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Fréchet spaces with quotients failing the bounded approximation property

by

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Abstract. It is proved that every Fréchet–Montel space not isomorphic to ω has a quotient space which does not have the bounded approximation property.

A separable Fréchet space E is said to have the *bounded approximation property* if there is a sequence $A_n: E \to E$ $(n \in N)$ of continuous linear operators with $\dim A_n(E) < +\infty$ and $\lim_n A_n x = x$ for all $x \in E$. It was a problem of Grothendieck ([4], p. 136) to find a nuclear Fréchet space without the bounded approximation property. This was solved in [2]. A simplified construction was given in [9].

In [3] it was also shown by means of the structure theory of nuclear Fréchet spaces that the space (s) of rapidly decreasing sequences has quotients without the bounded approximation property. This was extended to arbitrary stable power series spaces in [8]. In this paper we show that every Fréchet–Montel space (hence every nuclear Fréchet space) not isomorphic to ω has a quotient failing the bounded approximation property.

The basic idea for all of these constructions is an observation of A. Pełczyński that a Fréchet space which admits continuous norm but is not countably normed cannot have the bounded approximation property (see [2]). This leads to certain necessary conditions and the examples are constructed so as to violate these conditions. Our proof is based on the condition given in [9] which is simpler than the one used in [2] and [3] and easily follows from the bounded approximation property. In Proposition 5 we show that in fact both of these conditions are equivalent to countably normedness.

So our results also say that countably normedness which is clearly hereditary under subspaces is, even in the presence of continuous norms, in a very strict sense not hereditary under quotients.

We use standard notation (see [2], [6], [7] e.g.).

From [9] we take the following lemma.

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1. LEMMA. *If E has the bounded approximation property and a continuous norm, then the following holds:*

There exists $k_0 \in N$ such that for each $k \geqslant k_0$ there exists $j > k$ such that every $\|\ \|_j$-Cauchy sequence in E which is $\|\ \|_{k_0}$-null is also $\|\ \|_k$-null.

We use Lemma 1 to obtain a condition for the existence of quotients without the bounded approximation property. We will use this formulation to obtain our main results.

2. PROPOSITION. *Let G be a Fréchet space and $(\|\ \|_k)_{k\in N}$ an increasing sequence of norms which defines the topology of G. Suppose that G has a vector subspace L which is closed in the normed space $(G, \|\ \|_1)$ and satisfies the following condition:*

For each $k < j$ there exists x in the completion of $(G, \|\ \|_j)$ such that x is the $\|\ \|_k$-limit of a sequence in L but not the $\|\ \|_{k+1}$-limit of any sequence in L.

Then G/L does not have the bounded approximation property.

Proof. Let E be the quotient G/L with topology defined by $(|\ |_k)_{k\in N}$ where $|x+L|_k = \inf_{y\in L} \|x+y\|_k$. Since L is $\|\ \|_1$-closed it follows that $|\ |_1$ is a norm.

We will show that the condition in Lemma 1 is violated. In fact, take any k_0 and $k = k_0+1$. Applying our assumption to $k<j$, we have a sequence (x_n) in G which is $\|\ \|_j$-Cauchy, so that (x_n+L) is $|\ |_j$-Cauchy; also (x_n+L) is $|\ |_{k_0}$-null but not $|\ |_k$-null.

Our next step is to formulate the condition of Proposition 2 in case G is a nuclear Köthe space $K = K(a)$. Here $a = (a_{n,k})$ is an infinite matrix of positive numbers increasing in k and satisfying

$$\sum_n a_{n,k}\, a_{n,k+1} < +\infty \quad \text{for all } k \in N.$$

Then we define

$$K(a) = \{\xi = (\xi_n):\ \|\xi\|_k = \sum_n |\xi_n|\, a_{n,k} < +\infty \ \text{ for all } k\}$$

and $(\|\ \|_k)$ is an increasing sequence of norms defining a topology such that $K(a)$ is a nuclear Fréchet space.

The space $K = K(a)$ is continuously embedded into the Cartesian product ω of countably many copies of the scalar field in such a way that for any vector subspace H of K we can consider the completion H_k of $(H, \|\ \|_k)$ as a vector subspace of ω. The condition of Proposition 2 may then be expressed as follows:

$$K_j \cap L_k \ \text{ is not contained in } \ L_{k+1}.$$

In order to obtain this condition for every $k<j$ we will consider N to be decomposed into infinitely many infinite subsets $M^{k,j}$. It suffices to find for every $k<j$ a subspace $L^{k,j}$ of $K^{k,j} = \{\xi = (\xi_n) \in K:\ \xi_n = 0 \text{ for } n \notin M^{k,j}\}$ such that $L^{k,j}$ is $\|\ \|_1$-closed and $K^{k,j}_j \cap L^{k,j}_k$ is not contained in $L^{k,j}_{k+1}$. The space $L = \{\xi = (\xi_n) \in K:\ \xi|_{M^{k,j}} \in L^{k,j}$ for all $k, j\}$ will then satisfy the condition of Proposition 2.

Since in the construction of $L^{k,j}$ we may assume without any loss of generality that $k = 2$, $j = 4$ it will be enough to construct for every Köthe space $K = K(a)$ a $\|\ \|_1$-closed subspace $L \subset K$ such that $K_4 \cap L_2$ is not contained in L_3.

Now we are able to give the main technical result in the paper.

3. PROPOSITION. *Any nuclear Köthe space which admits a continuous norm has a quotient without the bounded approximation property.*

Proof. We put $b_{n,k} = a_{2n-1,k}$, $c_{n,k} = a_{2n,k}$ and by making a diagonal transformation we may assume that

$$\sum_n \frac{b_{n,k}}{b_{n,k+1}} < +\infty, \qquad k = 2, 3; \quad (c_{n,4})_n \in l_1; \quad \lim_n c_{n,5} = +\infty.$$

We have up to isomorphism

$$K = \{(x, y):\ \|(x, y)\|_k = \sum_n |x_n|\, b_{n,k} + \sum_n |y_n|\, c_{n,k} < +\infty \ \text{ for all } k\}$$

and we define

$$L = \{(x, y) \in K:\ y_n = \sum_{\nu=1}^{n} b_{\nu,3}\, x_\nu \ \text{ for all } n\}.$$

In view of the above discussion we must show that L is closed in K w.r.t. $\|\ \|_1$ and that $K_4 \cap L_2$ is not contained in L_3.

It is clear from the defining relation that L is coordinatewise closed so it is closed w.r.t. $\|\ \|_1$.

Let $\lambda = (\lambda_n)$ be any element of l_1 with $\lambda_n > 0$ for all n and write

$$x_n = \frac{\lambda_n}{b_{n,4}}, \quad y_n = \sum_{\nu=1}^{n} \lambda_\nu \frac{b_{\nu,3}}{b_{\nu,4}}, \qquad A = \sum_{n=1}^{\infty} \lambda_n \frac{b_{n,3}}{b_{n,4}}.$$

It is clear from our assumptions that $(x, y) \in K_4$. We will show that it is in L_2, i.e., that it can be approximated in $\|\ \|_2$ by elements of L.

Given n_0, set

$$\xi_n = \begin{cases} x_n, & n \leqslant n_0, \\ -\dfrac{1}{b_{n_0+1,3}} \sum\limits_{\nu=1}^{n_0} x_\nu b_{\nu,3}, & n = n_0+1, \\ 0, & n > n_0. \end{cases} \qquad \eta_n = \sum_{\nu=1}^{n} \xi_\nu b_{\nu,3},$$

We obtain

$$\|(x,y)-(\xi,\eta)\|_2 = \sum_{n=n_0+2}^{\infty} |x_n|\, b_{n,2} + \sum_{n=n_0+1}^{\infty} |y_n|\, c_{n,2} + \left| x_{n_0+1} + \frac{1}{b_{n_0+1,3}} \eta_{n_0} \right| b_{n_0+1,2}$$

$$\leqslant \sum_{n=n_0+1}^{\infty} \lambda_n \frac{b_{n,2}}{b_{n,4}} + A \sum_{n=n_0+1}^{\infty} c_{n,2} + \frac{b_{n_0+1,2}}{b_{n_0+1,3}} A,$$

which goes to 0 as n_0 goes to ∞. Thus $(x,y) \in K_4 \cap L_2$.

Finally we must show that $(x,y) \notin L_3$. First we observe that the finitely non-zero vectors in L are $\|\ \|_3$-dense in L. In fact, given $(u,v) \in L$ define (ξ,η) exactly as above with x, y replaced by u, v. Clearly, (ξ,η) is a finitely non-zero vector in L and

$$\|(u,v)-(\xi,\eta)\|_3 = \sum_{n=n_0+2}^{\infty} |u_n|\, b_{n,3} + \sum_{n=n_0+1}^{\infty} |v_n|\, c_{n,3} + \left| u_{n_0+1} + \frac{1}{b_{n_0+1,3}} \eta_{n_0} \right| b_{n_0+1,3}$$

$$= \sum_{n=n_0+2}^{\infty} |u_n|\, b_{n,3} + \|(u,v)\|_3 \sum_{n=n_0+1}^{\infty} c_{n,3} + |v_{n_0+1}|.$$

The first two terms go to 0 as n_0 goes to ∞ because of the assumptions and the last because $(u,v) \in K$ and $\lim_n c_{n,5} = \infty$. Notice that $|v_n| \leqslant \|(u,v)\|_3$ for $(u,v) \in L$ and all n.

Thus it suffices to show that (x,y) cannot be approximated in $\|\ \|_3$ by finitely non-zero vectors in L. Since x, y have no non-zero coordinates, this cannot be done with such vectors having the number of non-zero terms bounded. Thus it must be shown that (x,y) cannot be approximated in $\|\ \|_3$ by vectors ξ, η of the form

$$\xi_n = \begin{cases} x_n, & n \leqslant n_0, \\ -\dfrac{1}{b_{n_0+1,3}} \displaystyle\sum_{\nu=1}^{n_0} x_\nu b_{\nu,3}, & n = n_0+1, \\ 0, & n > n_0. \end{cases} \qquad \eta_n = \sum_{\nu=1}^{n} \xi_\nu b_{\nu,3},$$

with arbitrarily large n_0.

Suppose then that we did have $\|(x,y)-(\xi,\eta)\|_3 \leqslant \varepsilon$. Then

$$\sum_{n=1}^{n_0} |x_n - \xi_n|\, b_{n,3} \leqslant \varepsilon, \qquad \left| x_{n_0+1} + \frac{1}{b_{n_0+1,3}} \eta_{n_0} \right| b_{n_0+1,3} \leqslant \varepsilon$$

so

$$\varepsilon \geqslant |\eta_{n_0}| - |x_{n_0+1}|\, b_{n_0+1,3} \geqslant |y_{n_0}| - |y_{n_0} - \eta_{n_0}| - \lambda_{n_0+1} \frac{b_{n_0+1,3}}{b_{n_0+1,4}}$$

$$= |y_{n_0}| - \left| \sum_{\nu=1}^{n_0} (x_\nu - \xi_\nu)\, b_{\nu,3} \right| - \lambda_{n_0+1} \frac{b_{n_0+1,3}}{b_{n_0+1,4}}$$

$$\geqslant |y_{n_0}| - \varepsilon - \lambda_{n_0+1} \frac{b_{n_0+1,3}}{b_{n_0+1,4}}$$

and letting n_0 go to ∞ we obtain $2\varepsilon \geqslant A$ which leads to a contradiction.

The result of Proposition 3 extends immediately to the class of all Fréchet spaces admitting a nuclear Köthe space with continuous norm as quotient. These spaces have been characterized among the separable Fréchet spaces by Bellenot and Dubinsky [1]. Putting their results (in particular [1], Cor. 4) together with Proposition 3, we obtain the main result of this paper.

4. Theorem. *Every Fréchet–Montel space not isomorphic to ω has a quotient without the bounded approximation property.*

We will now show that the condition in Lemma 1 which we used for our construction and also the condition used in [3] (see (iii) below) are in fact equivalent to countably normedness.

We recall that a Fréchet space E is called *countably normed* if its topology can be defined by an increasing sequence of norms $(\|\ \|_k)_{k\in N}$ such that the unique extensions $A_k\colon E_{k+1} \to E_k$ of the identity are injective, where E_k is the completion of the normed space $(E, \|\ \|_k)$, or equivalently: such that any sequence (x_n) in E which is Cauchy with respect to one of these norms $\|\ \|_k$ and null with respect to another $\|\ \|_j$ is also null with respect to $\|\ \|_k$. The main value of the conditions (ii) and (iii) below is that they are formulated in terms of an arbitrary fundamental system of norms.

5. Proposition. *Let E be a Fréchet space which admits a continuous norm and let $(\|\ \|_k)_{k\in N}$ be an increasing sequence of norms which define the topology of E. Denote by E_k the completion of the normed space $(E, \|\ \|_k)$ and by $A_k\colon E_{k+1} \to E_k$ the unique extension of the identity. Then the following conditions are equivalent:*

(i) *E is countably normed.*

(ii) *There exists $k_0 \in N$ such that for each $k \geqslant k_0$ there exists $j > k$ such that every $\|\ \|_j$-Cauchy sequence in E which is $\|\ \|_{k_0}$-null is also $\|\ \|_k$-null.*

(iii) *There exists $k_0 \in N$ such that for each $x \in \bigcap_{k=k_0}^{\infty} A_{k_0} \ldots A_k(E_{k+1})$ there exists a sequence (x_k) such that $x_{k_0} = x$ and $x_k = A_k x_{k+1}$ for all $k \geqslant k_0$.*

Proof. (i) $\Rightarrow$ (ii). Let $(|\ |_k)_{k\in N}$ be a sequence of norms defining the topology of

E and satisfying the definition of countably normedness. Choose k_0 such that $|\ |_1$ is dominated by $\|\ \|_{k_0}$ and give $k \geqslant k_0$. Choose k' such that $\|\ \|_k$ is dominated by $|\ |_{k'}$. Then choose j such that $|\ |_{k'}$ is dominated by $\|\ \|_j$. The conclusion is then clear.

(ii) $\Rightarrow$ (iii). For every $k \geqslant k_0$ choose $j = j_k$ according to (ii). We may assume (j_k) strictly increasing. If $x \in \bigcap_{k=k_0}^{\infty} A_{k_0} \ldots A_k(E_{k+1})$ then we have for every $k > k_0$ $x_k \in E_k$ such that $x = A_{k_0} \ldots A_{k-1} x_k$. Now condition (ii) says exactly that $A_{k_0} \ldots A_{k-1}$ is injective on $A_k \ldots A_{j_k - 1}(E_{j_k})$. Hence $A_k \ldots A_{j_k-1}(y_{j_k}) = A_k \ldots A_{j-1}(y_j)$ for all $j \geqslant j_k$, in particular for $j = j_{k+1}$. So we can take $x_k = A_k \ldots A_{j_k-1}(y_{j_k})$.

(iii) $\Rightarrow$ (i). We may assume $k_0 = 1$. Let $F_k = A_1 \ldots A_k(E_{k+1})$ with the quotient norm induced from E_{k+1}. $F = \bigcap_{k=1}^{\infty} F_k$ with the projective topology is a countably normed Fréchet space. It is a continuously imbedded subspace of E_1. Let $P_k\colon E \to E_k$ be the canonical map. We will show that $P_1\colon E \to E_1$ maps E 1-1 onto F. Since E and F are Fréchet spaces the closed graph theorem then implies that E and F are isomorphic which proves the assertion.

P_1 is 1-1, because $\|\cdot\|_1$ is a norm.

Since $P_1 = A_1 \ldots A_k P_{k+1}$ for all k we have $\operatorname{im} P_1 \subset F$. For $y \in F$ there exists by assumption a sequence (x_k) such that $x_1 = y$ and $x_k = A_k x_{k+1}$ for all k. Since E is complete, it follows that there exists $x \in E$ with $P_1 x = y$. Hence we have $F \subset \operatorname{im} P_1$ and the proof is complete.

The following result, which is a consequence of the above proof has some application (see [5]).

6. Corollary. *If E is a projective limit of a sequence of maps of Hilbert spaces and E is countably normed then E is isomorphic to a projective limit of a sequence of injective maps of Hilbert spaces.*

Proof. If E is countably normed then by Proposition 5 and the hypothesis, we have condition (iii) of Proposition 5 with each E_k a Hilbert space. Therefore the F_k in the proof of (iii) $\Rightarrow$ (i) are Hilbert spaces.

Since the proof of Theorem 4 was based on condition (ii) in Proposition 5, we can state the following

7. Theorem. *Every Fréchet–Montel space not isomorphic to ω has a quotient which has a continuous norm but is not countably normed.*

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