$ and the condition*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2 [\log\log(\max(i, k)+4)]^2 < \infty$$

is satisfied, then

$$\varepsilon_{MN}^{\alpha\beta}(x) = \left\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N} [\sigma_{mn}^{\alpha-1,\beta-1}(x) - \sigma_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2}$$
$$= o_x\{1\} \quad \text{a.e. as} \quad \min(M, N) \to \infty.$$

On the other hand, taking Theorems 1, 2, 3 and 4 for granted, we can deduce two interesting theorems on the so-called *strong (C, α, β)-summability* of series (1.1).

THEOREM 5. *If $\alpha > \frac{1}{2}$ and condition* (2.3) *is satisfied, then*

$$\left\{\frac{1}{M+1}\sum_{m=0}^{M} [\sigma_{mn}^{\alpha-1,0}(x) - f(x)]^2\right\}^{1/2} = o_x\{1\} \quad \text{a.e. as} \quad \min(M, n) \to \infty.$$

THEOREM 6. *If $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$ and condition* (2.2) *is satisfied, then*

$$\left\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N} [\sigma_{mn}^{\alpha-1,\beta-1}(x) - f(x)]^2\right\}^{1/2} = o_x\{1\} \quad \text{a.e.}$$
$$\text{as} \quad \min(M, N) \to \infty.$$

For example, Theorem 6 immediately follows from Theorems 2 and 4 if we take into account that

$$\Big\{\sum_{m=0}^{M}\sum_{n=0}^{N} [\sigma_{mn}^{\alpha-1,\beta-1}(x) - f(x)]^2\Big\}^{1/2}$$
$$\leqslant \Big\{\sum_{m=0}^{M}\sum_{n=0}^{N} [\sigma_{mn}^{\alpha-1,\beta-1}(x) - \sigma_{mn}^{\alpha\beta}(x)]^2\Big\}^{1/2} + \Big\{\sum_{m=0}^{M}\sum_{n=0}^{N} [\sigma_{mn}^{\alpha\beta}(x) - f(x)]^2\Big\}^{1/2}.$$

We note that Theorem 6 in the special case $\alpha = \beta = 1$ was proved in [7] using another method.

4. Auxiliary results. In this section we treat the (C, α, β)-means of the numerical series

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u_{ik},$$

the u_{ik} are real numbers, defined by

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{A_m^{\alpha} A_n^{\beta}}\sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{\alpha} A_{n-k}^{\beta} u_{ik} \quad (m, n = 0, 1, \ldots;\ \alpha > -1,\ \beta > -1).$$

We remind some identities and inequalities well-known in the literature. For all α and γ,

$$A_m^{\alpha+\gamma+1} = \sum_{i=0}^{m} A_i^{\alpha} A_{m-i}^{\gamma} \tag{4.1}$$

(see, e.g. [10], p. 77, formula (1.10)). Hence the representations

$$\sigma_{mn}^{\alpha+\gamma,0} = \frac{1}{A_m^{\alpha+\gamma}}\sum_{i=0}^{m} A_{m-i}^{\gamma-1} A_i^{\alpha} \sigma_{in}^{\alpha 0}$$

and

$$\sigma_{mn}^{\alpha+\gamma,\beta+\delta} = \frac{1}{A_m^{\alpha+\gamma} A_n^{\beta+\delta}}\sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{\gamma-1} A_{n-k}^{\delta-1} A_i^{\alpha} A_k^{\beta} \sigma_{ik}^{\alpha\beta} \tag{4.2}$$

easily follow (cf. the corresponding formula for single series in [10], p. 78, the first formula at the top).

We often need the following estimate, as well. There exist two positive constants C_1 and C_2 such that

$$C_1 \leqslant \frac{A_m^{\alpha}}{m^{\alpha}} \leqslant C_2 \quad (m = 1, 2, \ldots;\ \alpha > -1) \tag{4.3}$$

(see, e.g. [2], p. 69, formula (25), or [10], p. 77, formula (1.18)). This helps us to obtain the following two Tauberian type results.

LEMMA 1. *If* $\alpha > -\frac{1}{2}$, $\varepsilon > 0$ *and*

$$\frac{1}{M+1}\sum_{m=0}^{M}[\sigma_{mn}^{\alpha 0}]^2 \to 0 \quad \textit{as} \quad \min(M, n)\to\infty,$$

then

$$\sigma_{Mn}^{\alpha+1/2+\varepsilon,0}\to 0 \quad \textit{as} \quad \min(M, n)\to\infty.$$

Furthermore, if

$$\frac{1}{M+1}\sum_{m=0}^{M}[\sigma_{mn}^{\alpha 0}]^2 \leqslant B^2 \quad (M, n = 0, 1, \ldots)$$

with a positive number B, then there exists a constant C depending only on α *and* ε *such that*

$$|\sigma_{Mn}^{\alpha+1/2+\varepsilon,0}| \leqslant CB \quad (M, n = 0, 1, \ldots).$$

LEMMA 2. *If* $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $\varepsilon > 0$, $\eta > 0$ *and*

$$\text{(4.4)} \qquad \frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}[\sigma_{mn}^{\alpha\beta}]^2 \to 0 \quad \textit{as} \quad \min(M, N)\to\infty,$$

then

$$\text{(4.5)} \qquad \sigma_{MN}^{\alpha+1/2+\varepsilon,\beta+1/2+\eta}\to 0 \quad \textit{as} \quad \min(M, N)\to\infty.$$

Furthermore, if

$$\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}[\sigma_{mn}^{\alpha\beta}]^2 \leqslant B^2 \quad (M, N = 0, 1, \ldots)$$

with a positive number B, then there exists a constant C depending only on α, β, ε, *and* η *such that*

$$|\sigma_{MN}^{\alpha+1/2+\varepsilon,\beta+1/2+\eta}| \leqslant CB \quad (M, N = 0, 1, \ldots).$$

The corresponding result for single series was established by Zygmund [9], pp. 360–361. Keeping his proof in mind, the proofs of Lemmas 1 and 2 are routines. For the sake of completeness, we show here how condition (4.4) implies statement (4.5). To this end, by (4.2),

$$\sigma_{MN}^{\alpha+1/2+\varepsilon,\beta+1/2+\eta} = \frac{1}{A_M^{\alpha+1/2+\varepsilon}A_N^{\beta+1/2+\eta}}\sum_{m=0}^{M}\sum_{n=0}^{N}A_{M-m}^{-1/2+\varepsilon}A_{N-n}^{-1/2+\eta}A_m^{\alpha}A_n^{\beta}\sigma_{mn}^{\alpha\beta}.$$

Hence, using the Cauchy inequality,

$$|\sigma_{MN}^{\alpha+1/2+\varepsilon,\beta+1/2+\eta}| \leqslant \frac{1}{A_M^{\alpha+1/2+\varepsilon}A_N^{\beta+1/2+\eta}}\Big\{\sum_{m=0}^{M}\sum_{n=0}^{N}[\sigma_{mn}^{\alpha\beta}]^2\Big\}^{1/2}\times$$
$$\times\Big\{\sum_{m=0}^{M}[A_{M-m}^{-1/2+\varepsilon}A_m^{\alpha}]^2\sum_{n=0}^{N}[A_{N-n}^{-1/2+\eta}A_n^{\beta}]^2\Big\}^{1/2}.$$

Taking into account (4.1), (4.3) and (4.4), it is not hard to deduce that

$$|\sigma_{MN}^{\alpha+1/2+\varepsilon,\beta+1/2+\eta}| = O\Big\{\frac{1}{(M+1)^{\alpha+1/2+\varepsilon}(N+1)^{\beta+1/2+\eta}}\Big\}\times$$
$$\times o\{(M+1)^{1/2}(N+1)^{1/2}\}\,O\{(M+1)^{\alpha+\varepsilon}(N+1)^{\beta+\eta}\} = o\{1\}$$
$$\text{as} \quad \min(M, N)\to\infty,$$

which is (4.5) to be proved.

We will make use of the following representations, too:

$$\text{(4.6)} \qquad \sigma_{mn}^{\alpha-1,\beta} - \sigma_{mn}^{\alpha\beta} = \frac{1}{\alpha A_m^{\alpha}A_n^{\beta}}\sum_{i=1}^{m}\sum_{k=0}^{n}A_{m-i}^{\alpha-1}A_{n-k}^{\beta}\,iu_{ik} \quad (\alpha > 0,\ \beta > -1)$$

and

$$\text{(4.7)} \qquad \sigma_{mn}^{\alpha-1,\beta-1} - \sigma_{mn}^{\alpha-1,\beta} - \sigma_{mn}^{\alpha,\beta-1} + \sigma_{mn}^{\alpha\beta}$$
$$= \frac{1}{\alpha\beta A_m^{\alpha}A_n^{\beta}}\sum_{i=1}^{m}\sum_{k=1}^{n}A_{m-i}^{\alpha-1}A_{n-k}^{\beta-1}\,iku_{ik} \quad (\alpha > 0,\ \beta > 0).$$

Both representations follow from the identities

$$A_m^{\alpha-1} = \frac{\alpha}{\alpha+m}A_m^{\alpha} \quad \text{and} \quad A_{m-i}^{\alpha} = \frac{\alpha+m-i}{\alpha}A_{m-i}^{\alpha-1}.$$

Finally, we present the following two useful inequalities:

$$\text{(4.8)} \qquad \sum_{m=i}^{\infty}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2 = O\Big\{\frac{1}{i}\Big\} \quad (i = 1, 2, \ldots;\ \alpha > 1/2)$$

and

$$\text{(4.9)} \qquad \sum_{m=i}^{\infty}\frac{1}{m}\left[1-\frac{A_{m-i}^{\alpha}}{A_m^{\alpha}}\right]^2 = O\{1\} \quad (i = 1, 2, \ldots;\ \alpha > 0).$$

The first inequality is well-known in the literature (see, e.g. [2], p. 110), while the second one can be proved in the following way. We start with the identity

$$A_m^{\alpha} - A_{m-1}^{\alpha} = A_m^{\alpha-1}$$

(see, e.g. [10], p. 77, formula (1.12)). This immediately implies

$$A_m^\alpha - A_{m-i}^\alpha = \sum_{j=m-i+1}^{m} A_j^{\alpha-1} \quad (i = 1, 2, \ldots, m).$$

By (4.3) there exists a constant C, depending on α, such that

$$0 \leqslant 1 - \frac{A_{m-i}^\alpha}{A_m^\alpha} \leqslant \frac{C}{m^\alpha} \sum_{j=m-i+1}^{m} j^{\alpha-1} \leqslant C \frac{m^\alpha - (m-i)^\alpha}{m^\alpha}$$

$$(i = 1, 2, \ldots, m;\ m = 1, 2, \ldots).$$

Since for $0 < \alpha \leqslant 1$ we trivially have

$$1 - \left(1 - \frac{i}{m}\right)^\alpha \leqslant \frac{i}{m},$$

while for $\alpha \geqslant 1$, using a convexity argument,

$$\frac{m^\alpha - (m-i)^\alpha}{m^\alpha} \leqslant \frac{\alpha m^{\alpha-1} i}{m^\alpha} = \alpha \frac{i}{m},$$

we can infer that

$$1 - \frac{A_{m-i}^\alpha}{A_m^\alpha} = O\left\{\frac{i}{m}\right\} \quad (i = 1, 2, \ldots, m;\ m = 1, 2, \ldots;\ \alpha > 0).$$

Now,

$$\sum_{m=i}^{\infty} \frac{1}{m} \left[1 - \frac{A_{m-i}^\alpha}{A_m^\alpha}\right]^2 = O\{1\} \sum_{m=i}^{\infty} \frac{i^2}{m^3} = O\{1\},$$

proving (4.9).

5. Proof of Theorem 1. The proof is done on the basis of Theorem 3, which will be proved in Section 7, and of the following consequence of Lemma 1.

COROLLARY 1. *If* $\alpha > -\frac{1}{2}$, $\varepsilon > 0$ *and*

$$(5.1) \quad \left\{\frac{1}{M+1} \sum_{m=0}^{M} [\sigma_{mn}^{\alpha 0}(x) - f(x)]^2\right\}^{1/2} = o_x\{1\} \quad \textit{a.e. as} \quad \min(M, n) \to \infty,$$

then

$$(5.2) \quad \sigma_{Mn}^{\alpha+1/2+\varepsilon, 0}(x) - f(x) = o_x\{1\} \quad \textit{a.e. as} \quad \min(M, n) \to \infty.$$

In fact, setting

$$u_{00} = a_{00}\varphi_{00}(x) - f(x) \quad \text{and} \quad u_{ik} = a_{ik}\varphi_{ik}(x) \quad (i^2 + k^2 > 0),$$

Corollary 1 immediately follows from Lemma 1.

After these preliminaries, the proof of (3.1) is quite simple. By Theorem C, (3.1) holds for $\alpha = 1$. Hence, by Theorem 3, we obtain (5.1) for $\alpha = 0$. Thus, by Corollary 1, we get (5.2) for $\alpha = 0$. Applying again Theorem 3, we find (5.1) for $\alpha = -\frac{1}{2} + \varepsilon$. Hence, again by Corollary 1, we obtain (5.2) for $\alpha = 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, this is equivalent to (3.1).

6. Proof of Theorem 2. The proof relies on Theorem 4 which will be proved in Section 8, and on the following consequence of Lemma 2.

COROLLARY 2. *If* $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$, $\varepsilon > 0$, $\eta > 0$ *and*

$$(6.1) \quad \left\{\frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} [\sigma_{mn}^{\alpha\beta}(x) - f(x)]^2\right\}^{1/2} = o_x\{1\} \quad \textit{a.e.}$$

$$\textit{as} \quad \min(M, N) \to \infty,$$

then

$$(6.2) \quad \sigma_{MN}^{\alpha+1/2+\varepsilon, \beta+1/2+\eta}(x) - f(x) = o_x\{1\} \quad \textit{a.e. as} \quad \min(M, N) \to \infty.$$

Now, by Theorem B, relation (3.2) holds for $\alpha = \beta = 1$. Hence, by Theorem 4, we get (6.1) for $\alpha = \beta = 0$. Thus, by Corollary 2, we obtain (6.2) for $\alpha = \beta = 0$. Using again Theorem 4, we find (6.1) for $\alpha = -\frac{1}{2} + \varepsilon$ and $\beta = -\frac{1}{2} + \eta$. Hence, by Corollary 2, we get (6.2) for $\alpha = 2\varepsilon$ and $\beta = 2\eta$. Since ε and η are arbitrary positive numbers, this is equivalent to (3.2) to be proved.

7. Proof of Theorem 3. The proof is made in four steps.

(i) If M is a positive integer, then $2^{p-1} < M \leqslant 2^p$ with some nonnegative integer p. (For simplicity in notation, we neglect the case $M = 0$.) Since

$$\delta_{Mn}^\alpha(x) \leqslant \sqrt{2}\, \delta_{2^p, n}^\alpha(x),$$

it is enough to prove that

$$\delta_{2^p, n}^\alpha(x) = o_x\{1\} \quad \text{a.e. as} \quad p \to \infty,$$

uniformly in n.

(ii) We can perform one more reduction. It is clear that

$$(7.1) \quad [\delta_{2^p, n}^\alpha(x)]^2 = \sum_{r=-2}^{p-1} 2^{r-p+2} \cdot \frac{1}{2^{r+2}} \sum_{m=2^r+1}^{2^{r+1}} [\sigma_{mn}^{\alpha-1, 0}(x) - \sigma_{mn}^{\alpha 0}(x)]^2,$$

where we make the following convention: for $r = -2$ and -1 by 2^r we mean -1 and 0, respectively. Also, if we prove that

$$(7.2) \quad {}^1\delta_{pn}^\alpha(x) = \left\{\frac{1}{2^p} \sum_{m=2^p+1}^{2^{p+1}} [\sigma_{mn}^{\alpha-1, 0}(x) - \sigma_{mn}^{\alpha 0}(x)]^2\right\}^{1/2}$$

$$= o_x\{1\} \quad \text{a.e. as} \quad p \to \infty,$$

uniformly in n, then we are done. For convenience, we assume that $p \geqslant 0$ from now on.

(iii) First, we prove (7.2) for the special case $n=2^q$:

$$(7.3)\qquad {}^1\delta^{\alpha}_{p,2^q}(x)=o_x\{1\}\quad \text{a.e. as}\quad p\to\infty,$$

uniformly in q, $q\geqslant -1$.

To this effect, by (4.6) and the Cauchy inequality,

$$(7.4)\qquad [{}^1\delta^{\alpha}_{p,2^q}(x)]^2=\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\Big[\sum_{i=1}^{m}\sum_{k=0}^{2^q}\frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m}ia_{ik}\varphi_{ik}(x)\Big]^2$$

$$\leqslant\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\sum_{t=-2}^{q-1}(t+3)^2\Big[\sum_{i=1}^{m}\sum_{k=2^t+1}^{2^{t+1}}\frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m}ia_{ik}\varphi_{ik}(x)\Big]^2\times$$

$$\times\sum_{t=-2}^{q-1}\frac{1}{(t+3)^2},$$

with the same agreement concerning 2^t for $t=-2$ and -1 as we made after (7.1). This inequality motivates the following definition:

$$F^2(x)=\sum_{p=0}^{\infty}\sum_{t=-2}^{\infty}\frac{(t+3)^2}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\Big[\sum_{i=1}^{m}\sum_{k=2^t+1}^{2^{t+1}}\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}ia_{ik}\varphi_{ik}(x)\Big]^2.$$

If the termwise integrated series is finite, then B. Levi's theorem implies $F(x)\in L^2$ and thus relation (7.3) is proved. But this is the case as the following computation shows:

$$\int F^2(x)\,d\mu(x)=\sum_{p=0}^{\infty}\sum_{t=-2}^{\infty}\frac{(t+3)^2}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\sum_{i=1}^{m}\sum_{k=2^t+1}^{2^{t+1}}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2 i^2a^2_{ik}$$

$$\leqslant\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\sum_{i=1}^{m}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2 i^2a^2_{ik}[\log 8(k+1)]^2$$

$$\leqslant 2\sum_{m=2}^{\infty}\sum_{k=0}^{\infty}\sum_{i=1}^{m}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2 ia^2_{ik}[\log 8(k+1)]^2$$

$$\leqslant 2\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}ia^2_{ik}[\log 8(k+1)]^2\sum_{m=i}^{\infty}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2<\infty,$$

the last series being finite due to (4.8) and (3.3).

(iv) Now, let $2^q<n\leqslant 2^{q+1}$ with some $q\geqslant 1$. Since

$${}^1\delta^{\alpha}_{pn}(x)\leqslant{}^1\delta_{p,2^q}(x)+\Big\{\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\Big[\sum_{i=1}^{m}\sum_{k=2^q+1}^{n}\frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m}ia_{ik}\varphi_{ik}(x)\Big]^2\Big\}^{1/2},$$

we can estimate as follows:

$$(7.5)\qquad \max_{2^q<n\leqslant 2^{q+1}}{}^1\delta^{\alpha}_{pn}(x)\leqslant{}^1\delta^{\alpha}_{p,2^q}(x)+M^{\alpha}_{pq}(x),$$

where

$$[M^{\alpha}_{pq}(x)]^2=\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\max_{2^q<n\leqslant 2^{q+1}}\Big[\sum_{i=1}^{m}\sum_{k=2^q+1}^{n}\frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m}ia_{ik}\varphi_{ik}(x)\Big]^2.$$

Applying the Rademacher–Men'shov inequality (see [2], p. 79 or [4], Theorem 3) separately for each fixed m, we get

$$\alpha^2\int[M^{\alpha}_{pq}(x)]^2\,d\mu(x)\leqslant\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}(\log 2^{q+1})^2\sum_{i=1}^{m}\sum_{k=2^q+1}^{2^{q+1}}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2 i^2a^2_{ik}.$$

Consequently, again by (4.8) and (3.3),

$$\alpha^2\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\int[M^{\alpha}_{pq}(x)]^2\,d\mu(x)$$

$$\leqslant\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}(\log 2^{q+1})^2\sum_{i=1}^{m}\sum_{k=2^q+1}^{2^{q+1}}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2 i^2a^2_{ik}$$

$$\leqslant\sum_{p=0}^{\infty}\sum_{k=3}^{\infty}\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\sum_{i=1}^{m}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2 i^2a^2_{ik}[\log 2k]^2$$

$$\leqslant 2\sum_{m=2}^{\infty}\sum_{k=3}^{\infty}\sum_{i=1}^{m}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2 ia^2_{ik}[\log 2k]^2$$

$$\leqslant 2\sum_{i=1}^{\infty}\sum_{k=3}^{\infty}ia^2_{ik}[\log 2k]^2\sum_{m=i}^{\infty}\Big[\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\Big]^2<\infty.$$

Hence B. Levi's theorem implies that

$$(7.6)\qquad M^{\alpha}_{pq}(x)=o_x\{1\}\quad\text{a.e. as}\quad \max(p,q)\to\infty.$$

Combining (7.3), (7.5) and (7.6) we find (7.2) to be proved.

8. Proof of Theorem 4. By the triangle inequality,

$$\varepsilon^{\alpha\beta}_{MN}(x)\leqslant\Big\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}[\sigma^{\alpha-1,\beta-1}_{mn}(x)-\sigma^{\alpha-1,\beta}_{mn}(x)-\sigma^{\alpha,\beta-1}_{mn}(x)+$$

$$+\sigma^{\alpha\beta}_{mn}(x)]^2\Big\}^{1/2}+\Big\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}[\sigma^{\alpha-1,\beta}_{mn}(x)-\sigma^{\alpha\beta}_{mn}(x)]^2\Big\}^{1/2}+$$

$$+\Big\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}[\sigma^{\alpha,\beta-1}_{mn}(x)-\sigma^{\alpha\beta}_{mn}(x)]^2\Big\}^{1/2}$$

$$={}^1\varepsilon^{\alpha\beta}_{MN}(x)+{}^2\varepsilon^{\alpha\beta}_{MN}(x)+{}^3\varepsilon^{\alpha\beta}_{MN}(x).$$

Thus, Theorem 4 will be proved by Propositions 1 and 2 below.

PROPOSITION 1. *If $\alpha > \frac{1}{2}$, $\beta > \frac{1}{2}$ and condition* (1.2) *is satisfied, then*

$$^1\varepsilon_{MN}^{\alpha\beta}(x) = o_x\{1\} \quad \text{a.e. as } \max(M, N) \to \infty.$$

Proof of Proposition 1. Let $2^{p-1} < M \leqslant 2^p$ and $2^{q-1} < N \leqslant 2^q$ with certain nonnegative integers p and q. The cases $M = 0$ or $N = 0$ are neglected here, simply because of difficulties in notation. Since

$$^1\varepsilon_{MN}^{\alpha\beta}(x) \leqslant 2\, {}^1\varepsilon_{2^p,2^q}^{\alpha\beta}(x),$$

it is enough to prove

(8.1) $$^1\varepsilon_{2^p,2^q}^{\alpha\beta}(x) = o_x\{1\} \quad \text{a.e. as} \quad \max(p, q) \to \infty.$$

By representation (4.7),

$$\int [{}^1\varepsilon_{2^p,2^q}^{\alpha\beta}(x)]^2\, d\mu(x) = \frac{1}{\alpha^2\beta^2(2^p+1)(2^q+1)} \sum_{m=1}^{2^p}\sum_{n=1}^{2^q}\sum_{i=1}^{m}\sum_{k=1}^{n} \left[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\right]^2 \left[\frac{A_{n-k}^{\beta-1}}{A_n^\beta}\right]^2 i^2k^2a_{ik}^2$$

$$= \frac{1}{\alpha^2\beta^2(2^p+1)(2^q+1)} \sum_{i=1}^{2^p}\sum_{k=1}^{2^q} i^2k^2a_{ik}^2 \sum_{m=i}^{2^p}\left[\frac{A_{m-1}^{\alpha-1}}{A_m^\alpha}\right]^2 \sum_{n=k}^{2^q}\left[\frac{A_{n-k}^{\beta-1}}{A_n^\beta}\right]^2.$$

Taking into account inequality (4.8), hence

$$\int [{}^1\varepsilon_{2^p,2^q}^{\alpha\beta}(x)]^2\, d\mu(x) = \frac{O\{1\}}{2^p 2^q}\sum_{i=1}^{2^p}\sum_{k=1}^{2^q} ika_{ik}^2.$$

Performing double summation yields

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\int [{}^1\varepsilon_{2^p,2^q}^{\alpha\beta}(x)]^2\, d\mu(x) = O\{1\}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2^p2^q}\sum_{i=1}^{2^p}\sum_{k=1}^{2^q} ika_{ik}^2$$

$$= O\{1\}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} ika_{ik}^2 \sum_{p\geqslant \log i}\frac{1}{2^p}\sum_{q\geqslant\log k}\frac{1}{2^q} = O\{1\}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 < \infty.$$

The application of B. Levi's theorem provides (8.1) to be proved.

PROPOSITION 2. *If $\alpha > \frac{1}{2}$, $\beta > 0$ and the condition*

(8.2) $$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2[\log\log(k+4)]^2 < \infty$$

is satisfied, then

(8.3) $$^2\varepsilon_{MN}^{\alpha\beta}(x) = o_x\{1\} \quad \text{a.e. as} \quad M \to \infty,$$

uniformly in N.

The symmetric counterpart of Proposition 2 reads as follows: If $\alpha > 0$, $\beta > \frac{1}{2}$ and the condition

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2[\log\log(i+4)]^2 < \infty$$

is satisfied, then

$$^3\varepsilon_{MN}^{\alpha\beta}(x) = o_x\{1\} \quad \text{a.e. as} \quad N \to \infty,$$

uniformly in M.

Proof of Proposition 2. Since

$$^2\varepsilon_{MN}^{\alpha\beta}(x) \leqslant 2\, {}^2\varepsilon_{2^p,2^q}^{\alpha\beta}(x)$$

for $2^{p-1} < M \leqslant 2^p$ and $2^{q-1} < N \leqslant 2^q$ with $p, q \geqslant 0$, instead of (8.3) we have only to prove

(8.4) $$^2\varepsilon_{2^p,2^q}^{\alpha\beta}(x) = o_x\{1\} \quad \text{a.e. as} \quad p \to \infty,$$

uniformly in q.

We can again insert one more simplifying step. Clearly,

$$[{}^2\varepsilon_{2^p,2^q}^{\alpha\beta}(x)]^2 = \sum_{r=-2}^{p-1}\sum_{t=-2}^{q-1} 2^{r+t-p-q+4} \times$$

$$\times \frac{1}{2^{r+2}2^{t+2}}\sum_{m=2^r+1}^{2^{r+1}}\sum_{n=2^t+1}^{2^{t+1}} [\sigma_{mn}^{\alpha-1,\beta}(x) - \sigma_{mn}^{\alpha\beta}(x)]^2,$$

with the same convention concerning 2^r and 2^t for $r, t = -2$ and -1 as we made after (7.1). Thus, in order to prove (8.4) it is sufficient to prove

(8.5) $$^4\varepsilon_{pq}^{\alpha\beta}(x) = \left\{\frac{1}{2^p2^q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}[\sigma_{mn}^{\alpha-1,\beta}(x) - \sigma_{mn}^{\alpha\beta}(x)]^2\right\}^{1/2} = o_x\{1\}$$

$$\text{a.e. as} \quad p \to \infty,$$

uniformly in q. Here, again for the sake of simplicity in notation, we assume that $p, q \geqslant 0$.

Using representation (4.6), we can split ${}^4\varepsilon_{pq}^{\alpha\beta}(x)$ into three parts as follows:

$$ {}^4\varepsilon_{pq}^{\alpha\beta}(x) \leqslant \Big\{\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\Big[\sum_{i=1}^{m}\sum_{k=0}^{2^q}\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^\alpha}ia_{ik}\varphi_{ik}(x)\Big]^2\Big\}^{1/2}+ $$
$$ +\Big\{\frac{1}{2^p2^q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\Big[\sum_{i=1}^{m}\sum_{k=2^q+1}^{n}\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^\alpha}ia_{ik}\varphi_{ik}(x)\Big]^2\Big\}^{1/2}+ $$
$$ +\Big\{\frac{1}{2^p2^q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\Big[\sum_{i=1}^{m}\sum_{k=1}^{n}\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^\alpha}\Big(1-\frac{A_{n-k}^\beta}{A_n^\beta}\Big)ia_{ik}\varphi_{ik}(x)\Big]^2\Big\}^{1/2} $$
$$ = {}^1\delta_{p,2^q}^\alpha(x)+{}^2\delta_{pq}^\alpha(x)+{}^5\varepsilon_{pq}^{\alpha\beta}(x), $$

where ${}^1\delta_{pn}^\alpha(x)$ was already defined in (7.2) (as to the representation of ${}^1\delta_{p,2^q}^\alpha(x)$ see the first equality in (7.4)), while ${}^2\delta_{pq}^\alpha(x)$ and ${}^5\varepsilon_{pq}^{\alpha\beta}(x)$ are first introduced here. On the basis of this decomposition, the next three lemmas will complete the proof of (8.5) and that of Proposition 2.

LEMMA 3. *If $\alpha>\frac12$ and condition (8.2) is satisfied, then*

(8.6) $$ {}^1\delta_{p,2^q}^\alpha(x)=o_x\{1\}\quad \textit{a.e. as}\quad p\to\infty, $$

uniformly in q.

Proof. The statement of Lemma 3 is an easy consequence of (7.2). To see this, we set

$$ a_{it}^* = \{\sum_{k=2^t+1}^{2^{t+1}} a_{ik}^2\}^{1/2}\qquad (i=0,1,\ldots;\ t=-2,-1,0,\ldots) $$

and

$$ \varphi_{it}^*(x)=\begin{cases}\dfrac{1}{a_{it}^*}\displaystyle\sum_{k=2^t+1}^{2^{t+1}}a_{ik}\varphi_{ik}(x) & \text{if}\quad a_{it}^*\neq 0,\\ \varphi_{i,2^t+1}(x) & \text{if}\quad a_{it}^*=0\end{cases} $$

(keeping in mind the convention made after (7.1)). Obviously, $\{\varphi_{it}^*(x): i=0,1,\ldots;\ t=-2,-1,0,\ldots\}$ is an orthonormal system and by (8.2)

$$ \sum_{i=0}^\infty\sum_{t=-2}^\infty [a_{it}^*]^2[\log(t+4)]^2<\infty. $$

So we can apply Theorem 3 and obtain (7.2) which in this case says

(8.7) $$ {}^1\Delta_{pq}^\alpha(x)=o_x\{1\}\quad \text{a.e. as}\quad p\to\infty, $$

uniformly in q, where

$$ {}^1\Delta_{pq}^\alpha(x)=\Big\{\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\Big[\sum_{i=1}^{m}\sum_{t=-2}^{q}\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^\alpha}ia_{it}^*\varphi_{it}^*(x)\Big]^2\Big\}^{1/2}={}^1\delta_{p,2^q+1}^\alpha(x). $$

By this, (8.7) is equivalent to (8.6) to be proved.

LEMMA 4. *If $\alpha>\frac12$ and condition (1.2) is satisfied, then*

(8.8) $$ {}^2\delta_{pq}^\alpha(x)=o_x\{1\}\quad \textit{a.e. as}\quad \max(p,q)\to\infty. $$

Proof. It is fairly simple. By (4.8) and (1.2),

$$ \alpha^2\sum_{p=0}^\infty\sum_{q=0}^\infty\int[{}^2\delta_{pq}^\alpha(x)]^2d\mu(x) $$
$$ =\sum_{p=0}^\infty\sum_{q=0}^\infty\frac{1}{2^p2^q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\sum_{i=1}^{m}\sum_{k=2^q+1}^{n}\Big[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\Big]^2 i^2a_{ik}^2 $$
$$ \leqslant 2\sum_{m=2}^\infty\sum_{q=0}^\infty\frac{1}{2^q}\sum_{n=2^q+1}^{2^{q+1}}\sum_{i=1}^{m}\sum_{k=2^q+1}^{n}\Big[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\Big]^2 ia_{ik}^2 $$
$$ \leqslant 2\sum_{m=2}^\infty\sum_{q=0}^\infty\sum_{i=1}^{m}\sum_{k=2^q+1}^{2^{q+1}}\Big[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\Big]^2 ia_{ik}^2 $$
$$ =2\sum_{m=2}^\infty\sum_{k=2}^\infty\sum_{i=1}^{m}\Big[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\Big]^2 ia_{ik}^2 $$
$$ \leqslant 2\sum_{i=1}^\infty\sum_{k=2}^\infty ia_{ik}^2\sum_{m=i}^\infty\Big[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\Big]^2<\infty. $$

Applying B. Levi's theorem, we get (8.8) to be proved.

LEMMA 5. *If $\alpha>\frac12$, $\beta>0$ and condition (1.2) is satisfied, then*

(8.9) $$ {}^5\varepsilon_{pq}^{\alpha\beta}(x)=o_x\{1\}\quad \textit{a.e. as}\quad \max(p,q)\to\infty. $$

Proof. An easy calculation gives that

$$ \alpha^2\sum_{p=0}^\infty\sum_{q=0}^\infty\int[{}^5\varepsilon_{pq}^{\alpha\beta}(x)]^2d\mu(x) $$
$$ =\sum_{p=0}^\infty\sum_{q=0}^\infty\frac{1}{2^p2^q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\sum_{i=1}^{m}\sum_{k=1}^{n}\Big[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\Big]^2\Big[1-\frac{A_{n-k}^\beta}{A_n^\beta}\Big]^2 i^2a_{ik}^2 $$
$$ \leqslant 2\sum_{m=2}^\infty\sum_{q=0}^\infty\frac{1}{2^q}\sum_{n=2^q+1}^{2^{q+1}}\sum_{i=1}^{m}\sum_{k=1}^{n}\Big[\frac{A_{m-i}^{\alpha-1}}{A_m^\alpha}\Big]^2\Big[1-\frac{A_{n-k}^\beta}{A_n^\beta}\Big]^2 ia_{ik}^2 $$

$$\leqslant 4 \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{n}\sum_{i=1}^{m}\sum_{k=1}^{n}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2\left[1-\frac{A_{n-k}^{\beta}}{A_n^{\beta}}\right]^2 i a_{ik}^2$$

$$\leqslant 4 \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} i a_{ik}^2 \sum_{m=i}^{\infty}\left[\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right]^2 \sum_{n=k}^{\infty}\frac{1}{n}\left[1-\frac{A_{n-k}^{\beta}}{A_n^{\beta}}\right]^2 < \infty,$$

the last inequality is thanks to (4.8), (4.9) and (1.2). Hence B. Levi's theorem yields (8.9).

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Fourier series and Hilbert transforms with values in UMD Banach spaces

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Abstract. Let B be a Banach space with the unconditional martingale differences property and let T be the circle group. It is shown that if in addition B has an unconditional basis then the Fourier series of $f\in L_B^p(T)$, $p>1$, converges to f a.e.

§ 1. Introduction. The Banach spaces B for which the Hilbert transform $H\colon L^p\to L^p$ admits a bounded B-valued extension to L_B^p, $1<p<\infty$, were recently characterized by a condition called *ζ-convexity* (see [4] and [2]). The class of all such spaces is also denoted as UMD, due to the fact that the unconditionality of martingale differences holds for B-valued random variables if and only if B is ζ-convex.

It is natural to ask to which extent the most important estimates of harmonic analysis carry over to the B-valued setting, $B\in\mathrm{UMD}$. Since the rotation method still applies, the singular integral operators falling under the scope of this method have B-valued extensions which are bounded in L_B^p. A different class of singular integral operators is considered in [3] but the proof requires that the space $B\in\mathrm{UMD}$ has an unconditional basis. With the same restriction, we aim to extend here the pointwise convergence theorems for Fourier series ([5] and [10]) to the B-valued setting. Thus, it is shown in Section 3 that the Fourier series of $f\in L_B^p(T)$, $p>1$, converges to $f(x)$ a.e., and in Section 4, that the lacunar sequences of partial sum operators converge to $f(x)$ a.e. if $f\in H_B^1(T)$. These are exactly the same results which hold for the scalar case. Finally, Section 5 contains an interesting stability property of UMD spaces.

§ 2. Notation and basic lemmas. Throughout the paper, B will denote a Banach space in the class UMD. The UMD-constant for B will be the least C such that the inequality

$$\int \|\tilde{H}f(x)\|_B^2\,dx \leqslant C^2 \int \|f(x)\|_B^2\,dx$$