On the capacity of a continuum with a non-dense orbit under a hyperbolic toral automorphism

by

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Abstract. In this paper we compute an upper and lower estimation for the capacity of a continuum (connected compact set) lying in the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ whose orbit under a hyperbolic toral automorphism is not dense in $\mathbb{T}$. Also estimations of capacity in Poincaré’s sense are considered.

Introduction. The main results.

1. First we define capacity. Let $(X, d)$ be a compact metric space and let $A$ be any subset in $X$. Cover it with finitely many balls \( \{B(x_i, r_i)\}_{i=1}^n \) with centres in $A$ of radii $r_i \leq \varepsilon$. By $I(A, \varepsilon)$ denote the minimal possible $k$. The number

\[
C_A = \lim \sup_{\varepsilon \to 0} \frac{\log I(A, \varepsilon)}{-\log \varepsilon}
\]

is called the capacity of the set $A$. Observe that $\dim_H A \leq C_A$, where $\dim_H$ is the Hausdorff dimension and that

\[
\text{if } a_i \searrow 0, \lim_{i \to \infty} \sup_{i} \frac{\log I(A, a_i)}{-\log a_i} < +\infty, \quad \text{then}
\]

\[
C_A = \lim_{i \to \infty} \frac{\log I(A, a_i)}{-\log a_i}.
\]

2. Denote by $\pi: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the standard covering projection. A hyperbolic toral automorphism is a map $f: \mathbb{T} \to \mathbb{T}$ which has a linear lift $\tilde{f}: \mathbb{R} \to \mathbb{R}$ without eigenvalues of modulus 1. It is clear that there exists a minimal number $r \geq 1$ such that either the eigenvalues of $f$ are real and positive or they are not roots of real numbers. By $\tilde{f}$ we denote $\tilde{f}$. We define

\[
E_1 = \bigcup_{j=0}^{\infty} (\tilde{f} - j\lambda \text{id})^{-1}(0)
\]

if an eigenvalue $\lambda$ of $\tilde{f}$ is real and

\[
E_1 = \bigcup_{j=0}^{\infty} (\tilde{f} - j\lambda \text{id})^{-1}(0) \cup \bigcup_{j=0}^{\infty} (\tilde{f} - j\lambda \text{id})^{-1}(0) \cap \mathbb{R},
\]
where \( \hat{f} : \mathbb{C}^* \to \mathbb{C}^* \) is the complexification of \( \tilde{f} \), if \( \lambda \) is complex. The linear subspace \( E = \bigoplus_{|\lambda| < 1} E_{|\lambda|} \) is called the contracting eigenspace for \( \tilde{f} \) and \( E^* = \bigoplus_{|\lambda| > 1} E_{|\lambda|} \) is called the expanding eigenspace for \( \tilde{f} \).

3. Now, let \( K \) be an arbitrary continuum lying in \( \mathbb{T}^n \) and let \( \tilde{K} \) be an arbitrary subset of its lift. Consider a cost \( c + \sum_{k \in A} V_k(\tilde{K}) \), where \( A \) consists of eigenvalues of \( \tilde{f} \), \( V_k(\tilde{K}) \in E_k \) is a linear subspace invariant under \( \tilde{f} \) such that \( \tilde{K} = c + \sum_{k \in A} V_k(\tilde{K}) \) and \( \dim \sum_{k \in A} V_k(\tilde{K}) \) is the least possible. These properties define the subspace \( \sum_{k \in A} V_k(\tilde{K}) \) uniquely.

4. Now, we recall that R. Mañé proved in [5] the following theorem. Let \( \alpha : (a, b) \to \mathbb{T}^n \) be a rectifiable nonconstant path and let \( f : \mathbb{T}^n \to \mathbb{T}^n \) be a hyperbolic toral automorphism. Then the closure of the orbit of \( \alpha((a, b)) \) under \( f \) contains a coset of a toral subgroup invariant under some power of \( f \).

([5] improves Frank's result, where the paths were \( C^2 \), see [2].)

We prove a related result:

Theorem 1. Let \( f : \mathbb{T}^n \to \mathbb{T}^n \) be a hyperbolic toral automorphism. Let \( K \subset \mathbb{T}^n \) be a set such that its lift \( \tilde{K} \subset \mathbb{R}^n \) contains a non-one-point continuum \( \tilde{K} = c + \sum_{k \in A} V_k(\tilde{K}), (|\lambda_1| \leq \ldots \leq |\lambda_A|) \). Denote \( V_k(\tilde{K}) = V_k \). If one of the following cases holds:

(a) \( \dim V_k = 1, |\lambda_k| > 1, |\lambda_{k-1}| < 1, \)
(b) \( \dim V_k = 1, |\lambda_k| < 1, |\lambda_{k+1}| > 1, \)
(c) \( \dim V_k = 1, |\lambda_k|, |\lambda_{k+1}| > 1 \) and \( C_k < 2 - \frac{\log|\lambda_{k-1}|}{\log|\lambda_k|} \)
(d) \( \dim V_k = 1, |\lambda_k|, |\lambda_{k+1}| < 1 \) and \( C_k < 2 - \frac{\log|\lambda_{k+1}|}{\log|\lambda_k|} \)
then the closure of the orbit of \( K \) under \( f \) contains a coset of a toral subgroup invariant under some power of \( f \).

Theorem 1 concerns in fact compact sets which are not zero-dimensional (i.e., countable or unions of a Cantor set and a countable set). The problem is that every such set must contain a non-one-point continuum, see [1].

5. Przytycki [7] has constructed for any Anosov diffeomorphism \( f \) of an \( n \)-dimensional torus \( \mathbb{T}^n \) an invariant subset of arbitrary dimension between 1 and \( n-2 \). Paper [7] develops the Hancock idea, see [3], [4]. We shall use this construction. We recall it in the case where the diffeomorphism \( f \) is algebraic and \( \dim \mathbb{E} = 1 \). Here is an outline: One may assume that the orthogonal projection \( P : \mathbb{E}^* \to \mathbb{E}^{* - 1} = \{ x \in \mathbb{E}^* ; x_n = 0 \} \) is an isomorphism. Fix \( k (1 \leq k \leq n-2) \). One may consider \( \mathbb{E}^{* - 1} \) as the union of \( (n-1) \)-dimensional cuboids \[ \{ x = (x_1, \ldots, x_{n-1}) ; m_1 \leq x_1 \leq m_1 + 1, \ldots, m_{n-1} \leq x_{n-1} \leq m_{n-1} + 1, m_n \in \mathbb{Z} \} \]
with edges of length 1. Denote by \( \mathcal{K} \) the union of \( (n-k-2) \)-dimensional skeleton of these cuboids. Let \( D \) be a \( k \)-dimensional disc embedded by \( g \) into \( \mathbb{E}^* \). There exists a continuous mapping \( g_k : \mathbb{E}^* \to \mathbb{E}^* \) such that \( g_k \) is \( C_{S^1} \) close to \( g \) and \( g_k(D) \) is disjoint from \( P^{-1}(B(\mathcal{K}, r)) \), where \( B(\mathcal{K}, r) = \{ x \in \mathbb{E}^{* - 1} ; g(\mathcal{K}, x) < r \} \) and \( C_0 \) is a constant coefficient. Let \( d > 1, a > 0 \) satisfy the condition

\[
\alpha \| f^d v \| > \lambda^d \| v \|, \quad v \in \mathbb{E}^*.
\]

There exists a positive integer \( q \) which satisfies the inequality

\[
1 - a C_0 \sum_{i=1}^m (1/2^i) > 0.
\]

Assume that a continuous mapping \( g_k : D \to \mathbb{E}^* \) such that \( f^m \circ g_k(D) \cap P^{-1}(B(\mathcal{K}, r)) = \emptyset \),

then is defined. There exists a continuous mapping \( h : D \to \mathbb{E}^* \) such that \( h(D) \cap P^{-1}(B(\mathcal{K}, r)) = \emptyset \) and \( h \) is \( C_{S^1} \) close to \( f^{m+1} \circ g_k \). Define \( g_{k+1} = f^m \circ h \circ g_k \). By (1) there exists a continuous mapping \( G = \lim_{i \to \infty} g_i \). Is a continuum. In [7] Przytycki proved that

\[
k = \dim (\tau(G(D))) = \dim (G(D)) = \dim (\{ x \in \bigcup_{j=-\infty}^{\infty} f^j \circ \pi \circ G(D) \}).
\]

In this paper we prove the following theorem:

Theorem 2. Fix \( 1 \leq k \leq n-1 \). Let \( f : \mathbb{T}^n \to \mathbb{T}^n \) be a hyperbolic toral automorphism. Let \( \lambda_1, \ldots, \lambda_k \) be arbitrary eigenvalues for \( \tilde{f} \) such that \( 1 < |\lambda| \leq \ldots \leq |\lambda| \) and let \( V_1 \subset \mathbb{E}_{|\lambda|}, \ldots, V_k \subset \mathbb{E}_{|\lambda|} \) be arbitrary \( f \)-invariant, linear subspaces in \( \mathbb{R}^n \), \( \sum_{j=2}^k \dim V_j \leq k \), \( \sum_{j=2}^k \dim V_j = k + k' > k \). Then for every \( \varepsilon > 0 \)

there exists a \( k \)-dimensional continuum \( K \subset \mathbb{T}^n \) such that \( \dim (\{ x \in \bigcup_{j=-\infty}^{\infty} f^j(K) \}) = k \) and

\[
(C_k - \varepsilon + \log|\lambda|^{-1}) \left( \frac{\dim V_k - k}{\log|\lambda|} + \sum_{j=2}^k \dim V_j \log|\lambda| \right) = 0.
\]
Remark. In constructing \( K \) one can consider, instead of the whole \( V \), an invariant subspace \( V' \subset V \) such that \( \dim(V' \oplus V) = k + k' > k \), where \( k' = 1 \) if \( \lambda_1 \) is real and \( k' = 2 \) if \( \lambda_1 \) is complex (not real). In the case \( k' = 2 \) one can consider only \( dim V' \geq 2 \). (Otherwise, if \( dim V' = 1 \), one could consider \( \oplus \overline{V'} \) only, having \( \dim(\overline{V'}) = k + 1 \). These changes of \( V \) clearly improve the estimations (i), (ii) for an appropriate \( K \). It is immediately computable that, after these improvements of the spaces \( V \) (for \( k' = 1 \) or \( k' = 2 \) and \( dim V' \geq 2 \)), estimation (ii) is better than (i).

For example, in the case of \( T^4, \lambda_1 < 1 < \lambda_2 < \lambda_3 \), for every curve \( \gamma \) with non-dense orbit, its capacity, by Theorem 1, satisfies

\[
C_{\gamma} \geq 2 - \frac{\log \lambda_2}{\log \lambda_3}.
\]

This estimation is the best possible because, by Theorem 2, there exists a curve \( \gamma_0 \) with non-dense orbit where \( C_{\gamma_0} \leq 2 + \frac{\log \lambda_2}{\log \lambda_3} \). Can this be removed?

If \( \lambda_2 = \lambda_3 \) then the method used in the proof of Theorem 1 gives nothing interesting. In view of Theorem 2 we really cannot have any estimation of type \( C_{\gamma} \geq 1 + \text{const} > 1 \). So we cannot prove anything more than non-rectifiability.

6. We shall use the following simple

**Geometrical Lemma.** For all integer numbers \( n \geq 1 \), \( 1 \leq p \leq n \), and real numbers \( \gamma, \eta > 0 \), there exists a constant \( C > 0 \) such that, if \( W \) is a \( p \)-dimensional parallelepiped lying in \( \mathbb{R}^n \) with the edges of lengths \( a_1, \ldots, a_p \geq \eta \), then \( I(W, \eta) \leq C a_1 \cdots a_p \).

**Proofs.**

1. Proof of Theorem 1. Consider first the case where \( |\lambda_{k+1} - 1| > 0 \). Fix \( 0 < \theta < |\lambda_{k+1} - 1|/2 \). Suppose to the contrary that the closure of the orbit of \( K \) under \( f \) does not contain any coset of a toral subgroup invariant under \( f^m \) for every \( m \geq 1 \). The definition of the linear space \( \bigoplus_{p} V \) immediately implies the existence of at least one pair \((a, b)\) of points lying in \( \mathcal{R} \) such that

\[
b - a = v_{1} + v, \quad v \in \bigoplus_{p} E_{\lambda_1}, \quad 0 \neq v \in V.
\]

The group \( \mathcal{G}(\mathcal{V}) \) is of course a toral subgroup invariant under \( f^\gamma \). Therefore, according to the contrary assumption, for every \( x \in \mathbb{R}^n \) there exists at least one point \( x \in V', x \in V \) and \( \epsilon(x) > 0 \) such that

\[
f^n(K) \cap B(x, \epsilon(x)) = \emptyset \quad \text{for every} \quad j > 0.
\]

Choose points \( x_1, \ldots, x_m \in \mathbb{R}^n \) such that \( \bigcup_{j=1}^{m} B(x_j, \epsilon(x_j)) = \mathbb{T} \). So, for every \( x \in \mathbb{R}^n \), there exists a point \( x_j \in \mathbb{R}^n \) \((1 \leq j \leq m)\) such that \( \Phi(x, x_j) \leq \frac{1}{2} \epsilon(x_j) \). Thus, for \( \epsilon = \min(\epsilon(x_1), \ldots, \epsilon(x_m)) \),

\[
f^K \cap B(x, \epsilon(x)) = \emptyset \quad \text{for every} \quad j > 0, \quad x \in \mathbb{R}^n.
\]

(after a new, improved choice of \( \epsilon \)).

Denote by \( d_j \) the infimum of the lengths of rectifiable curves joining the points \( a \) and \( b \) in \( \bigoplus_{p} V \), whose \( j \)-th image under \( f \) is disjoint from \( B(x + x, e + Z^\eta) \) for all \( x \in \mathbb{R}^n \). It is clear that there exists a constant \( C_1 > 0 \) such that

\[
\|f^j\| \leq C_1 (\lambda_{\mu} + \theta)^j \|u\| \quad \text{for every} \quad u \in F \quad \text{and} \quad j > 0.
\]

Let \( \delta_j = u/C_1 (\lambda_{\mu} + \theta)^j \) \((j > 0, 1, \ldots)\). By (1) and (2)

\[
f^j(B(K, \delta_j) \cap (\bigoplus_{p} V')) \cap (B(x + x, e) + Z^\eta) = \emptyset \quad \text{for} \quad j > 0, \quad x \in \mathbb{R}^n.
\]

Now, we consider an arbitrary \( \delta_j \)-net in \( \mathcal{R} \). Since \( \mathcal{R} \) is connected, there exists a broken line with the vertices chosen from our \( \delta_j \)-net and the distances between the successive vertices not greater than \( 2\delta_j \). Obviously, this broken line lies in a coset \( \bigoplus_{p} V \) and due to (3) its image under \( f^j \) is disjoint from \( B(x + x, e) + Z^\eta \) for all \( x \in \mathbb{R}^n \). So

\[
d_j < 2\delta_j I(\mathcal{R}, \delta_j).
\]

This implies

\[
I(\mathcal{R}, \delta_j) \geq \frac{d_j}{2\delta_j} = \frac{C_1 (\lambda_{\mu} + \theta)^j}{2e}
\]

and

\[
\frac{\log I(\mathcal{R}, \delta_j) + \log (C_1 (\lambda_{\mu} + \theta)^j/2e)}{\log (C_1 (\lambda_{\mu} + \theta)^j/2e)}
\]

but, as

\[
\lim_{j \to \infty} \frac{\log (C_1 (\lambda_{\mu} + \theta)^j/2e)}{\log (C_1 (\lambda_{\mu} + \theta)^j/2e)} = 1,
\]
we obtain

\begin{equation}
C_{\varphi} \geq 1 + \limsup_{j \to \infty} \frac{\log d_j}{\log (\lambda^j + \theta)}.
\end{equation}

Since the torus \( T^n \) is compact, there exists a constant \( C_{\varphi} > 0 \) such that every segment of the straight line \( x + V_\varphi \) with length \( \geq \frac{1}{4} C_{\varphi} \), intersects the set \( B(\varphi, \epsilon/3) + Z^* \) for all \( x \in \mathbb{R}^n \), \( \varphi \in V_\varphi \).

Since the directions of the vectors \( f^j b - f^j a \) tend to the direction of the straight line \( V_\varphi \) due to \( \lambda_{\varphi - 1} < \lambda_\varphi \) and due to \( \theta \), for an integer \( j \) large enough every segment of the straight line \( x + R(f^j b - f^j a) \) with length \( C_{\varphi} \), intersects the set \( B(\varphi, \epsilon/2) + Z^* \) for all \( x \in \mathbb{R}^n \), \( \varphi \in V_\varphi \).

For every \( j \geq 0 \) denote by \( l_j \) an arbitrary rectifiable curve in \( \varphi \)

\begin{equation}
\int l_j \cap (B(\varphi + \epsilon, \epsilon) + Z^*) = \emptyset \quad \text{for all} \quad x \in \mathbb{R}^n.
\end{equation}

Denote by \( P_j \) the projection of \( \varphi \) onto the line \( R(f^j b - f^j a) \), along \( \frac{1}{\epsilon} \varphi \).

In the curve \( f^j l_j \) choose points \( f^j a = y_1, \ldots, y_{k_0}, y_{k_0 + 1}, \ldots, y_{k_0 + 2} = f^j b \) such that \( \|P_j(y_{i+1} - y_i)\| = C_{\varphi} \) for \( i = 1, \ldots, t(j) \) and \( \|P_j(y_{i+1} - y_i)\| \leq C_{\varphi} \). Let \( y_{i+1} = y_i + P_j(y_{i+1} - y_i) \) \( i = 1, \ldots, t(j) \). Since the interval \( y_1 y_{i+1} \) has the length \( C_{\varphi} \), it intersects \( B(y_i, \epsilon/2) + Z^* \) for every \( 1 \leq i \leq t(j) \) and \( j \) large enough. Thus by (5) there exists a point \( z_i \) in the segment of the curve \( f^j l_j \) between the points \( y_i \) and \( y_{i+1} \) and a point \( z \in y_i + R(f^j b - f^j a) \) such that \( z_i - z \in \mathbb{R}^n \), \( \|z_i - z\| = \epsilon/2 \).

Since \( + R(f^j b - f^j a) \), \( V_\varphi \rightarrow 0 \) and \( \lambda_{\varphi} \) is the eigenvalue for \( f \) corresponding to \( V_\varphi \), there exists a constant \( C_{\varphi} > 0 \) such that

\[\|f^k u\| \geq C_{\varphi} (\lambda_{\varphi} - \theta)^k \|u\| \quad \text{for every} \quad k \geq 0, \quad u \in R(b - a).\]

Observe that \( \sum_{k=0}^{\infty (\log (\lambda_{\varphi} - \theta))_k} \|f^k u\| = \|f^j b - f^j a\| \); hence

\[C_{\varphi} (\lambda_{\varphi} - \theta)^j \|b - a\| = \|f^j b - f^j a\| \geq C_{\varphi} (\lambda_{\varphi} - \theta)^j \|b - a\| \cdot \frac{\log (\lambda_{\varphi} - \theta)}{\log (\lambda_{\varphi} - \theta) + \theta}.\]

So

\[t(j) \geq \frac{C_{\varphi}}{\lambda_{\varphi} - \theta} \|b - a\| - 1 \quad \text{and} \quad \|f^{-j} x - f^{-j} y\| \leq \frac{C_{\varphi}}{\lambda_{\varphi} - \theta}.\]

There exists a constant \( C_{\varphi} > 0 \) such that, for every \( u \in \mathbb{R}^n \), \( j \geq 0 \),

\[\|f^{-j} u\| \leq C_{\varphi} (\|\lambda_{\varphi - 1} + \theta\|)^j \|u\|\]

By the triangle inequality, for every \( i = 1, \ldots, t(j) \) the length of the part of \( l_j \) between \( y_i \) and \( y_{i+1} \) is not less than

\[\|f^{-j} y_i - f^{-j} y_{i+1}\| \geq \|f^{-j} y_i - f^{-j} y_{i+1}\| - \|f^{-j} y_i - f^{-j} y_{i+1}\| = C_{\varphi} \|\lambda_{\varphi - 1} + \theta\|^{-j} \|f^{-j} y_i - f^{-j} y_{i+1}\|.\]

Taking the sum over \( i \) and the infimum over the rectifiable curves, we obtain by (6)

\[d_j \geq C_{\varphi} \|\lambda_{\varphi - 1} + \theta\|^j \|b - a\| \cdot \frac{C_{\varphi} \|\lambda_{\varphi - 1} + \theta\|^{-j} \|b - a\|}{C_{\varphi} \|\lambda_{\varphi - 1} + \theta\|^{-j} \|b - a\| + C_{\varphi} \|\lambda_{\varphi - 1} + \theta\|^{-j} \|b - a\|}\]

As \( \|\lambda_{\varphi - 1} + \theta\| > 1 \), \( \|\lambda_{\varphi - 1} - \lambda_{\varphi} - \theta\| > 1 \), there exists a constant \( C_{\varphi} > 0 \) independent of \( j \) such that for \( j \) large enough

\[d_j \geq C_{\varphi} \|\lambda_{\varphi - 1} + \theta\| \|b - a\|.\]

Together with (4) this implies that

\[C_{\varphi} \geq 1 + \limsup_{j \to \infty} \frac{j(\log (\lambda_{\varphi} - \theta) - \log (\|\lambda_{\varphi - 1} + \theta\|) + \log C_{\varphi})}{j \log (\lambda_{\varphi} + \theta)} \]

\[= 1 + \frac{\log (\lambda_{\varphi} - \theta) - \log (\|\lambda_{\varphi - 1} + \theta\|) + \log C_{\varphi}}{\log (\lambda_{\varphi} + \theta)} \]

Since \( \theta \) can be an arbitrarily small positive number, we have

\[C_{\varphi} \geq 2 \frac{\log (\lambda_{\varphi - 1})}{\log \lambda_{\varphi}} \quad \text{hence} \quad C_{\varphi} \geq 2 \frac{\log (\lambda_{\varphi - 1})}{\log \lambda_{\varphi}}.\]

This contradicts the assumptions of our theorem.
The case \(|a_{i-1}| < 1\) is much simpler. Let \(P_x\) denote the projection of \(\bigcap_{y \in V_x} V_y\) onto \(V_x\) along \(\bigcap_{y \neq x} V_y\). The Hausdorff distances between the sets \(\mathcal{C}(f_x(R), P_x(f_x(R)))\) tend exponentially to 0, while the lengths of the curves \(P_x(f_x(R))\) tend to \(\infty\). Hence, in \(\mathcal{T}^n, \mathcal{C}(\pi(V_x)) \subset \mathcal{C}(\bigcup_{x \in V} f_x(R))\) (we use again the fact stated directly after formula (4)).

Proof of Geometrical Lemma. We can cover each edge of length \(a_i\) (1 \(\leq i \leq p\)) by \([a_i, \frac{\gamma}{\gamma}] + 1\) segments of length \(\leq \frac{\gamma}{\gamma}\). Thus we can cover our parallelepipeds with

\[
\prod_{i=1}^{p} \left( \left\lfloor \frac{2a_i}{\gamma} \right\rfloor + 1 \right) \leq \prod_{i=1}^{p} \left( \frac{2a_i}{\gamma} + \frac{a_i}{\eta} \right) = \left( \frac{2 + \frac{1}{\gamma}}{\gamma} \right)^p \prod_{i=1}^{p} a_i
\]

parallelepipeds with the edges of length \(\leq \frac{\gamma}{2}\), because \(a_i \leq \eta\) (1 \(\leq i \leq p\)). Since each parallelepiped with the edges of length \(\leq \frac{\gamma}{2}\) lies in a ball of radius \(\frac{\gamma}{2}\), the Lemma is true if we set \(C = (2 + 1/\gamma)^p\).

Proof of Theorem 2. Assume \(\dim E = 1\). We mention at the end of the proof how to get rid of that with the use of the ideas from [7]. Fix \(\epsilon, \theta > 0\). Wishing to construct a set satisfying the properties from Theorem 2, we must state precisely the definition of the mappings \(g_i: D \to E, i = 1, 2, \ldots\) from [5, Introduction]. Let \(V = \bigcup_{i=1}^{p} V_i\). We shall regard as a "good" parallelepiped each \(k\)-dimensional parallelepiped lying in \(V\) each edge of which is parallel to some vector from \(\bigcup_{i=1}^{p} V_i\) and is of length not less than \(r/4\).

Let \(v = (v_1, \ldots, v_{k+1})\) be a basis in \(V\) such that \(v_i \in \bigcup_{i=1}^{p} V_i, i = 1, \ldots, k+1\). Let \(v\) be a "triangulation" of \(V\) by parallelepipeds with edges of length exactly \(r/4\), parallel to vectors from \(v\).

Denote by \(k(W)\) the \(k\)-dimensional skeleton of a parallelepiped \(W\). Having \(g_i: D \to V\) defined, first perturb \(f^{(i+1)} \circ g_i\) to \(h\) by projecting \(f^{(i+1)} \circ g_i(D) \cap P^{-1}(B(x', r))\) onto \(P^{-1}(Fr B(x', r))\) as in [7]. Next approximate \(h\) to \(h\) so that

\[h(D) = \bigcap_{W \in \mathcal{C}} h(W), W \in \mathcal{C}, \forall \mathcal{O} \neq 0)\).

(Do this by consecutive projections onto the skeletons \(\bigcup_{i=1}^{p} (W, i = k + k, \ldots, k)\). Observe that there exists a constant \(C_2 > 0\) such that

\[g(k, f^{(i+1)} \circ g_i) \leq C_1 r \quad \text{and} \quad h(D) \cap P^{-1}(B(x', r/2)) = \emptyset.
\]

Now, if \(Q \subset V\) is a "good" polyhedron, i.e., if it is a union of a finite number of "good" parallelepipeds, then, setting in the Geometrical Lemma

\[\eta = r/4, \gamma = r/2, \text{we get}
\]

\[I(Q, 1/2) \leq \sum_{j=1}^{n} I(W_j, 1/2) \leq C \sum_{j=1}^{n} d^{a_j} \ldots d^{b_j},
\]

where \(Q = \bigcup_{j=1}^{n} W_j\) and each \(W_j (1 \leq j \leq m)\) is a "good" parallelepiped with edges of length \(d^{a_j} \ldots d^{b_j}\), respectively.

We denote by \(I(Q)\) the infimum over all admissible right-hand sums. Therefore

\[(1) \quad I(Q, 1/2) \leq CI(Q).
\]

If \(r\) is large enough, then for every \(u \in E^r\), \(\|f^u u\| \geq \|u\|\); so, if \(W\) is a "good" parallelepiped, then \(f^u(W)\) is also a "good" parallelepiped with edges whose product of lengths is not greater than

\[C_2 \left( \frac{\|\lambda_j\| + \|\theta\|^{\text{max}} \frac{1}{k} \right) \prod_{i=2}^{k+1} (\|\lambda_j\| + \|\theta\|^{\text{max}})^{\frac{1}{k}}, \prod_{i=1}^{k} a_i = C_2 A^k \prod_{i=1}^{k} a_i,
\]

where \(C_2 > 0\) is a constant independent of \(k\), and \(a_i\) is the length of the edges of \(W\). Thus, if \(Q\) is a "good" polyhedron, then

\[(2) \quad I(f^u(Q)) \leq CI(Q) C_2 A^k.
\]

Let \(Q(X) = \bigcup_{S \in E} S, S = B(f^u(X), (C_1 + 1)r)\), where \(X \subset V\). Observe here that \(Q(X)\) is a "good" polyhedron.

Now, if \(W\) is a "good" parallelepiped, then, since \(f^u(W)\) is also a "good" parallelepiped, the number of parallelepipeds \(S \in E, S = B(f^u(W), (C_1 + 1)r)\) is not greater than \(C_3 \prod_{i=1}^{k} a_i\), where \(a_i\) is the length of the edges of \(f^u(W)\) and \(C_3\) is a constant coefficient. Thus

\[I(Q(W)) \leq C_2 (r/4)^{k+1} C_2 \prod_{i=1}^{k} a_i.
\]

Consequently, if \(R\) is a "good" polyhedron, then

\[(3) \quad I(Q(R)) \leq C_2 (r/4)^{k+1} I(f^u(R))\]

Since, for every \(j = 0, 1, \ldots, e^{(i+1)} \circ g_{j+1}(D) \subset Q(f^u \circ g_i(D))\), we have

\[f^u \circ g_j(D) \subset Q(g_j(D)), \quad j = 0, 1, \ldots,
\]

and by (2) and (3) for \(h = 2C_2 C_2 (r/4)^{k+1}\)

\[I(Q(g_j(D))) \leq h^k A^k I(g_j(D)).
\]
By (1) this implies that for \( j = 0, 1, \ldots \)
\[
I(\bar{f}^j \circ \varrho_0(D), 1) \leq I(\bar{f}^j(\varrho_0(D)), 1/2) \leq C I(\bar{f}^j(\varrho_0(D))) \leq C h^j A^j I(\varrho_0(D)).
\]
From the construction of \( G \) we have \( \varrho(\bar{f}^j \circ G, \bar{f}^j \circ \varrho_0) \leq r \). Therefore
\[
I(\bar{f}^j \circ G, 1) \leq C h^j A^j I(\varrho_0(D)).
\]
There exists a constant \( C_4 > 0 \) such that
\[
\| \bar{f}^j \| \leq C_4(\lambda|J| - \theta)^{-m}\| \bar{f} \| \quad \text{for every } e \in V, m \geq 0;
\]
so
\[
I(G(D), C_4(1 + 2r)(\lambda|J| - \theta)^{-m}) \leq C h^j A^j I(\varrho_0(D)).
\]
Thus, by (1) from the Introduction,
\[
C_{\text{con}} \leq \lim_{j \to \infty} \frac{\log I(G(D), C_4(1 + 2r)(\lambda|J| - \theta)^{-m})}{\log(C_4(1 + 2r)(\lambda|J| - \theta)^{-m})} \\
= \lim_{j \to \infty} \frac{\log I(\varrho_0(D)) + \log h + \log \left( \sum_{i=1}^t \dim V_i \log (\lambda|J| + \theta) \right)}{\log(C_4(1 + 2r) + \sum_{i=1}^t \log(\lambda|J| + \theta))} \\
= \frac{\log h}{\log(\lambda|J| - \theta)} + \frac{\log \left( \sum_{i=1}^t \dim V_i \log (\lambda|J| + \theta) \right)}{\log(\lambda|J| - \theta)}.
\]
Since \( \theta > 0 \) is arbitrarily small, for \( q \) large enough
\[
C_{\text{con}} \leq \varepsilon + k \cdot \varepsilon \frac{\log \left( \sum_{i=1}^t \dim V_i \log (\lambda|J| + \theta) \right)}{\log(\lambda|J| - \theta)}
\]
and consequently
\[
C_{\text{con}} \leq \varepsilon + k \cdot \varepsilon \frac{\log \left( \sum_{i=1}^t \dim V_i \log (\lambda|J| + \theta) \right)}{\log(\lambda|J| - \theta)}
\]
If \( \dim E' > 1 \) to keep \( \text{dim}_{\text{ns}} (\bar{\omega} \cup \bigcup_{j=0}^\infty f^j(\varrho(D))) \) finite, one can repeat the corresponding construction from \([\text{[13]}\).

Namely, having \( \bar{f}^{\text{ng}(k)} \circ \varrho_0 \), one can change it to \( \bar{h}^{\text{ng}(k)}(D) \) that is disjoint from a family of dispersed balls in \( E' \), with a fixed (may be, large) radius. Next one can change it to \( k' \) and \( \lambda' \) as before. Large balls need large \( q \), in particular to have convergence in formula (2) from the Introduction. Of course, \( Q \) (X) and the constants \( C_2 r \), \( C_3 r \) must be changed adequately.

Estimations for the Pesin capacity

1. Let \( M \) be a smooth Riemannian manifold, let \( v \) be a Riemannian measure on \( M \), let \( X \subseteq M \) be an arbitrary Borel subset. Denote by \( \mathcal{S} = (U_i)_{i \in I} \) an arbitrary family of open subsets in \( M \) satisfying the following condition:

\( \mathcal{P} \) For every \( \varepsilon > 0 \) there exists a subfamily \( \{ V_i \}_{i \in I} \) of \( \mathcal{S} \) such that \( \bigcup_{i \in I} V_i \supseteq X \) and, for \( i \in I \), \( \text{diam}(V_i) < \varepsilon \).
Analogously to the definition of the dimension of the set \( X \) relative to the manifold \( M \) and family \( \mathcal{V} \), we can define the capacity of a compact set \( X \) relative to the manifold \( M \) and family \( \mathcal{V} \) as follows:

\[
(M, \mathcal{V})-\operatorname{Cap}_X = \dim M \lim_{\varepsilon \to 0} \sup_{\varepsilon > 0} -\log v(\mathcal{V}, \varepsilon)
\]

where \( I(X, \mathcal{V}, \varepsilon) \) denotes the smallest number of sets in \( \mathcal{V} \) with diameter \( \leq \varepsilon \) covering the set \( X \) and \( v(\mathcal{V}, \varepsilon) = \sup \{v(U) : U \in \mathcal{V}, \text{diam}(U) \leq \varepsilon \} \).

2. Let \( N \) be a smooth Riemannian manifold, \( f : N \to N \) a diffeomorphism, \( M \subset N \) a smooth immersed submanifold, and \( X \subset M \) any Borel subset. Fix \( \delta > 0 \) and consider the family

\[
\mathcal{V}_{f, \delta} = \{ U \subset M : \exists x \in U \exists \varepsilon > 0, x \notin f^{-\varepsilon}(B_{\rho_M}(f^*(x), \delta)) \}.
\]

Suppose that for sufficiently small \( \delta > 0 \) and for the family \( \mathcal{V}_{f, \delta} \) condition (P) holds. We call the numbers

\[
C_{X,f} = \limsup_{\varepsilon \to 0} ((M, \mathcal{V}_{f, \delta})-\operatorname{Cap}_X),
\]

\[
C_{X,f} = \liminf_{\varepsilon \to 0} ((M, \mathcal{V}_{f, \delta})-\operatorname{Cap}_X)
\]

the upper and the lower capacity relative to the mapping \( f \), respectively. If \( C_{X,f} = C_{X,f} \), then we shall call this number the capacity relative to the mapping \( f \) and denote it by \( C_{X,f} \).

3. Now we prove the following

**Theorem 3.** Let \( f : T^* \to T^* \) be a hyperbolic toral automorphism. Let \( K \subset T^* \) be a set such that its lift \( \hat{K} \subset \mathbb{R}^n \) contains a \( k \)-dimensional \( (1 \leq k \leq n) \) continuum \( \hat{K} = c + \bigoplus_{i=1}^l V_i \), where \( V_i \subset \mathbb{R}^n (1 \leq i \leq l) \) are \( \hat{f} \)-invariant linear subspaces in \( \mathbb{R}^n \), \( \lambda_i \) are the eigenvalues for \( \hat{f} \) and \( 1 < |\lambda_i| < \ldots < |\lambda_l| \). If \( \sum_{i=1}^l \dim V_i < k \), then \( \sum \dim V_i = k + \kappa \) and the projection of \( \hat{K} \) onto \( \bigoplus_{i=1}^l V_i \) along \( V_i \) has a non-empty interior, then, taking in the definition of thePesin capacity \( N = T^* \), \( M = \pi(c + \bigoplus_{i=1}^l V_i) \), the capacity \( C_{K,f} \) exists and

\[
C_{K,f} \geq (k + k) \sum_{i=1}^l \dim V_i \log |\lambda_i|.
\]

**Proof.** It is clear that in this case the family \( \mathcal{V}_{f, \delta} \) satisfies condition (P) for every \( \delta > 0 \) and our theorem is equivalent to the same theorem if we replace \( T^* \) by \( \mathbb{R}^* \), \( \pi(c + \bigoplus_{i=1}^l V_i) \) by \( c + \bigoplus_{i=1}^L V_i \), \( K \) by \( \hat{K} \), \( f \) and \( \hat{f} \).

Observe that here the map \( \delta \to (c + \bigoplus_{i=1}^l V_i, \mathcal{V}_{f, \delta})-\operatorname{Cap}_K \) is constant; hence, \( C_{K,f} \) exists and, of course, it is equal to

\[
(k + k) \limsup_{\varepsilon \to 0} \frac{\log I(X, \mathcal{V}_{f, \delta}, \mu, \varepsilon)}{-\log \varepsilon}
\]

for every \( \delta > 0 \),

where \( \mu \) is the Lebesgue measure on \( c + \bigoplus_{i=1}^l V_i \) and \( I(X, \mathcal{V}_{f, \delta}, \mu, \varepsilon) \) is the smallest number of sets in \( \mathcal{V} \) with measure \( \mu \) less than or equal to \( \varepsilon \), covering the set \( X \).

Now, fix \( \delta, \theta > 0 \). Let \( U_j = f^{-1}(B_{\rho_M}(f(x), \delta)) \in \mathcal{U}_{f, \delta} \), \( x \in M = c + \bigoplus_{i=1}^l V_i \). Since \( \mu(U_j) = (\det(\hat{f} \bigoplus V_i))^{-1} \Gamma^\delta \theta^{-k} \), where \( \Gamma \) is the Lebesgue measure of a unit ball in \( \bigoplus_{i=1}^l V_i \), it follows that if \( \mu(U_j) \leq \varepsilon \) then

\[
\frac{-\log(\varepsilon(T^* \delta \theta^{-k}))}{\log \det(\hat{f} \bigoplus V_i)}
\]

and this implies that

\[
I(\hat{K}, \mathcal{V}_{f, \delta}, \mu, \varepsilon) \geq I(\hat{f} \bigoplus V_i, \mu, \varepsilon) \geq C_{K,f} \sum \dim V_i \log |\lambda_i| (\hat{K}, \delta).
\]

From the assumptions of our theorem there exists a constant \( C_1 > 0 \) such that for \( j > 0 \)

\[
I(\hat{f} \bigoplus V_i, \delta) \geq C_1 (|\lambda_i| - \theta)^{\delta k_{V_i}} = \sum_{i=1}^l (|\lambda_i| - \theta)^{\delta k_{V_i}}.
\]

By (1) this implies

\[
C_{K,f} \geq (k + k) \sum \dim V_i \log |\lambda_i|.
\]

Since \( \theta \) is arbitrarily small, we obtain the required estimation.

4. We also prove the following
Theorem 4. The continuum $K = \pi(G(D))$ from Theorem 2 satisfies the following inequality:

$$C_{K,f} \leq \varepsilon + (k+k') \frac{\dim V_1 \log |\lambda_1| + \sum_1 \delta \dim V_i \log |\lambda_i|}{\sum_1 \dim V_i \log |\lambda_i|}.$$  

Proof. Let $V_i = f^{-q+1}\{y \in M_i | (f^n(y),1+2r)\in W_{r_i+2r}, \ (x \in M = f^{i+1}V_i), \}$ where $q$ and $r$ denote the numbers from the proof of Theorem 2.

Obviously, for

$$I(\zeta) = \left[ -\frac{\log (\zeta/\Gamma(1+2r \chi^A_\xi))}{\log \det (f^q \otimes V)} \right] + 1, \quad \mu(V_{10}) \leq \zeta.$$  

Therefore, by (4) from the proof of Theorem 2,

$$I(G(D), W_{r_i+2r}, \mu, \zeta) \leq I(f^{q+1} \circ G(D), 1+2r) \leq C_{G(D)} I(\theta_0(D))$$

and

$$C_{G(D,f)} \leq (k+k') \frac{\log h}{\log \det (f^q \otimes V)} \frac{\log A}{\log \det (f^q \otimes V)}.$$  

Thus for $q$ large enough we obtain our estimation because

$$\log A = (\dim V_1 - k) \log |\lambda_1| + \sum_1 \delta \dim V_i \log |\lambda_i|$$

and

$$\log \det (f^q \otimes V) = \sum_1 \dim V_i \log |\lambda_i|.$$  

Remark. In view of Theorem 4 we see that the estimation from Theorem 3 is the best possible. This is done for every $k$. Recall that for classical capacity we obtained the best possible estimations only for the curve $(n = 1)$.

One might think that the Pesin capacity is more adequate for such estimations.

However, the estimation in Theorem 3 is valid for every $k$-dimensional continuum $K$ including continua with dense orbits. So the Pesin capacity is less sensitive than the classical one. It does not detect the "fractal" shape of curves with non-dense orbits.

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References


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