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Solution to a problem of S. Rolewicz

by

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Abstract. The strong dual of a Banach space containing an isomorphic copy of $l^1(\Gamma)$ has a non-empty convex closed and bounded subset such that every point is a support point.

Introduction. In [8], S. Rolewicz asks if there exists in a non-separable Banach space a non-empty convex closed and bounded subset such that every point is a support point. In particular, he is interested in the space $M([0, 1])$ of Lebesgue-measurable functions on $[0, 1]$. In this note we give an answer to this case, proving that if E is a Banach space containing an isomorphic copy of $l^1(\Gamma)$, there exists in the strong dual a non-empty convex closed and bounded subset such that every point is a support point. We do not know an answer to the first question.

Let E be a Banach space over the field $\boldsymbol{R}$ of the real numbers. Let E^* be the topological dual of E. We denote by $\|\cdot\|$ the norm in E, as well as the strong norm in E^*. Let A be a convex and closed subset of E. A point $x_0 \in A$ is a *support point* of A if there exists a $\varphi \in E^*$ such that

(1) $\varphi(x) \leqslant \varphi(x_0)$, $\forall x \in A$ and

(2) $\exists x \in A$ such that $\varphi(x) < \varphi(x_0)$.

This is the definition which appears in [8]. It coincides with what in [2], p. 21, is called a *proper support point*. We shall say that a subset A of a Banach space E has *property* $(*)$ if A is a non-empty convex closed and bounded set such that every $a \in A$ is a support point of A.

Those problems arose in connection with the result of S. Rolewicz in [8] which asserts that every non-empty separable convex and closed subset of a Banach space has a point which is not a support point.

I. The case $C([0, 1])$. Let $C([0, 1])$ be the Banach space of real continuous functions defined on $[0, 1]$. If $x \in [0, 1]$, we denote by δ_x the element of $C([0, 1])^*$ defined by $\delta_x(f) = f(x)$, $\forall f \in C([0, 1])$. $C([0, 1])^*$ is a non-separable Banach space which can be identified with the space of Radon measures on $[0, 1]$. Let B be the closed unit ball of $C([0, 1])$ and B° that of

$C([0, 1])^*$. The set of extreme points of B° is $\mathrm{Ext}(B^\circ) = \{\delta_x, -\delta_x, x \in [0, 1]\}$. A measure $\mu \in C([0, 1])^*$ is *atomic* when it is concentrated on a countable subset of $[0, 1]$, that is, there exists a countable set $D \subset [0, 1]$ such that $\mu([0, 1] \setminus D) = 0$.

PROPOSITION 1. *There exists in* $C([0, 1])^*$ *a subset with property* $(*)$.

Proof. Let $A = \{\mu\colon \mu \in C([0, 1])^*, \ \|\mu\| = 1, \ \mu \geqslant 0, \ \mu \text{ atomic}\} \subset C([0, 1])^*$. Obviously, A is a convex set. We shall prove that it is closed: Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence in A which converges to $\mu \in C([0, 1])^*$. Then $\|\mu\| = 1$ and $\mu \geqslant 0$. There exists a countable set $D_n \subset [0, 1]$ such that $\mu_n([0, 1] \setminus D_n) = 0$, $n = 1, 2, \ldots$ For every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\|\mu - \mu_n\| < \varepsilon$, $n = n_0, n_0+1, n_0+2, \ldots$ We have $\|\mu - \mu_n\| = \sup\{\sum_{i=1}^{\infty} |(\mu - \mu_n)(A_i)|\colon \{A_i\}_{i=1}^{\infty}$ partition of $[0, 1]$ in a countable number of pairwise disjoint Borel subsets$\}$. In particular,

$$\{\bigcup_{n=1}^{\infty} D_n, \ \bigcap_{n=1}^{\infty} \mathrm{C}D_n\}$$

is one of these partitions. Therefore

$$|(\mu - \mu_n)(\bigcup_{m=1}^{\infty} D_m)| + |(\mu - \mu_n)(\bigcap_{m=1}^{\infty} \mathrm{C}D_m)| < \varepsilon, \qquad n = n_0, n_0+1, n_0+2, \ldots$$

Hence

$$|(\mu - \mu_n)(\bigcap_{m=1}^{\infty} \mathrm{C}D_m)| < \varepsilon, \qquad n = n_0, n_0+1, n_0+2, \ldots$$

but

$$\mu_n(\bigcap_{m=1}^{\infty} \mathrm{C}D_m) = 0, \qquad n = 1, 2, 3, \ldots$$

Therefore

$$|\mu(\bigcap_{m=1}^{\infty} \mathrm{C}D_m)| < \varepsilon, \qquad \forall \varepsilon > 0.$$

It follows that

$$\mu(\bigcap_{m=1}^{\infty} \mathrm{C}D_m) = 0.$$

Observe that the closed convex hull of $\mathrm{Ext}(B^\circ)$ is different from B°. Incidentally, it follows from (10) that $C([0, 1])$ contains an isomorphic copy of l^1 and $C([0, 1])$ is not sequentially dense in $C([0, 1])^{**}$.

Let us now prove that every point of A is a support point. Let $\mu_0 \in A$. Therefore, there exists a countable subset D of $[0, 1]$ such that $\mu_0([0, 1] \setminus D) = 0$. We define a continuous linear functional L on $C([0, 1])^*$ by the formula $L(\mu) = \mu(D)$, $\forall \mu \in C([0, 1])^*$. Obviously, $\|L\| = 1 = \mu_0([0, 1])$. $L(\mu_0) = \mu_0(D) = 1$. Moreover, if μ belongs to A, $L(\mu) = \mu(D) \leqslant 1 = \mu([0, 1])$. Obviously, $\inf\{L(\mu)\colon \mu \in A\} < 1$. Therefore, L is a support functional of A and μ_0 is a support point of A. ∎

Note. If a Banach space E has a subset with property $(*)$, it is evident that every isomorphic Banach space has also a subset with property $(*)$. In view of Milyutin's theorem [5], if $C(S)$ is the Banach space of continuous real functions over an uncountable compact metric space S, $C(S)^*$ has a subset with property $(*)$.

II. The case $l^\infty(\Gamma)$.

PROPOSITION 2. *Let* Γ *be an infinite set. There exists in* $l^\infty(\Gamma)$ *a subset with property* $(*)$.

Proof. Let us first assume that Γ is countable. $C([0, 1])$ is a separable Banach space and therefore it is isometric to a suitable quotient space of l^1. Thus, l^∞ has a closed subspace which is isometric to $C([0, 1])^*$. By Proposition 1, there exists in $C([0, 1])^*$ a subset A with property $(*)$. Obviously, A can be identified with a subset of l^∞ with property $(*)$.

If Γ is uncountable, l^∞ is isometric to a closed subspace of $l^\infty(\Gamma)$. The result now follows from the first part. ∎

Now, let μ be a sigma-finite measure. $L^1(\mu)$ is separable if and only if $L^\infty(\mu)$ is isomorphic to $l^\infty(\Gamma)$ ([6], [7] and [9]). Therefore, Proposition 2 gives an answer to the question raised by S. Rolewicz about $M([0, 1])$ mentioned at the beginning.

III. A general situation.

The following lemma is rather obvious. Let us prove it for the sake of completeness:

LEMMA. *Let* E *and* F *be two Banach spaces. Let* $T\colon E \to F$ *be a continuous linear mapping from* E *into* F. *Let us suppose that there exists a non-empty convex closed subset* A *of* F *such that every point of* A *is a support point and* $A \subset T(E)$. *Then* $T^{-1}(A)$ *is a subset of* E *with the same properties as* A.

Proof. Let $x_0 \in T^{-1}(A)$. Therefore $T(x_0) = y_0 \in A$, and hence there exists a continuous linear functional $\varphi \in F^*$ such that

(1) $\varphi(y_0) \geqslant \varphi(y)$, $\forall y \in A$ and

(2) $\exists y_1 \in A$ such that $\varphi(y_0) > \varphi(y_1)$.

Let $\psi = \varphi \circ T \in E^*$. Let $x_1 \in A$ be a point such that $T(x_1) = y_1$. Then

(1) $\psi(x_0) \geqslant \psi(x)$, $\forall x \in A$ and

(2) $\psi(x_0) > \psi(x_1)$.

Hence x_0 is a support point of $T^{-1}(A)$. ∎

THEOREM. *Let* E *be a Banach space containing an isomorphic copy of* $l^1(\Gamma)$. *Then there exists in* E^* *a subset with property* $(*)$.

Proof. E^* has a quotient space isomorphic to $l^\infty(\Gamma)$. Let q be the canonical mapping from E^* onto this quotient. Using Proposition 2, we can find a subset A_1 of this quotient space with property $(*)$. By the Lemma, it easily follows that $A = q^{-1}(A_1) \cap (M+\varepsilon) B^\circ$ is a subset of E^* with property $(*)$, where ε is an arbitrary positive number, B° is the closed unit ball of E^*, and $\|x\| \leqslant M$, $\forall x \in A_1$. ■

Corollary 1. *In $l^\infty(\Gamma)^*$ there exists a subset with property $(*)$.*

Proof. $l^\infty(\Gamma)$ contains an isomorphic copy of l^1. ■

Corollary 2. *Let E be a Banach space with a quotient space which is isomorphic to $l^\infty(\Gamma)$. Then, there exists in E^* a subset with property $(*)$.*

Proof. E^* contains a closed subspace which is isomorphic to $l^\infty(\Gamma)^*$. ■

Note 2. Let E be a Banach space which does not contain an isomorphic copy of $l^1(\Gamma)$. It is not always true that every convex closed and bounded subset of E^* has a point which is not a support point. For example, take $E = l^2(\Gamma)$, Γ a non-countable set. In $l^2(\Gamma)$ there exists a subset with property $(*)$ [8]. $l^2(\Gamma)$ is not a separable space. The following question arises immediately:

Problem. Let E be a separable Banach space which does not contain an isomorphic copy of l^1. Is it true that every convex closed and bounded subset of E^* has a point which is not a support point?

If the answer is negative, the example must be a separable Banach space which does not contain an isomorphic copy of l^1 and such that E^* is not a separable space. There exists some examples of this situation ([1], [3] and [4]).

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