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On p-absolutely summing operators acting on Banach lattices

by

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Abstract. Necessary and sufficient conditions are given for certain classes of concave operators acting on Banach lattices to be contained in the ideal of p-absolutely summing operators.

Introduction. It is the purpose of this paper to investigate some connections between the ideal property (in the sense of Pietsch [16], see also Nielsen [13]) of the class of majorizing operators and several concepts of summing operators.

Section 1 contains almost all the necessary definitions and basic facts. Section 2 is mainly devoted to the motivation of our study. In particular, we show that the ideal property of majorizing operators is closely related to the properties of p-regular and p-lattice summing operators (these notions were introduced by Nielsen and the present author [14]). In Section 3 we give necessary and sufficient conditions on Banach lattices X and Banach spaces E for all 1-concave operators from X into E to be p-absolutely summing. We should mention that in many cases the sufficient conditions are known (cf. [13]), even in a more general context. By the main result of Section 3 they become the best possible assumptions in almost all situations except some extreme cases. There is a still open problem what happens in these cases and our knowledge is far from being satisfactory.

In Section 4 we deal with spaces on which p-concave and p-absolutely summing operators coincide. For some p and certain rank spaces we obtain a new characterization of abstract M-spaces, i.e., of Banach lattices isomorphic to a sublattice of $C(K)$. The remaining cases are also discussed and we give a small contribution to the theory of Hilbert–Schmidt spaces introduced by Jarchow [5].

1. Preliminaries and notation. For the background we refer to [2], [16] and [17]. Throughout the paper all vector spaces are assumed to be over the reals. For our convenience the letters E, F will stand for Banach spaces and X, Y for Banach lattices. E^* denotes the topological dual of E with the pairing $\langle \cdot, \cdot \rangle$; p' denotes the dual number to p, $1 \leqslant p \leqslant \infty$, i.e., $1/p + 1/p' = 1$. Given $\mathbf{x}$

$=(x_1,\ldots,x_n)\in E^n$, we put

$$w_p(\mathbf{x})=\sup\{(\sum|\langle x',x_i\rangle|^p)^{1/p}\colon\ x'\in E^*,\ \|x'\|\leqslant 1\},$$

$$l_p(\mathbf{x})=(\sum\|x_i\|^p)^{1/p}.$$

If $\mathbf{x}=(x_1,\ldots,x_n)\in X^n$ then, following Krivine [7], we define

$$(\sum|x_i|^p)^{1/p}=\sup\{\sum a_i x_i\colon\ a_i\in R,\ \sum|a_i|^{p'}\leqslant 1\}$$

and we put

$$k_p(\mathbf{x})=\|(\sum|x_i|^p)^{1/p}\|.$$

As usual, we mean $(\sum|x_i|^p)^{1/p}=\sup|x_i|$ if $p=\infty$.

The expressions introduced above correspond to three types of the summability of vectors and yield several concepts of summing operators. Let us denote by $B(E,F)$ the space of all operators (i.e., bounded linear operators) from E into F.

An operator $T\in B(E,F)$ is said to be *p-absolutely summing* if there is a $C>0$ such that

$$l_p(T\mathbf{x})\leqslant Cw_p(\mathbf{x})$$

for all $n\in N$ and all $\mathbf{x}\in E^n$.

An operator $T\in B(X,F)$ is called *p-concave* if there is a $C>0$ such that

$$l_p(T\mathbf{x})\leqslant Ck_p(\mathbf{x})$$

for all $n\in N$ and $\mathbf{x}\in X^n$.

An operator $T\in B(E,X)$ is, by definition, *p-lattice summing* if for some $C>0$

$$k_p(T\mathbf{x})\leqslant Cw_p(\mathbf{x})$$

for all $n\in N$ and $\mathbf{x}\in E^n$.

We say that an operator $T\in B(X,Y)$ is *p-regular* if there is a $C>0$ such that

$$k_p(T\mathbf{x})\leqslant Ck_p(\mathbf{x})$$

for all $n\in N$ and $\mathbf{x}\in X^n$.

We denote by $\Pi_p(E,F)$, $K_p(X,F)$, $\Lambda_p(E,X)$ and $P_p(X,Y)$ the Banach spaces of p-absolutely summing, p-concave, p-lattice summing and p-regular operators equipped with the norms π_p, $\varkappa_p$, λ_p, ϱ_p, respectively, which are, by definition, the smallest constants C in appropriate inequalities.

The roots of these concepts go back to Grothendieck [4], who studied 1-absolutely summing operators. p-absolutely summing operators were introduced and investigated by Pietsch [15]. The definition and a study of p-concave operators is due to Krivine [7]. Yanovskii [21] invented 1-lattice summing operators and their extensions to arbitrary p is due to Nielsen and the present author [14]. The p-regularity of operators appeared in substance in Nielsen [13], being a generalization of the notion of regular operators, i.e., of operators possessing the modulus derived from Kantorovich [6]. We should mention that in some cases the above concepts arise in the works of many authors. For example, 1-concave and ∞-lattice summing operators were investigated by Schlotterbeck (cf. [17]) under the names of *cone absolutely summing* and *majorizing operators*. In the sequel we shall use these notions interchangeably.

We say that X is *q-concave*, $1\leqslant q\leqslant\infty$, if the identity I_X is q-concave and that X is *p-convex* if X^* is p'-concave or, equivalently, if for some $C>0$ $k_p(\mathbf{x})\leqslant Cl_p(\mathbf{x})$ for all n, $\mathbf{x}\in X^n$. We put

$$p(X)=\sup\{p\colon\ 1\leqslant p\leqslant\infty,\ X \text{ is } p\text{-convex}\},$$

$$q(X)=\inf\{q\colon\ 1\leqslant q\leqslant\infty,\ X \text{ is } q\text{-concave}\}.$$

E is said to be *finitely representable* in F (shortly: E f.r. F) if there is a $C>0$ such that for any finite-dimensional subspace $E_0\subset E$ one can find a subspace $F_0\subset F$ which is C-isomorphic to E_0.

Similarly we define X to be *latticially finite-representable* in Y (shortly: X l.f.r. Y) by replacing "subspace" by "sublattice" and also by using "isomorphism" in the sense of lattice structure. We denote by $(e_i^{(p)})$, $1\leqslant p\leqslant\infty$, the standard basis of l_p if $p<\infty$ or of c_0. The cones of positive elements in X and $B(X,Y)$ will be denoted by X_+ and $B_+(X,Y)$. Finally, we establish a useful convention: given two classes of operators M and N, we write $M(E,F)\subseteq N(F^*,E^*)$ if, for any $T\in M(E,F)$, $T^*\in N(F^*,E^*)$, and $M(F,G)\circ N(E,F)$ for the space $\{TS\colon\ S\in N(E,F),\ T\in M(F,G)\}$.

2. Auxiliary results on *p*-regular operators and on the ideal property. Recall that an operator $T\in B(X,Y)$ is called *regular* if it maps order bounded sets in X into order bounded sets in Y^{**} or, equivalently, $T=T_1-T_2$, where $T_1,T_2\in B_+(X,Y^{**})$ (cf. [17]). *p-regular operators* extend this concept.

PROPOSITION 2.1. *Let* $1\leqslant p\leqslant\infty$.

(1) $P_p(X,Y)\subseteq P_q(X,Y)$ *and* $\varrho_q(T)\leqslant\varrho_p(T)$ *if* $1\leqslant p\leqslant q\leqslant 2$ *or* $2\leqslant q\leqslant p\leqslant\infty$,

(2) $P_p(X,Y)=P_{p'}(Y^*,X^*)$,

(3) $B(X,Y)=P_2(X,Y)$ *and* $\varrho_2(T)\leqslant K_G\|T\|$ (K_G — *the Grothendieck constant*).

(4) *The following properties of* $T\in B(X,Y)$ *are equivalent*:

(i) T *is 1-regular*,

(ii) T *is* ∞*-regular*,

(iii) T *and* T^* *are regular*.

Proof. (2) follows from the identity (cf. [2], p. 47)

$$k_p(\mathbf{x}) = \sup \sum \langle x_i', x_i \rangle / k_{p'}(\mathbf{x}'), \qquad \mathbf{x} = (x_1, \ldots, x_n) \in X^n, \tag{5}$$

where the supremum is taken over all $\mathbf{x}' = (x_1', \ldots, x_n') \in X^{*n}$.

(3) is due to Krivine [7] and carries the same information as the Grothendieck inequality.

By applying the properties of functions with q-stable distribution,

$$\Big(\int_0^1 \Big|\sum x_i f_i\Big|^p\Big)^{1/p} = c_{p,q}\Big(\sum |x_i|^q\Big)^{1/q}, \qquad x_1, \ldots, x_n \in X,$$

where f_i are independent copies of a function f with the Fourier transform $\exp(-|t|^q)$, $1 < q \leqslant 2$ [7], we obtain the first part of (1) since an operator T is p-regular if

$$\Big\|\Big(\int_0^1 \Big|\sum T x_i f_i\Big|^p\Big)^{1/p}\Big\| \leqslant C \Big\|\Big(\int_0^1 \Big|\sum f_i x_i\Big|^p\Big)^{1/p}\Big\|$$

for all $\mathbf{x} \in X^n$, $n \in N$ and all $f_1, \ldots, f_n \in L_p(0, 1)$. The second part of (1) is then a consequence of (2).

In view of (2) and the fact that T and T^* are or are not regular simultaneously ([17], p. 23) it suffices to prove that (ii) shows T to be regular. Let $x \in X_+$. Hence $\{\sup_{u \in A} |Tu|: A \text{ finite}, A \subset [0, x]\}$ is a norm-bounded directed family by (ii). Therefore there exists a limit $Sx = \lim_A \sup_{u \in A} |Tu| \in Y^{**}$ ([17], Th. 5.10). S is additive and positively homogeneous on X_+ and can be extended to the whole of X. Moreover, $S \geqslant T$ and so we put $T_1 = S$, $T_2 = S - T$, which gives the desired decomposition of T (for similar arguments see e.g. [17]). ∎

We immediately conclude that if either $X = L_1(\mu)$ or $Y = C(K)$ then $B(X, Y) = P_1(X, Y)$. But there are examples of spaces with the above property such that X is not isomorphic to any L_1-space and Y is not isomorphic to any L_∞-space, either ([1], [18]). This phenomenon and Krivine's theorem (3) give reasons for an investigation of spaces X, Y such that $B(X, Y) = P_p(X, Y)$ for p other than 1 (or ∞) or 2. Some results in this direction were obtained by Nielsen [13]. Ibidem one can find a study of pairs of spaces E, X with the ideal property, i.e., such that (up to notations) the composition of a majorizing operator from E into X with an arbitrary operator in X is again majorizing. Although the class Λ_∞ is not an operator ideal in the sense of Pietsch [16], it preserves the ideal property of compositions under suitable geometric conditions imposed on E and X. We shall see below that the ideal property is closely related to the behaviour of p-regular operators.

PROPOSITION 2.2. *Let $1 \leqslant p, q \leqslant \infty$. Then the following assertions are equivalent*:

(1) $$B(X^*, L_p) \circ \Lambda_\infty(L_{q'}, X^*) \subseteq \Lambda_\infty(L_{q'}, L_p),$$

(2) $$K_1(X, L_q) \subseteq \Pi_p(X, L_q),$$

(3) $$B(X^*, L_p) = P_q(X^*, L_p).$$

Proof: The equivalence of the first two conditions follows from Nielsen [12]. Since all the properties above are of the finite-dimensional character, we may replace L_p and L_q by l_p and l_q, respectively. Then by using finite rank operators we can see that the above conditions are just reformulations of each other. ∎

Remark. The same method can be used to establish the following relation:

$$K_1(X, L_q) \subseteq \Lambda_p(X, L_q) \quad \text{if and only if} \quad B(X^*, L_p) = K_q(X^*, L_p).$$

COROLLARY. *The above features are "lattice superproperties" of X, i.e., they are hereditary with respect to taking Y such that Y l.f.r. X.*

It suffices to make use of the extension property of 1-concave operators as it appears in [17] (Proposition 3.9.).

The following result proves a useful tool for our purpose.

LEMMA 2.3. *Let $1 \leqslant p, q, r \leqslant \infty$. Then the following equivalent conditions*:

(1) $$K_1(l_r, l_q) \subseteq \Pi_p(l_r, l_q),$$

(2) $$K_1(l_p, l_{q'}) \subseteq \Pi_r(l_p, l_{q'})$$

imply one of the situations:

(3i) $$q = 2, \ 1 \leqslant p, r \leqslant \infty,$$

(3ii) $$1 \leqslant q < 2, \ r > q' \text{ or } r = q', \ p \geqslant q \text{ or } r < q', \ p = \infty,$$

(3iii) $$2 < q \leqslant \infty, \ q < p < \infty \text{ or } p = q, \ r \geqslant q' \text{ or } p < q, \ r = \infty.$$

Proof. (1) is equivalent to (2) by Nielsen's characterization of p-absolutely summing operators [12] and the fact that $K_1(l_r, l_q)^* = K_1(l_{r'}, l_{q'})$ (cf. [17]). Hence we may assume $1 \leqslant q < 2$ since case (3iii) is immediate. We infer that (1) yields $r \geqslant q'$, provided $p < \infty$. In fact, since $\Pi_p \subseteq \Lambda_2$ (see [14]), there is a $C > 0$ such that for all $\mathbf{x} \in (l_{r'}^n)^n$

$$\sup\{l_q(U\mathbf{x}')/\|U\|: U \in B(l_{r'}^n, l_2)\} = \lambda_2\Big(\sum x_i' \otimes e_i^{(q)}\Big) \leqslant C\varkappa_1\Big(\sum x_i' \otimes e_i^{(q)}\Big) = Ck_q(\mathbf{x}').$$

Now by taking $x_i' = e_i^{(r')}$ and the formal embedding $U\colon l_{r'}^m \to l_2$ we have

$$n^{1/q} \leqslant C n^{(1/r-1/2)_+} n^{1/r'},$$

where $a_+ = \max(a, 0)$, and this is possible if and only if $r \geqslant q'$. If $r = q'$ then (1) implies $\Lambda_\infty(l_{q'}, l_q) = K_1(l_{q'}, l_q) \subseteq \Pi_p(l_{q'}, l_q)$, which clearly gives $p \geqslant q$. ∎

In the above proof one can also apply Pietsch's concept of a limit order [16]. In fact, conditions (3) are sufficient for (1) or (2); however, we shall present a more general result in the next section.

LEMMA 2.4. *Let* $1 \leqslant q < 2$, $1 \leqslant p < \infty$. *If* $K_1(X, l_q) \subseteq \Pi_p(X, l_q)$ *then there is a* $C > 0$ *such that*

$$\|\sup |x_i|\| \leqslant C l_{q'}(\mathbf{x})$$

for all $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ *with disjoint components.*

Proof. Since $\Pi_p(X, l_q) \subseteq \Lambda_2(X, l_q)$, by the remark following Proposition 2.2 there is a $C > 0$ such that

$$l_q(U\mathbf{x}') \leqslant C\|U\| \, l_q(\mathbf{x}')$$

for all $U \in B(X^*, l_2)$ and all $\mathbf{x}' \in X^{*n}$. Let $x_1', \ldots, x_n' \in X^*$ be arbitrary disjoint vectors. Then there exist disjoint vectors $x_1, \ldots, x_n \in X$ with $1 \leqslant \|x_j\| \leqslant 2$ such that $\langle x_i', x_j \rangle = 0$ for $i \neq j$ and $\langle x_i', x_i \rangle = \|x_i'\|$ (cf. [3]). Since X is 2-convex by Corollary to Proposition 2.1 and Lemma 2.3, we have $\|U\| \leqslant 2c$, where $U = \sum x_i \otimes e_i^{(2)}$ and c is a 2-convexity constant of X. Moreover, $\|Ux_i'\| = \|x_i'\|$; hence

$$l_q(\mathbf{x}') \leqslant 2Cc \, \|\sup |x_i'|\|.$$

By duality [7] X satisfies the desired property. ∎

We do not know whether in the above assertion one could obtain the q-convexity of X. We guess that the answer is negative. Take, e.g., $p = 2$. The key point in our proof is the estimation of $\lambda_2(T)$ from below by $\lambda_\infty(T)$. It was done in fact for an arbitrary isomorphism T, and such an estimation for any $T \in B(X, l_q)$ is possible if and only if X is isomorphic to a Hilbert space ([14], Theorem 3.4.)!

3. A characterization result. In this part of the paper we characterize the property $K_1(X, L_q) \subseteq \Pi_p(X, L_q)$ for all possible values of p and q. We shall write $X \in AM$ if X is isomorphic to a sublattice of $C(K)$ [17].

THEOREM 3.1. *Let* $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$. *Let* X *be a Banach lattice such that* $p(X)$ *is attained in the following three cases*:

(i) $p(X) = q'$ *and* $1 < q < 2$,

(ii) $p(X) = q'$ *and* $2 < p = q$,

(iii) $p(X) = \infty$ *and* $q > 2$, $p < q$.

Then

I. $K_1(X, L_q) \subseteq \Pi_p(X, L_q)$

if and only if one of the following conditions is satisfied:

IIa. $q = 2$, X *arbitrary*,

IIb. $1 \leqslant q < 2$, $p(X) > q'$ *or* X *is* q'*-convex*, $p \geqslant q$,

IIc. $2 < q \leqslant \infty$, $p > q$ or $p = q$, X *is* q'*-convex or* $p < q$, $X \in AM$.

Proof. I ⇒ II. If $q > 1$ we show the implication by applying the Corollary following Proposition 2.2 and Lemma 2.3 with $r = p(X)$, since $l_{p(X)}$ l.f.r. X ([8], [11]). If $q = 1$ we rather use Lemma 2.4.

IIa ⇒ I. This follows by Proposition 2.2 and Proposition 2.1 (3).

IIb ⇒ I. Let $1 \leqslant q < 2$. If $p(X) > q'$ then $q(X^*) < q$ and hence $K_1(X, L_q) = \Lambda_\infty(L_{q'}, X^*) \subseteq \Pi_1(L_q, X^*)$ by [20]. By Kwapień's result [9] $\Pi_1(L_{q'}, X^*) \subseteq \Pi_1(X, L_q)$. Since $\Pi_1 \subseteq \Pi_p$ [15], the implication holds true in the first case. If $p \geqslant q$ and X is q'-convex, then $K_1(X, L_q) \subseteq \Lambda_\infty(X, L_q) \subseteq \Pi_p(X, L_q)$ by [17] and [15].

IIc ⇒ I. If $p > q$ we use the factorization of 1-concave operators through L_1 [17] and Lemma 1.4 in [13]. If $p = q$ and X is q'-convex, we proceed as in the case $q < 2$. Finally, if $X \in AM$ then 1-concave operators are 1-integral [15] and hence p-absolutely summing for all p, $1 \leqslant p < \infty$. ∎

Let us notice that a more general form of the assertion IIb ⇒ I can be found in [13] (Theorem 1.5) and by way of analogy also IIc ⇒ I can get a stronger form. However, we are mainly interested in the converse conclusion, and that is why we have the result in the form presented here. We infer also that, for a given p, $K_1(X, E) \subseteq \Pi_p(X, E)$ for any X if and only if $B(l_1, E) = \Pi_p(l_1, E)$, e.g., if E is a subspace of a quotient of L_q, $q < p$, cf. [13], when $p > 2$ or if E is isomorphic to a Hilbert space when $p \leqslant 2$. In fact, the case $p > 2$ follows immediately. In the case $p \leqslant 2$ we take X which is 2-concave, say $X = l_2$. Then from $K_1(l_2, E) \subseteq \Pi_p(l_2, E) \subseteq \Pi_2(l_2, E)$ we deduce by duality that $\Lambda_2(E, l_2) = \Pi_2(E, l_2) \subseteq \Lambda_\infty(E, l_2)$, and by applying Theorem 3.4 from [14] E turns out to be isomorphic to a Hilbert space.

Unfortunately, we are not able to avoid the restrictions of Theorem 3.1. The most interesting problem seems to be connected with (i). By Lemma 2.3, the condition $K_1(X, l_q) \subseteq \Pi_p(X, l_q)$, $p < q$, $q > 2$, yields $p(X) = \infty$ (in fact $p(Y) = \infty$ for any Banach lattice Y l.f.r. X). However, there are examples of spaces which have the above property without being AM-spaces ([1], [18]). Now we give some equivalent formulations of this property. For simplicity we restrict ourselves to the case $p = 2$.

PROPOSITION 3.2. *Let* $2 < q < \infty$ *and* $1/2 = 1/q + 1/s$. *Consider the statements*:

(1) $K_1(X, L_q) \subseteq \Pi_2(X, L_q)$;

(2) *there is a* $C>0$ *such that*

$$k_q(\mathbf{ax}) \leqslant C w_2(\mathbf{x}) \|\mathbf{a}\|_2 \quad \text{for all } n\in N,\ \mathbf{x}\in X^n,\ \mathbf{a}\in l_2^n;$$

(3) *there is a* $C>0$ *such that*

$$w_s(|\mathbf{x}|) \leqslant C w_2(\mathbf{x}) \quad \text{for all } n\in N,\ \mathbf{x}\in X^n;$$

(4) $$B_+(X, L_1) = \Pi_{s,2}(X, L_1);$$

(5) $$K_1(X, L_r) \subseteq \Pi_2(X, L_r) \quad \text{for all } r<q;$$

Then $(1)\Leftrightarrow(2)\Rightarrow(3)\Leftrightarrow(4)\Rightarrow(5)$.

$\Pi_{s,2}$ above denotes the ideal of $(s, 2)$-summing operators, i.e., of operators T such that $l_s(T\mathbf{x}) \leqslant C w_2(\mathbf{x})$ for all $\mathbf{x}$.

Proof. By the Nielsen characterization of 2-absolutely summing operators [12]

$$\lambda_\infty(TS) \leqslant C\|T\|\cdot\lambda_\infty(S)$$

for all $S\in \Lambda_\infty(l_q, l_2)$ and $T\in B(l_2, X)$, provided (1). Now (2) is a finite-dimensional reformulation of (1).

Since $w_{q'}(|\mathbf{x}|) \leqslant k_{q'}(\mathbf{x})$, hence $(2)\Rightarrow(3)$ follows by the Hölder inequality. We can associate an L_1-space $L(x')$ with any vector $x'\in X_+^*$, by taking the completion of $X/\ker x'$ under the norm $|||x||| = \langle x', |x|\rangle$ (cf. [17]). The natural embedding $S_{x'}\colon X\to L(x')$ is a positive operator and $\|S_{x'}\| = 1$. Since (4) depends only on the finite-dimensional lattice structure of L_1, we have $(3)\Leftrightarrow(4)$.

By Pietsch [16] (Th. 20.1.12) any $(s, 2)$-summing operator is $(r, 2)$-mixing for $r<q$. Since 1-concave operators factorize through L_1-space as follows: $T = UV$, $V\in B_+(X, L_1)$, $U\in B(L_1, L_r)$ ([17], Prop. IV.3.3), it follows by (4) that V is $(r, 2)$-mixing and U is q-absolutely summing ([13], Lemma 1.4). Therefore T is 2-absolutely summing ([16], Theorem 20.2.1). ■

4. p-concave and p-absolutely summing operators. The problem studied in this section is related to Maurey's result [10], which states that an operator from X into E is p-concave if and only if, for any positive operator S from $C(K)$ into X, TS is p-absolutely summing. Evidently, if $X\in AM$ then $K_p(X, E) = \Pi_p(X, E)$. For some special spaces E, e.g., $E = L_1$, the converse holds true as well [19]; also $K_1(X, L_q) = \Pi_1(X, L_q)$ if and only if $p(X) > q'$ (Theorem 3.1).

In this section we characterize those Banach lattices X for which $K_p(X, E) = \Pi_p(X, E)$ for some E.

LEMMA 4.1. *Let* $1\leqslant p<\infty$. *A Banach lattice* Y *is isomorphic to* L_1 *if and only if there is a* $C>0$ *such that*

(+) $$l_p(\mathbf{y}) \leqslant C w_p(\mathbf{y})$$

for all $\mathbf{y} = (y_1, \ldots, y_n)\in Y^n$ *with disjoint components.*

Proof. Let $Y = L_1$. Then for disjoint $y_1, \ldots, y_n\in Y$ we have

$$l_p(\mathbf{y}) = \sup_{\mathbf{a}\in R_+^n} \sum |a_i|\,\|y_i\|/\|a\|_{p'} = \sup_{\mathbf{a}} \big\|\sum a_i |y_i|\big\|/\|a\|_{p'} = w_p(\mathbf{y}).$$

Now assume (+) is satisfied. Then for $\mathbf{c}\in R_+^n$, for disjoint $y_1, \ldots, y_n\in Y$, we have

$$\begin{aligned} l_p(\mathbf{cy}) &\leqslant C\sup\{(\textstyle\sum c_i^p \langle y', y_i\rangle \|y_i\|^{p-1})^{1/p}\colon y'\geqslant 0,\ \|y'\|\leqslant 1\} \\ &\leqslant C\sup_i c_i \|y_i\|^{1-1/p} \|\textstyle\sum y_i\|^{1/p}. \end{aligned}$$

Put $c_i = \|y_i\|^{1/p-1}$. Then

$$\sum \|y_i\| \leqslant C^{1/p} \big\|\sum y_i\big\|.$$

Hence by Schlotterbeck's result ([17], Prop. IV.2.7) Y is isomorphic to L_1. ■

THEOREM 4.2. *Let* X *be a Banach lattice. Let* p, q *be such that* $1<p<\infty$, $q\neq 2$, $\max(q, 2)\geqslant p$. *Then*

$$K_p(X, L_q) = \Pi_p(X, L_q) \quad \textit{if and only if} \quad X\in AM.$$

Proof. We shall prove the "only if" part since the "if" part is evident.

We observe that there is a sequence $\mathbf{e} = (e_i)_{i=1}^\infty \subset L_q$ such that $\|e_i\| = 1$ and $a = w_{p'}(\mathbf{e}) < \infty$. In fact, we take either disjoint normalized vectors if $1<p\leqslant q$ or independent normalized p-stable functions if $1\leqslant q<p\leqslant 2$. From $K_p(X, L_q) = \Pi_p(X, L_q)$ we infer that for $T\in B_+(X, l_p)$ and $S\in B(l_p, L_q)$ we have $ST\in \Pi_p(X, L_q)$. Moreover, by the Banach–Steinhaus theorem there is a $C>0$ such that

(++) $$\pi_p(ST) \leqslant C\|S\|\cdot\|T\|.$$

Let $(x_j')_{j=1}^n \subset X_+^*$ be a sequence of disjoint vectors and let (x_k) be such that $1\leqslant \|x_k\| \leqslant 2$, $\langle x_j', x_k\rangle = 0$ for $j\neq k$ and $\langle x_j', x_j\rangle = \|x_j'\|$ [3]. By applying (++) to the operators $T = \sum_{j=1}^n x_j'\otimes e_j^{(p)}$, $S = \sum_{j=1}^n e_j^{(p')}\otimes e_j$ we infer that

(+++) $$l_p(\mathbf{x}') \leqslant 2Ca w_p(\mathbf{x}') w_p(\mathbf{x}).$$

In order to apply Lemma 4.1 and hence to finish the proof it suffices to show that X is p'-convex, which ensures that $w_p(\mathbf{x})$ is bounded. To this end note that X is of finite concavity since by Proposition 3.1, $p(X) > 1$. We use here the inclusion $K_1 \subset K_p$. Moreover, $p(X) = \infty$ if $2 < p < q$, i.e. $p(X) > p'$. For the remaining case, $p \leqslant 2$, let us assume that $s = p(X) \leqslant p'$. Then $s' = q(X^*)$ and by [11] $l_{s'}$ l.f.r. X^*. Hence there is a $C_1 > 0$ such that for each natural number n one can find disjoint vectors $x'_1, \ldots, x'_n \in X^*$ which are C_1-equivalent to $e_1^{(s')}, \ldots, e_n^{(s')}$. In particular, $w_p(\mathbf{x}') \leqslant C_1 n^{1/s'-1/p'}$. Let $x_1, \ldots, x_n \in X$ be as in $(+++)$. Since X is r-convex for all $r < s$, we have $w_p(\mathbf{x}) \leqslant C_2 n^{1/r-1/p'}$. One can choose r so that $1/r < 1/s + 1/p'$ as $s > 1$. Now it follows from $(+++)$ that

$$n^{1/p} \leqslant 2aCC_1 C_2 n^{1/r-1/p'} n^{1/s'-1/p'},$$

and thus $1 \leqslant 1/r + 1/s' - 1/p' < 1$, which is a contradiction. Therefore $p(X) > p'$, and hence X is p'-convex. ∎

Remark. By following the proof of Theorem 4.2 we conclude that

$$K_p(X, E) \subseteq \Pi_p(X, E) \text{ if and only if } X \in AM$$

provided one of the following conditions is satisfied:

(1) *There is a* $q \neq 2$, $p \leqslant \max(q, 2)$ *such that* l_q *f.r.* E,

(2) *There is a* $C > 0$ *such that for each* n *one can find* $e_1, \ldots, e_n \in E$ *with* $\|e_i\| = 1$ *and* $w_{p'}(\mathbf{e}) \leqslant C$, *and* $p(X) > 1$.

Let us point out three cases omitted in the above result.

I. $p < q = 2$,

II. $p \geqslant q = 2$.

Since $K_p(X, l_2) = K_2(X, l_2) = B(X, l_2)$ [10] and $\Pi_p(X, l_2) = \Pi_2(X, l_2)$, we have $K_p(X, l_2) = \Pi_p(X, l_2)$ if and only if $B(X, l_2) = \Pi_2(X, l_2)$. This defines the class of Hilbert–Schmidt spaces introduced and investigated by Jarchow [5].

III. $p > q > 2$.

By using similar arguments we deduce that the discussed property of X can be phrased as $B(X, l_q) = \Pi_p(X, l_q)$. Since $l_2 \subset L_q$, this shows X to be a Hilbert–Schmidt space.

We know three examples of Banach lattices satisfying I and II and III, namely L_1, L_∞, $L_1 \oplus L_\infty$. On the other hand, in any of these situations if X is of finite concavity or of convexity strictly stronger than 1 then X is isomorphic to L_1 or to a sublattice of L_∞, respectively. We conjecture that there are no other examples.

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