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On the capacity of a continuum with a non-dense orbit under a hyperbolic toral automorphism

by

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Abstract. In this paper we compute an upper and lower estimation for the capacity of a continuum (connected compact set) lying in the torus $T^n = R^n/Z^n$ whose orbit under a hyperbolic toral automorphism is not dense in T^n. Also estimations of capacity in Pesin's sense are considered.

Introduction. The main results.

1. First we define capacity. Let (X, ϱ) be a compact metric space and let A be any subset in X. Cover it with finitely many balls $\{B(x_i, r_i)\}_{i=1}^k$ with centres in A of radii $r_i \leqslant \varepsilon$. By $I(A, \varepsilon)$ denote the minimal possible k. The number

$$C_A = \limsup_{\varepsilon \to 0} \frac{\log I(A, \varepsilon)}{-\log \varepsilon}$$

is called the *capacity* of the set A. Observe that $\dim_H A \leqslant C_A$, where $\dim_H$ is the Hausdorff dimension and that

(1) *if* $\varepsilon_i \searrow 0$, $\limsup_{i \to \infty} (\varepsilon_i/\varepsilon_{i+1}) < +\infty$, *then*

$$C_A = \limsup_{i \to \infty} \frac{\log I(A, \varepsilon_i)}{-\log \varepsilon_i}.$$

2. Denote by $\pi\colon R^n \to R^n/Z^n$ the standard covering projection. A hyperbolic toral automorphism is a map $f\colon T^n \to T^n$ which has a linear lift $\tilde{f}\colon R^n \to R^n$ without eigenvalues of modulus 1. It is clear that there exists a minimal number $r \geqslant 1$ such that either the eigenvalues of $\tilde{f}^r$ are real and positive or they are not roots of real numbers. By $\tilde{f}$ we denote $\tilde{f}^r$. We define

$$E_\lambda = \bigcup_{j=0}^{\infty} (\tilde{f} - \lambda \mathrm{id})^{-j}(0)$$

if an eigenvalue λ of $\tilde{f}$ is real and

$$E_\lambda = \Big(\bigcup_{j=0}^{\infty} (\hat{f} - \lambda \mathrm{id})^{-j}(0) \cup \bigcup_{j=0}^{\infty} (\hat{f} - \bar{\lambda} \mathrm{id})^{-j}(0)\Big) \cap R^n,$$

where $\hat{f}\colon \mathbf{C}^n \to \mathbf{C}^n$ is the complexification of $\tilde{f}$, if λ is complex. The linear subspace $E^s = \bigoplus \{E_\lambda; |\lambda| < 1\}$ is called the *contracting eigenspace* for $\tilde{f}$ and $E^u = \bigoplus \{E_\lambda; |\lambda| > 1\}$ is called the *expanding eigenspace* for $\tilde{f}$.

3. Now, let K be an arbitrary continuum lying in $\mathbf{T}^n$ and let $\tilde{K}$ be an arbitrary subset of its lift. Consider a coset $c + \bigoplus_{\lambda\in\Lambda} V_\lambda(\tilde{K})$, where Λ consists of eigenvalues of $\tilde{f}$, $V_\lambda(\tilde{K}) \subset E_\lambda$ is a linear subspace invariant under $\tilde{f}$, such that $\tilde{K} \subset c + \bigoplus_{\lambda\in\Lambda} V_\lambda(\tilde{K})$ and $\dim\left(\bigoplus_{\lambda\in\Lambda} V_\lambda(\tilde{K})\right)$ is the least possible. These properties define the subspace $\bigoplus_{\lambda\in\Lambda} V_\lambda(\tilde{K})$ uniquely.

4. Now, we recall that R. Mañé proved in [5] the following

THEOREM. *Let $\alpha\colon (a, b) \to \mathbf{T}^n$ be a rectifiable nonconstant path and let $f\colon \mathbf{T}^n \to \mathbf{T}^n$ be a hyperbolic toral automorphism. Then the closure of the orbit of $\alpha((a, b))$ under f contains a coset of a toral subgroup invariant under some power of f.*

([5] improves Frank's result, where the paths were C^2, see [2].)

We prove a related result:

THEOREM 1. *Let $f\colon \mathbf{T}^n \to \mathbf{T}^n$ be a hyperbolic toral automorphism. Let $K \subset \mathbf{T}^n$ be a set such that its lift $\tilde{\tilde{K}} \subset \mathbf{R}^n$ contains a non-one-point continuum $\tilde{K} \subset c + \bigoplus_{i=p}^{q} V_{\lambda_i}(\tilde{K})$, $(|\lambda_p| \leqslant \ldots \leqslant |\lambda_q|)$. Denote $V_{\lambda_i}(\tilde{K}) = V_i$. If one of the following cases holds:*

(a) $\dim V_q = 1, \quad |\lambda_q| > 1, \quad |\lambda_{q-1}| < 1,$

(a′) $\dim V_p = 1, \quad |\lambda_p| < 1, \quad |\lambda_{p+1}| > 1,$

(b) $\dim V_q = 1, \quad |\lambda_q|, |\lambda_{q-1}| > 1 \quad$ *and* $\quad C_K < 2 - \dfrac{\log|\lambda_{q-1}|}{\log|\lambda_q|},$

(b′) $\dim V_p = 1, \quad |\lambda_p|, |\lambda_{p+1}| < 1 \quad$ *and* $\quad C_K < 2 - \dfrac{\log|\lambda_{p+1}|}{\log|\lambda_p|},$

then the closure of the orbit of K under f contains a coset of a toral subgroup invariant under some power of f.

Theorem 1 concerns in fact compact sets which are not zero-dimensional (i.e., countable or unions of a Cantor set and a countable set). The problem is that every such set must contain a non-one-point subcontinuum, see [1].

5. Przytycki [7] has constructed for any Anosov diffeomorphism f of an n-dimensional torus $\mathbf{T}^n$ invariant subsets of arbitrary dimension between 1 and $n-2$. Paper [7] develops the Hancock idea, see [3], [4]. We shall use this construction. We recall it in the case where the diffeomorphism f is algebraic and $\dim E^s = 1$. Here is an outline: One may assume that the orthogonal projection $P\colon E^u \to \mathbf{R}^{n-1} = \{x \in \mathbf{R}^n;\ x_n = 0\}$ is an isomorphism. Fix k $(1 \leqslant k \leqslant n-2)$. One may consider $\mathbf{R}^{n-1}$ as the union of $(n-1)$-dimensional cubes

$$\{x = (x_1, \ldots, x_{n-1})\colon m_1 \leqslant x_1 \leqslant m_1 + 1, \ldots, m_{n-1} \leqslant x_{n-1} \leqslant m_{n-1} + 1,\ m_i \in \mathbf{Z}\}$$

with edges of length 1. Denote by $\mathscr{K}$ the union of $(n-k-2)$-dimensional skeletons of these cubes. Let D be a k-dimensional disc embedded by g into E^u. There exists a continuous mapping $g_0\colon D \to E^u$ such that g_0 is $C_0 r$ close to g and $g_0(D)$ is disjoint from $P^{-1}(B(\mathscr{K}, r))$, where $B(\mathscr{K}, r) = \{x \in \mathbf{R}^{n-1}\colon \varrho(\mathscr{K}, x) < r\}$ and C_0 is a constant coefficient. Let $\lambda > 1$, $\alpha > 0$ satisfy the condition

(1) $$\alpha \|\tilde{f}^n v\| \geqslant \lambda^n \|v\|, \qquad v \in E^u.$$

There exists a positive integer q which satisfies the inequality

(2) $$1 - \alpha C_0 \sum_{i=1}^{\infty} (1/\lambda^q)^i > 0.$$

Assume that a continuous mapping $g_i\colon D \to E^u$ such that

$$\tilde{f}^{qi} \circ g_i(D) \cap P^{-1}(B(\mathscr{K}, r)) = \emptyset$$

is defined. There exists a continuous mapping $h\colon D \to E^u$ such that $h(D) \cap P^{-1}(B(\mathscr{K}, r)) = \emptyset$ and h is $C_0 r$ close to $\tilde{f}^{q(i+1)} \circ g_i$. We define $g_{i+1} = \tilde{f}^{-q(i+1)} \circ h$. By (1) there exists a continuous mapping $G = \lim_{i\to\infty} g_i$. $G(D)$ is a continuum. In [7] Przytycki proved that

$$k = \dim(\pi(G(D))) = \dim(G(D)) = \dim\Big(\mathrm{cl}\Big(\bigcup_{j=-\infty}^{\infty} f^j \circ \pi \circ G(D)\Big)\Big).$$

In this paper we prove the following

THEOREM 2. *Fix $1 \leqslant k \leqslant n-2$. Let $f\colon \mathbf{T}^n \to \mathbf{T}^n$ be a hyperbolic toral automorphism. Let $\lambda_1, \ldots, \lambda_l$ be arbitrary eigenvalues for $\tilde{f}$ such that $1 < |\lambda_1| \leqslant \ldots \leqslant |\lambda_l|$ and let $V_1 \subset E_{\lambda_1}, \ldots, V_l \subset E_{\lambda_l}$ be arbitrary $\tilde{f}$-invariant, linear subspaces in $\mathbf{R}^n$, $\sum_{j=2}^{l} \dim V_j \leqslant k$, $\sum_{j=1}^{l} \dim V_j = k + k' > k$. Then for every $\varepsilon > 0$ there exists a k-dimensional continuum $K \subset \mathbf{T}^n$ such that $\dim\Big(\mathrm{cl}\Big(\bigcup_{j=-\infty}^{\infty} f^j(K)\Big)\Big) = k$ and*

(i) $$C_K \leqslant \varepsilon + (\log|\lambda_1|)^{-1}\Big((\dim V_1 - k')\log|\lambda_1| + \sum_{j=2}^{l} \dim V_j \log|\lambda_j|\Big),$$

(ii) $$C_K \leqslant \varepsilon + k + k'\Big(1 - \frac{\log|\lambda_1|}{\log|\lambda_l|}\Big).$$

Remark. In constructing K one can consider, instead of the whole V_1, an invariant subspace $V_1' \subset V_1$ such that $\dim(V_1' \oplus \bigoplus_{2}^{l} V_j) = k+k'' > k$, where $k'' = 1$ if λ_1 is real and $k'' = 1$ or 2 if λ_1 is complex (not real). In the case $k'' = 2$ one can consider only $\dim V_l' \geq 2$. (Otherwise, if $\dim V_l = 1$, one could consider $\bigoplus_{1}^{l-1} V_j$ only, having $\dim(\bigoplus_{1}^{l-1} V_j) = k+1$.) These changes of V_1 (V_l) clearly improve the estimations (i), (ii) for an appropriate K. It is immediately computable that, after these improvements of the spaces V (for $k' = 1$ or $k' = 2$ and $\dim V_l \geq 2$), estimation (ii) is better than (i)!

For example, in the case of T^3, $\lambda_1 < 1 < \lambda_2 < \lambda_3$, for every curve γ with non-dense orbit, its capacity, by Theorem 1, satisfies

$$C_\gamma \geq 2 - \frac{\log \lambda_2}{\log \lambda_3}.$$

This estimation is the best possible because, by Theorem 2, there exists a curve γ_0 with non-dense orbit where $C_{\gamma_0} \leq \varepsilon + 2 - \dfrac{\log \lambda_2}{\log \lambda_3}$. Can this ε be removed?

If $\lambda_2 = \lambda_3$ then the method used in the proof of Theorem 1 gives nothing interesting. In view of Theorem 2 we really cannot have any estimation of type $C_K \geq 1+\text{const} > 1$. So we cannot prove anything more than non-rectifiability.

6. We shall use the following simple

Geometrical Lemma. *For all integer numbers $n \geq 1$, $1 \leq p \leq n$, and real numbers $\gamma, \eta > 0$, there exists a constant $C > 0$ such that, if W is a p-dimensional parallelepiped lying in R^n with the edges of lengths $a_1, \ldots, a_p \geq \eta$, then $I(W, \gamma) \leq C a_1 \ldots a_p$.*

Proofs.

1. Proof of Theorem 1. Consider first the case where $|\lambda_{q-1}| > 1$. Fix $0 < \theta < (\lambda_q - |\lambda_{q-1}|)/2$. Suppose to the contrary that the closure of the orbit of K under f does not contain any coset of a toral subgroup invariant under f^m for every $m \geq 1$. The definition of the linear space $\bigoplus_{p}^{q} V_i$ immediately implies the existence of at least one pair (a, b) of points lying in $\tilde{K}$ such that

$$(0) \qquad b - a = v_1 + v, \quad v_1 \in \bigoplus_{p}^{q-1} E_{\lambda_i}, \quad 0 \neq v \in V_q.$$

The group $\mathrm{cl}(\pi(V_q))$ is of course a toral subgroup invariant under f^r. Therefore, according to the contrary assumption, for every $x \in R^n$ there exists at least one point $\bar{x} \in V_q$ and $\varepsilon(x) > 0$ such that

$$f^j(K) \cap B(\pi(x)+\pi(\bar{x}), \varepsilon(x)) = \emptyset \quad \text{for every } j \geq 0.$$

Choose points $x_1, \ldots, x_m \in R^n$ such that $\bigcup_{j=1}^{m} B(\pi(x_j), \frac{1}{2}\varepsilon(x_j)) = T^n$. So, for every $x \in R^n$, there exists a point $x_j \in R^n$ $(1 \leq j \leq m)$ such that $\varrho(\pi(x), \pi(x_j)) \leq \frac{1}{2}\varepsilon(x_j)$. Thus, for $\varepsilon = \frac{1}{4}\min(\varepsilon(x_1), \ldots, \varepsilon(x_m))$,

$$(1) \qquad f^j(K) \cap B(\pi(x)+\pi(\bar{x}), 2\varepsilon) = \emptyset \quad \text{for every } j \geq 0,\ x \in R^n,$$

(after a new, improved choice of $\bar{x}$).

Denote by d_j the infimum of the lengths of rectifiable curves joining the points a and b in $c + \bigoplus_{p}^{q} V_i$, whose jth image under $\tilde{f}$ is disjoint from $B(x+\bar{x}, \varepsilon) + Z^n$ for all $x \in R^n$. It is clear that there exists a constant $C_1 > 0$ such that

$$(2) \qquad \|\tilde{f}^j u\| \leq C_1 (\lambda_q + \theta)^j \|u\| \quad \text{for every } u \in \bigoplus_{p}^{q} V_i \text{ and } j \geq 0.$$

Let $\delta_j = \varepsilon / C_1 (\lambda_q + \theta)^j$ $(j = 0, 1, \ldots)$. By (1) and (2)

$$(3) \quad \tilde{f}^j\big(B(\tilde{K}, \delta_j) \cap (c + \bigoplus_{p}^{q} V_i)\big) \cap \big(B(x+\bar{x}, \varepsilon) + Z^n\big) = \emptyset \quad \text{for} \quad j \geq 0,\ x \in R^n.$$

Now, we consider an arbitrary δ_j-net in $\tilde{K}$. Since $\tilde{K}$ is connected, there exists a broken line with the vertices chosen from our δ_j-net and the distances between the successive vertices not greater than $2\delta_j$. Obviously, this broken line lies in a coset $c + \bigoplus_{p}^{q} V_i$ and due to (3) its image under $\tilde{f}^j$ is disjoint from $B(x+\bar{x}, \varepsilon) + Z^n$ for all $x \in R^n$. So

$$d_j \leq 2\delta_j I(\tilde{K}, \delta_j).$$

This implies

$$I(\tilde{K}, \delta_j) \geq \frac{d_j}{2\delta_j} = \frac{d_j C_1 (\lambda_q + \theta)^j}{2\varepsilon}$$

and

$$\frac{\log I(\tilde{K}, \delta_j)}{-\log \delta_j} \geq \frac{\log(C_1(\lambda_q+\theta)^j/2\varepsilon) + \log d_j}{\log(C_1(\lambda_q+\theta)^j/\varepsilon)}$$

but, as

$$\lim_{j \to \infty} \frac{\log(C_1(\lambda_q+\theta)^j/2\varepsilon)}{\log(C_1(\lambda_q+\theta)^j/\varepsilon)} = 1,$$

we obtain

$$C_{\tilde K} \geqslant 1+\limsup_{j\to\infty}\frac{\log d_j}{j\log(\lambda_q+\theta)}. \tag{4}$$

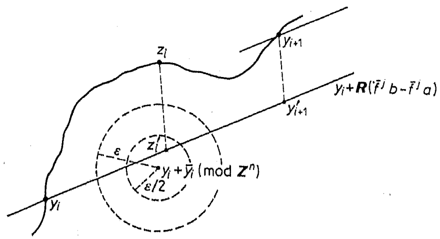
Since the torus T^n is compact, there exists a constant $C_2>0$ such that every segment of the straight line $x+V_q$ with length $\geqslant \frac{1}{2}C_2$ intersects the set $B(x+v,\ \varepsilon/3)+Z^n$ for all $x\in R^n$, $v\in V_q$.

Since the directions of the vectors $\bar f^j b-\bar f^j a$ tend to the direction of the straight line V_q due to $|\lambda_{q-1}|<\lambda_q$ and due to (0), for an integer j large enough every segment of the straight line $x+R(\bar f^j b-\bar f^j a)$ with length C_2 intersects the set $B(x+v,\ \varepsilon/2)+Z^n$ for all $x\in R^n$, $v\in V_q$.

For every $j\geqslant 0$ denote by l_j an arbitrary rectifiable curve in $c+\bigoplus_{p}^{q} V_i$ joining a and b such that

$$\bar f^j(l_j)\cap\big(B(x+\bar x,\ \varepsilon)+Z^n\big)=\emptyset \quad \text{for all } x\in R^n. \tag{5}$$

Denote by P_j the projection of $\bigoplus_{p}^{q} V_i$ onto the line $R(\bar f^j b-\bar f^j a)$, along $\bigoplus_{p}^{q-1} V_i$. In the curve $\bar f^j(l_j)$ choose points $\bar f^j a=y_1,\ldots,y_{t(j)},\ y_{t(j)+1},\ y_{t(j)+2}=\bar f^j b$ such that $\|P_j(y_{i+1}-y_i)\|=C_2$ for $i=1,\ldots,t(j)$ and $\|P_j(y_{t(j)+2}-y_{t(j)+1})\|\leqslant C_2$. Let $y'_{i+1}=y_i+P_j(y_{i+1}-y_i)\in y_i+R(\bar f^j b-\bar f^j a)$ $(1\leqslant i\leqslant t(j))$. Since the interval $\overline{y_i y'_{i+1}}$ has the length C_2, it intersects $B(y_i+\bar y_i,\ \varepsilon/2)+Z^n$ for every $1\leqslant i\leqslant t(j)$ and j large enough. Thus by (5) there exists a point z_i in the segment of the curve $\bar f^j(l_j)$ between the points y_i and y_{i+1} and a point $z'_i\in y_i+R(\bar f^j b-\bar f^j a)$ such that $z_i-z'_i\in\bigoplus_{\alpha=p}^{q-1} V_\alpha$, $\|z_i-z'_i\|\geqslant\varepsilon/2$.



Since $\measuredangle(R(\bar f^j b-\bar f^j a),\ V_q)\to 0$ and λ_q is the eigenvalue for $\bar f$ corresponding to V_q, there exists a constant $C_3>0$ such that

$$\|\bar f^k u\|\geqslant C_3(\lambda_q-\theta)^k\|u\| \quad \text{for every } k\geqslant 0,\ u\in R(b-a).$$

Observe that $\sum_{1}^{t(j)+1}\|y'_{i+1}-y_i\|=\|\bar f^j b-\bar f^j a\|$; hence

$$C_2\big(t(j)+1\big)\geqslant C_3(\lambda_q-\theta)^j\|b-a\|.$$

So

$$t(j)\geqslant C_2^{-1}C_3(\lambda_q-\theta)^j\|b-a\|-1 \tag{6}$$

and

$$\|\bar f^{-j}z'_i-\bar f^{-j}y_i\|\leqslant C_2C_3^{-1}(\lambda_q-\theta)^{-j}.$$

There exists a constant $C_4>0$ such that, for every $u\in\bigoplus_{p}^{q-1}V_\alpha$, $j\geqslant 0$, $\|\bar f^{-j}u\|\leqslant C_4(|\lambda_{q-1}|+\theta)^j\|u\|$.

By the triangle inequality, for every $i=1,\ldots,t(j)$ the length of the part of l_j between y_i and z_i is not less than

$$\begin{aligned}\|\bar f^{-j}y_i-\bar f^{-j}z_i\| &\geqslant \|\bar f^{-j}z_i-\bar f^{-j}z'_i\|-\|\bar f^{-j}z'_i-\bar f^{-j}y_i\|\\ &\geqslant C_4^{-1}(|\lambda_{q-1}|+\theta)^{-j}\varepsilon/2-C_2C_3^{-1}(\lambda_q-\theta)^{-j}.\end{aligned}$$

Taking the sum over i and the infimum over the rectifiable curves, we obtain by (6)

$$\begin{aligned}d_j &\geqslant \big(C_2^{-1}C_3(\lambda_q-\theta)^j\|b-a\|-1\big)\big(C_4^{-1}(|\lambda_{q-1}|+\theta)^{-j}\varepsilon/2-C_2C_3^{-1}(\lambda_q-\theta)^{-j}\big)\\ &=\frac{\varepsilon}{2}C_2^{-1}C_3C_4^{-1}\big((\lambda_q-\theta)/(|\lambda_{q-1}|+\theta)\big)^j-\|b-a\|+\\ &\quad +C_2C_3^{-1}(\lambda_q-\theta)^{-j}-\frac{\varepsilon}{2}C_4^{-1}(|\lambda_{q-1}|+\theta)^{-j}.\end{aligned}$$

As $(\lambda_q-\theta)/(|\lambda_{q-1}|+\theta)>1$, $|\lambda_{q-1}|$, $\lambda_q-\theta>1$, there exists a constant $C_5>0$ independent of j such that for j large enough

$$d_j\geqslant C_5\big((\lambda_q-\theta)/(|\lambda_{q-1}|+\theta)\big)^j.$$

Together with (4) this implies that

$$\begin{aligned}C_{\tilde K} &\geqslant 1+\limsup_{j\to\infty}\frac{j\big(\log(\lambda_q-\theta)-\log(|\lambda_{q-1}|+\theta)\big)+\log C_5}{j\log(\lambda_q+\theta)}\\ &=1+\frac{\log(\lambda_q-\theta)-\log(|\lambda_{q-1}|+\theta)}{\log(\lambda_q+\theta)}.\end{aligned}$$

Since θ can be an arbitrarily small positive number, we have

$$C_{\tilde K}\geqslant 2-\frac{\log|\lambda_{q-1}|}{\log\lambda_q} \quad\text{hence}\quad C_K\geqslant 2-\frac{\log|\lambda_{q-1}|}{\log\lambda_q}.$$

This contradicts the assumptions of our theorem.

The case $|\lambda_{q-1}|<1$ is much simpler. Let P_q denote the projection of $\bigoplus_p^q V_i$ onto V_q along $\bigoplus_p^{q-1} V_i$. The Hausdorff distances between the sets $\varrho_H(\tilde{f}^j(\tilde{K}), P_q(\tilde{f}^j(\tilde{K})))$ tend exponentially to 0, while the lengths of the curves $P_q(\tilde{f}^j(\tilde{K}))$ tend to ∞. Hence, in T^n, $\mathrm{cl}(\pi V_q)\subset \mathrm{cl}(\bigcup_{j=0}^{\infty} f^j(\tilde{K}))$ (we use again the fact stated directly after formula (4)).

Proof of Geometrical Lemma. We can cover each edge of length a_i $(1\leqslant i\leqslant p)$ by $[a_i/\frac{1}{2}\gamma]+1$ segments of length $\leqslant \frac{1}{2}\gamma$. Thus we can cover our parallelepiped with

$$\prod_{i=1}^{p}([2a_i/\gamma]+1)\leqslant \prod_{i=1}^{p}\left(\frac{2a_i}{\gamma}+\frac{a_i}{\eta}\right)=\left(\frac{2}{\gamma}+\frac{1}{\eta}\right)^p\prod_{i=1}^{p}a_i$$

parallelepipeds with the edges of length $\leqslant\gamma/2$, because $a_i\leqslant\eta$ $(1\leqslant i\leqslant p)$. Since each parallelepiped with the edges of length $\leqslant\gamma/2$ lies in a ball of radius $\leqslant\gamma$, the Lemma is true if we set $C=(2/\gamma+1/\eta)^p$.

Proof of Theorem 2. Assume $\dim E^s=1$. We mention at the end of the proof how to get rid of that with the use of an idea from [7]. Fix $\varepsilon,\theta>0$. Wishing to construct a set satisfying the properties from Theorem 2, we must state precisely the definition of the mappings $g_i\colon D\to E^u$, $i=1,2,\ldots$ from [5, Introduction]. Let $V=\bigoplus_{i=1}^{l} V_i$. We shall regard as a "good" parallelepiped each k-dimensional parallelepiped lying in V each edge of which is parallel to some vector from $\bigcup_{i=1}^{l} V_i$ and is of length not less than $r/4$.

Let $\mathscr{V}=(v_1,\ldots,v_{k+k'})$ be a basis in V such that $v_i\in\bigcup_{\alpha=1}^{l}V_\alpha$, $i=1,\ldots,k+k'$. Let τ be a "triangulation" of V by parallelepipeds with edges of length exactly $r/4$, parallel to vectors from $\mathscr{V}$.

Denote by $k(W)$ the k-dimensional skeleton of a parallelepiped W. Having $g_i\colon D\to V$ defined, first perturb $\tilde{f}^{q(i+1)}\circ g_i$ to h by projecting $\tilde{f}^{q(i+1)}\circ g_i(D)\cap P^{-1}(B(\mathscr{K},r))$ onto $P^{-1}(\mathrm{Fr}\,B(\mathscr{K},r))$ as in [7]. Next approximate h by h' so that

$$h'(D)\subset\bigcup\{k(W)\colon W\in\tau,\ W\cap h(D)\neq\emptyset\}.$$

(Do this by consecutive projections onto the skeletons $\bigcup l(W)$, $l=k+k'-1,\ldots,k$). Observe that there exists a constant $C_1>0$ such that

$$\varrho(h',\tilde{f}^{q(i+1)}\circ g_i)\leqslant C_1 r\quad\text{and}\quad h(D)\cap P^{-1}(B(\mathscr{K},r/2))=\emptyset.$$

Now, if $Q\subset V$ is a "good" polyhedron, i.e., if it is a union of a finite number of "good" parallelepipeds, then, setting in the Geometrical Lemma

$\eta=r/4$, $\gamma=1/2$, we get

$$I(Q,1/2)\leqslant\sum_{j=1}^{m}I(W_j,1/2)\leqslant C\sum_{j=1}^{m}a_1^{(j)}\ldots a_k^{(j)},$$

where $Q=\bigcup_{j=1}^{m}W_j$ and each W_j $(1\leqslant j\leqslant m)$ is a "good" parallelepiped with edges of length $a_1^{(j)},\ldots,a_k^{(j)}$.

We denote by $I(Q)$ the infimum over all admissible right-hand sums. Therefore

$$(1)\qquad I(Q,1/2)\leqslant CI(Q).$$

If q is large enough, then for every $u\in E^u$, $\|\tilde{f}^q u\|\geqslant\|u\|$; so, if W is a "good" parallelepiped, then $\tilde{f}^q(W)$ is also a "good" parallelepiped with edges whose product of lengths is not greater than

$$C_2\big((|\lambda_1|+\theta)^{\dim V_1-k'}\prod_{i=2}^{l}(|\lambda_i|+\theta)^{\dim V_i}\big)^q\prod_{i=1}^{k}a_i=C_2\Lambda^q\prod_{i=1}^{k}a_i,$$

where $C_2>0$ is a constant independent of j, and $\{a_i\}_{i=1}^k$ are the lengths of the edges of W. Thus, if Q is a "good" polyhedron, then

$$(2)\qquad I(\tilde{f}^q(Q))\leqslant I(Q)\,C_2\Lambda^q.$$

Let $Q(X)=\bigcup\{k(S)\colon S\in\tau,\ S\subset B(\tilde{f}^q(X),(C_1+\frac{1}{2})r)\}$, where $X\subset V$. Observe here that $Q(X)$ is a "good" polyhedron.

Now, if W is a "good" parallelepiped, then, since $\tilde{f}^q(W)$ also is a "good" parallelepiped, the number of parallelepipeds $S\in\tau$, $S\subset B(\tilde{f}^q(W),(C_1+\frac{1}{2})r)$ is not greater than $C_3\prod_{i=1}^{k}a_i$, where $\{a_i\}_{i=1}^k$ are the lengths of the edges of $\tilde{f}^q(W)$ and C_3 is a constant coefficient. Thus

$$I(Q(W))\leqslant(r/4)^k\,2\binom{k+k'}{k}C_3\prod_{i=1}^{k}a_i.$$

Consequently, if R is a "good" polyhedron, then

$$(3)\qquad I(Q(R))\leqslant 2C_3(r/4)^k\binom{k+k'}{k}I(\tilde{f}^q(R)).$$

Since, for every $j=0,1,\ldots$, $\tilde{f}^{q(j+1)}\circ g_{j+1}(D)\subset Q(\tilde{f}^{qj}\circ g_j(D))$, we have

$$\tilde{f}^{qj}\circ g_j(D)\subset Q^j(g_0(D)),\quad j=0,1,\ldots,$$

and by (2) and (3) for $h=2C_2C_3(r/4)^k\binom{k+k'}{k}$

$$I(Q^j(g_0(D)))\leqslant h^j\Lambda^{qj}I(g_0(D)).$$

By (1) this implies that for $j = 0, 1, \dots$

$$I(\tilde{f}^{qj} \circ g_j(D), 1) \leqslant I(Q^j(g_0(D)), 1/2) \leqslant CI(Q^j(g_0(D))) \leqslant Ch^j \Lambda^{qj} I(g_0(D)).$$

From the construction of G we have $\varrho(\tilde{f}^{qj} \circ G, \tilde{f}^{qj} \circ g_j) \leqslant r$. Therefore

(4) $$I(\tilde{f}^{qj} \circ G(D), 1+2r) \leqslant Ch^j \Lambda^{qj} I(g_0(D)).$$

There exists a constant $C_4 > 0$ such that

$$\|\tilde{f}^{-m} v\| \leqslant C_4 (|\lambda_1| - \theta)^{-m} \|v\| \quad \text{for every } v \in V, m \geqslant 0;$$

so

$$I(G(D), C_4(1+2r)(|\lambda_1|-\theta)^{-qj}) \leqslant Ch^j \Lambda^{qj} I(g_0(D)).$$

Thus, by (1) from the Introduction,

$$C_{G(D)} = \limsup_{j\to\infty} \frac{\log I(G(D), C_4(1+2r)(|\lambda_1|-\theta)^{-m})}{-\log(C_4(1+2r)(|\lambda_1|-\theta)^{-m})} \leqslant$$

$$\lim_{j\to\infty} \frac{\log I(g_0(D)) + j\log h + jq((\dim V_1 - k')\log(|\lambda_1|+\theta) + \sum_{i=2}^{l} \dim V_i \log(|\lambda_i|+\theta))}{-\log(C_4(1+2r)) + jq\log(|\lambda_1|-\theta)}$$

$$= \frac{1}{q}\frac{\log h}{(|\lambda_1|-\theta)} + \log^{-1}(|\lambda_1|-\theta)\Big((\dim V_1 - k')\log(|\lambda_1|+\theta) + \sum_{i=2}^{l} \dim V_i \log(|\lambda_i|+\theta)\Big).$$

Since this inequality holds for arbitrary $\theta > 0$ and since h is independent of q, for q large enough we obtain the estimation (i) from Theorem 2.

In proving (i), we estimated the number of "ellipsoids" $\tilde{f}^{-qi}(B_V(\cdot, 1+2r))$. Instead of that we could divide (cover) each such "ellipsoid" into cubes with edges of the length of the shortest axis of the "ellipsoid". We shall now prove estimation (ii) with the use of this idea..

There exists a number $\xi > 0$ such that every ball $B_V(x, 1+2r)$, $x \in V$, lies in a parallelepiped $\Pi(x)$ with edges of length exactly ξ, parallel to the vectors from $\mathscr{V}$.

Also there exist constants C_5, C_6 such that

$$\|\tilde{f}^{-j} w\| \leqslant C_5 (|\lambda_l| - \theta)^{-j} \|w\| \quad \text{for every } j \geqslant 0,\ w \in V_i,\ i = 1, \dots, l,$$

$$\|\tilde{f}^{-j} w\| \geqslant C_6 (|\lambda_l| + \theta)^{-j} \|w\| \quad \text{for every } j \geqslant 0,\ w \in V_l.$$

Thus we can cover each edge of the parallelepiped $\tilde{f}^{-qj}(\Pi(x))$ parallel to some vector from V_i by $[\xi C_5 (|\lambda_l|-\theta)^{-qj}/\xi C_6(|\lambda_l|+\theta)^{-qj}]+1$ segments of length $\leqslant C_5 \xi (|\lambda_l|-\theta)^{-qj}$; hence for j large enough we can cover our parallelepiped by

$$\prod_{i=1}^{l} \left[\frac{2C_5}{C_6}\left(\frac{|\lambda_l|+\theta}{|\lambda_l|-\theta}\right)^{qj}\right]^{\dim V_i}$$

parallelepipeds with edges of length $\leqslant \xi C_5 (|\lambda_l| - \theta)^{-qj}$. Therefore, by (4), for j large enough,

$$I(G(D), 2\xi C_5 (|\lambda_l|-\theta)^{-qj}) \leqslant C\left(\frac{2C_5}{C_6}\right)^{k+k'} I(g_0(D)) h^j \Lambda^{qj} \left(\prod_{i=1}^{l}\left(\frac{|\lambda_l|+\theta}{|\lambda_l|-\theta}\right)^{\dim V_i}\right)^{qj}.$$

By (1) from the Introduction it follows that

$$C_{G(D)} \leqslant \frac{1}{q}\frac{\log h}{\log(|\lambda_l|-\theta)} + \frac{(\dim V_1 - k')\log(|\lambda_1|+\theta) + \sum_{i=2}^{l} \dim V_i \log(|\lambda_i|+\theta) + (k+k')\log(|\lambda_l|+\theta)}{\log(|\lambda_l|-\theta)} - \frac{\sum_{i=1}^{l} \dim V_i \log(|\lambda_l|-\theta)}{\log(|\lambda_l|-\theta)}.$$

Since $\theta > 0$ is arbitrarily small, for q large enough

$$C_{G(D)} \leqslant \varepsilon + k + k'\left(1 - \frac{\log|\lambda_1|}{\log|\lambda_l|}\right)$$

and consequently

$$C_{\pi(G(D))} \leqslant \varepsilon + k + k'\left(1 - \frac{\log|\lambda_1|}{\log|\lambda_l|}\right).$$

If $\dim E^s > 1$ to keep still $\dim_{\text{top}}\Big(\text{cl}\big(\bigcup_{j=-\infty}^{\infty} f^j(\pi(G(D)))\big)\Big) = k$ (i.e., $\dim_{\text{top}}\Big(E^s \cap \text{cl}\big(\bigcup_{j=-\infty}^{\infty} f^j(\pi(G(D)))\big)\Big) = 0$), one can repeat the corresponding construction from [7].

Namely, having $\tilde{f}^{q(i+1)} \circ g_i$, one can change it to h'' such that $h''(D)$ is disjoint from a family of dispersed balls in E^u, with a fixed (may be, large) radius. Next one can change it to h and h' as before. Large balls need large q, in particular to have convergence as in formula (2) from the Introduction. Of course, $Q(X)$ and the constants $C_1 r$, C_3 must be changed adequately.

Estimations for the Pesin capacity

1. Let M be a smooth Riemannian manifold, let ν be a Riemannian measure on M, let $X \subset M$ be an arbitrary Borel subset. Denote by $\mathscr{U} = \{U_s\}_{s\in S}$ an arbitrary family of open subsets in M satisfying the following condition:

(P) *For every $\varepsilon > 0$ there exists a subfamily $\{V_i\}_{i\in I}$, $V_i \in \mathscr{U}$ of $\mathscr{U}$ such that $\bigcup_{i\in I} V_i \supset X$ and, for $i \in I$, $\text{diam}(V_i) \leqslant \varepsilon$.*

Analogously to the definition of the dimension of the set X relative to the manifold M and family $\mathcal{U}$, see [6], we can define the capacity of a compact set X relative to the manifold M and family $\mathcal{U}$ as follows:

$$(M, \mathcal{U})\text{-}C_X = \dim M \limsup_{\varepsilon \to 0} \frac{\log I(X, \mathcal{U}, \varepsilon)}{-\log v(\mathcal{U}, \varepsilon)},$$

where $I(X, \mathcal{U}, \varepsilon)$ denotes the smallest number of sets in $\mathcal{U}$ with diameter $\leqslant \varepsilon$ covering the set X and $v(\mathcal{U}, \varepsilon) = \sup\{v(U): U \in \mathcal{U}, \operatorname{diam}(U) \leqslant \varepsilon\}$.

2. Let N be a smooth Riemannian manifold, $f: N \to N$ a diffeomorphism, $M \subset N$ a smooth immersed submanifold, and $X \subset M$ any Borel subset. Fix $\delta > 0$ and consider the family

$$\mathcal{U}_{f,\delta} = \{U \subset M: \exists_{n \geqslant 0} \exists_{x \in X}\, U = f^{-n}(B_{f^n(M)}(f^n(x), \delta))\}.$$

Suppose that for sufficiently small $\delta > 0$ and for the family $\mathcal{U}_{f,\delta}$ condition (P) holds. We call the numbers

$$\bar{C}_{X,f} = \limsup_{\delta \to 0} ((M, \mathcal{U}_{f,\delta})\text{-}C_X),$$

$$\underline{C}_{X,f} = \liminf_{\delta \to 0} ((M, \mathcal{U}_{f,\delta})\text{-}C_X)$$

the *upper* and the *lower capacity relative to the mapping* f, respectively. If $\bar{C}_{X,f} = \underline{C}_{X,f}$, then we shall call this number the capacity relative to the mapping f and denote it by $C_{X,f}$.

3. Now we prove the following

THEOREM 3. *Let $f: T^n \to T^n$ be a hyperbolic toral automorphism. Let $K \subset T^n$ be a set such that its lift $\tilde{K} \subset R^n$ contains a k-dimensional $(1 \leqslant k \leqslant n)$ continuum $\tilde{K} \subset c + \bigoplus_1^l V_i$, where $V_i \subset E_{\lambda_i} (1 \leqslant i \leqslant l)$ are $\tilde{f}$-invariant linear subspaces in R^n, λ_i are the eigenvalues for $\tilde{f}$ and $1 < |\lambda_1| \leqslant \ldots \leqslant |\lambda_l|$. If $\sum_{i=2}^l \dim V_i \leqslant k$, $\sum_{i=1}^l \dim V_i = k + k'$ and the projection of $\tilde{K}$ onto $\bigoplus_{i=2}^l V_i$ along V_1 has a non-empty interior, then, taking in the definition of the Pesin capacity $N = T^n$, $M = \pi(c + \bigoplus_1^l V_i)$, the capacity $C_{K,f}$ exists and*

$$C_{K,f} \geqslant (k+k') \frac{\log|\lambda_1| (\dim V_1 - k') + \sum_2^l \dim V_i \log|\lambda_i|}{\sum_1^l \dim V_i \log|\lambda_i|}.$$

Proof. It is clear that in this case the family $\mathcal{U}_{f,\delta}$ satisfies condition (P) for every $\delta > 0$ and our theorem is equivalent to the same theorem if we replace T^n by R^n, $\pi(c + \bigoplus_1^l V_i)$ by $c + \bigoplus_1^l V_i$, K by $\tilde{K}$, and f by $\tilde{f}$.

Observe that here the map $\delta \mapsto (c + \bigoplus_1^l V_i, \mathcal{U}_{\tilde{f},\delta})\text{-}C_K$ is constant; hence $C_{K,f}$ exists and, of course, it is equal to

$$(k+k') \limsup_{\varepsilon \to 0} \frac{\log I(X, \mathcal{U}_{\tilde{f},\delta}, \mu, \varepsilon)}{-\log \varepsilon} \quad \text{for every } \delta > 0,$$

where μ is the Lebesgue measure on $c + \bigoplus_1^l V_i$ and $I(X, \mathcal{U}_{\tilde{f},\delta}, \mu, \varepsilon)$ is the smallest number of sets in $\mathcal{U}$ with measure μ less than or equal to ε, covering the set X.

Now, fix $\delta, \theta > 0$. Let $U_j = \tilde{f}^{-j}(B_{\tilde{f}^j(M)}(\tilde{f}^j(x), \delta)) \in U_{\tilde{f},\delta}$ $(x \in M = c + \bigoplus_1^l V_i)$. Since $\mu(U_j) = (\det(\tilde{f}|\bigoplus_1^l V_i))^{-j} \Gamma \delta^{k+k'}$, where Γ is the Lebesgue measure of a unit ball in $\bigoplus_{i=1}^l V_i$, it follows that if $\mu(U_j) \leqslant \varepsilon$ then

$$j \geqslant \frac{-\log(\varepsilon(\Gamma\delta^{n+k'}))}{\log \det(\tilde{f}|\bigoplus_1^l V_i)}$$

and this implies that

$$I(\tilde{K}, \mathcal{U}_{\tilde{f},\delta}, \mu, \varepsilon) \geqslant I(\tilde{f}^{[-\log(\varepsilon/\Gamma\delta^{k+k'})/\log\det(\tilde{f}|\bigoplus_1^l V_i)]}(\tilde{K}), \delta). \tag{1}$$

From the assumptions of our theorem there exists a constant $C_1 > 0$ such that for $j \geqslant 0$

$$I(\tilde{f}^j(\tilde{K}), \delta) \geqslant C_1 \Big((|\lambda_1| - \theta)^{\dim V_1 - k'} \prod_{i=2}^l (|\lambda_i| - \theta)^{\dim V_i}\Big)^j.$$

By (1) this implies

$$C_{\tilde{K},\tilde{f}} \geqslant (k+k') \frac{(\dim V_1 - k') \log(|\lambda_1| - \theta) + \sum_2^l \dim V_i \log(|\lambda_i| - \theta)}{\sum_1^l \dim V_i \log|\lambda_i|}.$$

Since θ is arbitrarily small, we obtain the required estimation.

4. We also prove the following

THEOREM 4. *The continuum* $K = \pi(G(D))$ *from Theorem* 2 *satisfies the following inequality*:

$$C_{K,f} \leqslant \varepsilon + (k+k')\frac{(\dim V_1 - k')\log|\lambda_1| + \sum_2^l \dim V_i \log|\lambda_i|}{\sum_1^l \dim V_i \log|\lambda_i|}.$$

Proof. Let $V_j = \tilde{f}^{-qj}\big(B_{\tilde{f}^j(M)}(\tilde{f}^{qj}(x),\, 1+2r)\big) \in \mathcal{U}_{\tilde{f},1+2r}$ $\big(x \in M = \bigoplus_1^l V_i\big)$, where q and r denote the numbers from the proof of Theorem 2.

Obviously, for

$$j(\zeta) = \left[-\frac{\log\big(\zeta/\Gamma(1+2r)^{k+k'}\big)}{\log\det\big(\tilde{f}^q|\bigoplus_1^l V_i\big)}\right] + 1, \qquad \mu(V_{j(\zeta)}) \leqslant \zeta.$$

Therefore, by (4) from the proof of Theorem 2,

$$\begin{aligned} I\big(G(D), \mathcal{U}_{\tilde{f},1+2r}, \mu, \zeta\big) &\leqslant I\big(\tilde{f}^{qj(\zeta)} \circ G(D),\, 1+2r\big) \\ &\leqslant Ch^{j(\zeta)qj(\zeta)}\, I\big(g_0(D)\big) \end{aligned}$$

and

$$C_{G(D),\tilde{f}} \leqslant (k+k')\frac{\log h}{\log\det(\tilde{f}^q|V)} + (k+k')\frac{\log \Lambda}{\log\det(\tilde{f}|V)}.$$

Thus for q large enough we obtain our estimation because

$$\log \Lambda = (\dim V_1 - k')\log|\lambda_1| + \sum_2^l \dim V_i \log|\lambda_i|$$

and

$$\log\det(\tilde{f}|V) = \sum_1^l \dim V_i \log|\lambda_i|.$$

Remark. In view of Theorem 4 we see that the estimation from Theorem 3 is the best possible. This is done for every k. Recall that for classical capacity we obtained the best possible estimations only for the curve $(n = 1)$.

One might think that the Pesin capacity is more adequate for such estimations.

However, the estimation in Theorem 3 is valid for every k-dimensional continuum K including continua with dense orbits! So the Pesin capacity is less sensitive than the classical one. It does not detect the "fractal" shape of curves with non-dense orbits.

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