largest value such that f vanishes identically on $[0, T_1]$. Then g must vanish on $[0, T-T_1]$.

Theorem I″ also follows from Mikusiński's Theorem. Indeed, taking partial Laplace transforms we have that, for each $z \in C^{N-1}$, the functions $\varphi_z(t_1)$ and $\theta_z(t_1)$ satisfy the assumption $(\varphi_z * \theta_z)(t_1) = 0$ for $0 \leqslant t_1 \leqslant T$. Thus for each $z \in C^{N-1}$, by Mikusiński's Theorem,

$$\varphi_z(\sigma)\theta_z(\tau) = 0 \quad \text{for all } \sigma \geqslant 0,\ \tau \geqslant 0,\ \sigma+\tau \leqslant T.$$

Let T_1 be the largest value such that $\varphi_z(\sigma)$ is zero for all $\sigma \leqslant T_1$ and all $z \in C^{N-1}$. If $T_1 < T$ there is a σ_1 larger than T_1 and arbitrarily close to T_1, and a $z' \in C^{N-1}$ such that $\varphi_{z'}(\sigma_1) \neq 0$. For all $\tau_1 \leqslant T-\sigma_1$ we have

$$\varphi_z(\sigma_1)\theta_z(\tau_1) = 0 \quad \text{for all } z.$$

Since the product of two entire functions of z is zero only if one of them is zero, it follows that $\theta_z(\tau_1) = 0$ for all z. Since $\sigma_1 > T_1$ can be taken arbitrarily close to $T_2 = T-T_1$,

$$\varphi_z(t_1) = 0, \quad 0 \leqslant t_1 \leqslant T_1,\ z \in C^{N-1},$$
$$\theta_z(t_1) = 0, \quad 0 \leqslant t_1 \leqslant T_2,\ z \in C^{N-1},$$

and $T_1 + T_2 = T$. By the uniqueness of the Laplace transform we have the result of Theorem I″.

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For a Banach space isomorphic to its square the Radon–Nikodým property and the Krein–Milman property are equivalent

by

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Abstract. We prove the result announced in the title.

0. Introduction. For the definition of the Radon–Nikodým property and the Krein–Milman property (abbreviated RNP and KMP) we refer to [5]. In the past ten years some effort has been made to prove or disprove the conjecture that these two notions are equivalent. For some classes of Banach spaces it has been shown that they are equivalent (see [7] and [3]).

There is an intermediate notion between RNP and KMP, namely the "Integral Representation Property" (abbr. IRP) [13]: A Banach space X (which we suppose to be separable to avoid measure-theoretic complications, in which we are not interested here) has IRP if for every bounded, closed, convex set $C \subseteq X$ and every $x \in C$ there is a probability measure μ on the extreme points of C such that x is the barycenter of μ.

A theorem due to Edgar shows that RNP $\Rightarrow$ IRP ([5], p. 145) and it is easy to see that IRP $\Rightarrow$ KMP. The converse implications are open.

E. Thomas [13] has shown that a Fréchet space X has RNP iff X^N has IRP. In the context of Banach spaces this may be formulated as follows: A Banach space X has RNP iff $l^2(X)$ (or any other appropriate space of sequences in X) has IRP.

It was observed in [11] that if X is isomorphic to its square then $l^2(X)$ has IRP iff X has IRP, i.e. in this case IRP and RNP are equivalent.

In the present paper we obtain – inspired by the argument of E. Thomas but using quite a different reasoning and an observation due to H. P. Rosenthal – analogous results for KMP in place of IRP.

The aim of the introductory Section 1 is to establish Corollary 1.3; this result – or at least some variant of it – seems to be known to people working in the field and its content can be derived from a construction of J. Bourgain [1]. However, we have preferred to use an approach due to C. Stegall [12]. This approach seems at the same time elementary and powerful; it also shows that the pathologies arising in the absence of RNP need not to be "constructed" but are already contained in any nonrepresentable operator from $L^1[0, 1]$ to X.

In Section 2 we establish that X has RNP iff $l^2(X)$ has KMP.

In Section 3 we introduce the notion of 3/4-embedding (which has nothing to do with Vienna waltz). It is weaker than the notion of embedding but stronger than the notion of semi-embedding, introduced by Lotz, Peck and Porta ([9], [2]). This somewhat strange notion furnishes a convenient framework for constructing the necessary maps from $l^2(X)$ into X – using the ideas of [11] – to prove: If $X\times X$ 3/4-embeds into X, then X has RNP if it has KMP (Corollary 3.7). This contains the result announced in the title.

In the final Section 4 we state some open questions.

After writing up the paper, we learned that Proposition 2.1 has already beeen explicitly formulated and published by A. Ho [6]. We learned this from the comprehensive monograph of R. Bourgin ([4], p. 418), where one may also find a variant of Corollary 1.3, which is due to James [8], and the results about IRP established in [11] and [13]. Still we believe that our presentation of these results, which are needed to establish the theorem announced in the title, is interesting in its own right.

Finally we also learned that in [10] the notion of 3/4-embeddings is investigated and they are characterised (among other results) as the injective Tauberian operators.

My thanks go to H. P. Rosenthal, R. C. James, E. Thomas, P. Mankiewicz and C. P. Stegall for some very stimulating discussions on the subject of this article.

1. Let (Ω, Σ, P) be a probability space. A bounded linear operator T from $L^1(\Omega, \Sigma, P)$ into a Banach space X is called *representable* if there is a function $F\in L^\infty(\Omega, \Sigma, P; X)$ such that for every $f\in L^1(\Omega, \Sigma, P)$

$$Tf = \int f(\omega)F(\omega)\,dP(\omega).$$

For $A\in\Sigma$ we say that T is *representable on* A if $T\circ R_A$ is representable, where $R_A\colon L^1\to L^1$ is the operator of multiplication by the characteristic function χ_A. It is clear that there is a set Ω_0 such that

$$P(\Omega_0) = \sup\{P(A)\colon\ T \text{ is representable on } A\},$$

and that T is representable iff $P(\Omega_0) = 1$.

Hereafter, if $E\in\Sigma$ with $P(E) > 0$ we put

$$\mathcal{L}_E^T = \{Tf\colon f = f\chi_E \geqslant 0 \text{ and } \int f\,dP = 1\}.$$

Also we set $B(0,\varepsilon) = \{x\colon \|x\|\leqslant\varepsilon\}$ and if $(x_j^*)_{j=1}^\infty$ is a normalized norming sequence in X^*,

$$B_m(0,\varepsilon) = \{x\colon \langle x, x_j^*\rangle \leqslant \varepsilon,\ j = 1,\ldots,m\}.$$

The following theorem, on which our approach is based, is due to C. Stegall ([12], p. 21) and it is remarkable how easily it may be proved, "starting from nothing". We shall denote by co the convex (not closed convex!) hull of a set.

THEOREM 1.1. *Suppose $T\colon L^1(\Omega,\Sigma,P)\to X$ is not representable. Then for any $\delta > 0$ there is $E_0\in\Sigma$ disjoint from Ω_0 with $P(E_0)+P(\Omega_0) > 1-\delta$ and an $\varepsilon > 0$ such that for every weakly compact set $W\subseteq X$ and any $E\subseteq E_0$*

$$\mathcal{L}_E^T = \operatorname{co}\left[\mathcal{L}_E^T\setminus(W+B(0,\varepsilon))\right].$$

Moreover, if X is separable and $(x_j^*)_{j=1}^\infty$ is a norming sequence of norm one elements in X^*, we may find not only the above but, for every weakly compact W and for every $0<\varepsilon'<\varepsilon$, an arbitrarily large set $F\subseteq E_0$ and $m\in N$ such that for any $E\subseteq F$

$$\mathcal{L}_E^T = \operatorname{co}\left[\mathcal{L}_E^T\setminus(W+B_m(0,\varepsilon'))\right]. \ \blacksquare$$

We shall not use the full strength of the theorem; in the subsequent corollary we apply it to a one-point set instead of a general weakly compact set. In the rest of the section we assume X to be separable.

COROLLARY 1.2. *In the setting of Theorem 1.1, let $x_0\in X$ and $W = \{x_0\}$. Then we may find a partition $\Pi = (D_1,\ldots,D_m)$ of F such that for every atom $D_j\in\Pi$*

$$\bar{\mathcal{L}}_{D_j}^T \subseteq \{x\in X\colon \langle x-x_0, x_j^*\rangle \geqslant \varepsilon'\},\qquad j=1,\ldots,m,$$

$\bar{\mathcal{L}}_{D_j}^T$ denoting the closure of $\mathcal{L}_{D_j}^T$.

Proof (Stegall). We may assume that x_0 is the origin. Let $h_j = T^*x_j^*$, $C_j = \{\omega\in F\colon h_j(\omega) > \varepsilon'\}$ and

$$C = F\setminus\bigcup_{j=1}^m C_j.$$

We shall show that $P(C) = 0$. Indeed, if this is not the case, fix any function $f = f\chi_C \geqslant 0$ with $\int f\,dP = 1$ and note that for every $j = 1,\ldots,m$

$$\varepsilon' \geqslant \int_\Omega f(\omega)h_j(\omega)\,dP(\omega) = \langle f, T^*x_j^*\rangle = \langle Tf, x_j^*\rangle.$$

Hence $\mathcal{L}_C^T \subseteq \{0\}+B_m(0,\varepsilon')$, a contradiction to Theorem 1.1.

Defining $D_1 = C_1$, $D_2 = C_2\setminus C_1$, $D_3 = C_3\setminus(C_1\cup C_2)$, etc. we obtain a partition of F and for any $j = 1,\ldots,m$ and $f = f\chi_{D_j}\geqslant 0$ with $\int f\,dP = 1$ we have

$$\varepsilon' < \int_\Omega f(\omega)h_j(\omega)\,dP(\omega) = \langle Tf, x_j^*\rangle,$$

thus proving the corollary. ■

COROLLARY 1.3. *Suppose again $T\colon L^1(\Omega,\Sigma,P)\to X$ is not representable. Then for any $\delta > 0$ there is $A\in\Sigma$ with $P(A)+P(\Omega_0) > 1-\delta$ and an increasing sequence $(\Sigma_n)_{n=0}^\infty$ of finite partitions of A such that for any decreasing sequence $(A^n)_{n=1}^\infty$ of atoms in Σ_n we have*

$$\bigcap_{n=1}^\infty \bar{\mathcal{L}}_{A^n}^T = \emptyset.$$

Proof. Let $(x_n)_{n=1}^{\infty}$ be dense in X. Apply Corollary 1.2 to find $F_n \in \Sigma$ such that $P(F_n)+P(\Omega_0) > 1-\delta/2^n$ and partitions Π_n, $\Pi_n = \{D_1^n, \ldots, D_{m(n)}^n\}$ of F_n such that

$$\mathscr{L}_{D_j^n}^T \subseteq \{x \in X\colon \langle x - x_n, x_j^* \rangle \geqslant \varepsilon'\}, \quad j = 1, \ldots, m(n).$$

Let $A = \bigcap_{n=1}^{\infty} F_n$, $\Sigma_0 = \{A\}$ and let Σ_n be the partition of A generated by the traces of $\Pi_1, \ldots, \Pi_n$ on A.

Given a decreasing sequence $(A^n)_{n=1}^{\infty}$ of atoms of Σ_n and an arbitrary $\bar{x} \in X$ find n_0 such that $\|\bar{x} - x_{n_0}\| < \varepsilon'/2$. As A^{n_0} is contained in $D_j^{n_0}$ for some $1 \leqslant j \leqslant m(n_0)$,

$$\mathscr{L}_{A^{n_0}}^T \subseteq \{x \in X\colon \langle x - x_{n_0}, x_j^* \rangle \geqslant \varepsilon'\},$$

hence $\bar{x} \notin \bigcap_{n=1}^{\infty} \mathscr{L}_{A^n}^T$. ∎

Remark. If $^*\mathscr{L}_E^T$ denotes the weak star closure of $\mathscr{L}_E^T$ in X^{**} then it is easily seen that under the assumption of the above corollary

$$\bigcap_{n=1}^{\infty} {}^*\mathscr{L}_{A^n}^T$$

has distance from X greater than or equal to ε'. It follows – using the argument in the proof of Proposition 2.1 below and the fact that the convex hull of finitely many σ^*-compact sets is σ^*-compact – that the extreme points of $^*\mathscr{L}_A^T$ have also distance from X greater than or equal to ε'; in particular, the extreme points of $^*\mathscr{L}_A^T$ are contained in $X^{**} \setminus X$. Hence we have reproved a theorem, due to J. Bourgain, that this extreme point phenomenon arises in every Banach space failing RNP (see also [12]).

2. A necessary and sufficient condition for RNP. In this section we show that X has RNP iff $l^2(X)$ has KMP. The proof is based on the following easy but crucial observation which has been pointed out to me by H. P. Rosenthal (oral communication) and which was noted by A. Ho [6].

In the situation of Corollary 1.3 fix an atom A_j^n in Σ_n. We call the atoms $A_{j(1)}^{n+1}, \ldots, A_{j(q)}^{n+1}$ in Σ_{n+1} which are contained in A_j^n the *successors* of A_j^n. Clearly

$$A_j^n = \bigcup_{p=1}^{q} A_{j(p)}^{n+1}, \qquad \mathscr{L}_{A_j^n}^T = \mathrm{co}\Big[\bigcup_{p=1}^{q} \mathscr{L}_{A_{j(p)}^{n+1}}^T\Big]$$

and

$$\mathscr{L}_{A_j^n}^T = \overline{\mathrm{co}}\Big[\bigcup_{p=1}^{q} \mathscr{L}_{A_{j(p)}^{n+1}}^T\Big]$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull.

Proposition 2.1. *In the setting of Corollary 1.3, assume in addition that for every $n = 0, 1, 2, \ldots$ and every atom A_j^n in Σ_n and its successors $A_{j(1)}^{n+1}, \ldots, A_{j(q)}^{n+1}$ in Σ_{n+1}*

$$(*) \qquad \mathscr{L}_{A_j^n}^T = \mathrm{co}\Big(\bigcup_{p=1}^{q} \mathscr{L}_{A_{j(p)}^{n+1}}^T\Big).$$

Then the closed, convex, bounded set $\mathscr{L}_A^T$ has no extreme point.

Proof. Assume x is extreme in $\mathscr{L}_A^T$. Applying $(*)$ to $n = 0$ we conclude from the extremality of x that there is an atom $A_{j_1}^1$ of Σ_1 such that

$$x \in \mathscr{L}_{A_{j_1}^1}^T.$$

Applying $(*)$ to $A_{j_1}^1$ we may find again – by the extremality of x – a successor $A_{j_2}^2 \in \Sigma_2$ of $A_{j_1}^1$ such that

$$x \in \mathscr{L}_{A_{j_2}^2}^T.$$

Continuing in an obvious way we find a decreasing sequence $(A_{j_n}^n)_{n=1}^{\infty}$ of atoms in Σ_n such that for every $n \in N$

$$x \in \mathscr{L}_{A_{j_n}^n}^T.$$

This contradicts Corollary 1.3. ∎

Remark 2.2. Condition $(*)$ means that the convex hull on the right-hand side of $(*)$ is already closed. How can one ensure that the convex hull of finitely many closed sets is closed? A sufficient condition is – intuitively speaking – that these sets lie in different subspaces of X, which are all separated by a "strictly positive angle". (This idea may also be found in [6].)

The idea of the present paper is to construct these "strictly positive angles" by brute force. The price which we pay for this construction is to go out of X into a sequence space on X, e.g. $l^2(X)$ which is defined by

$$l^2(X) = \{(x_n)_{n=1}^{\infty},\ x_n \in X\colon \|(x_n)\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\}.$$

Theorem 2.3. *A Banach space X has RNP iff $l^2(X)$ has KMP.*

Proof. If X has RNP, then $l^2(X)$ has RNP and therefore KMP. Conversely, suppose X fails RNP. We may assume X to be separable. There is an operator

$$T\colon L^1(\Omega, \Sigma, P) \to X$$

which is not representable. By Corollary 1.3 we may find $A \in \Sigma$, $P(A) > 0$ and an increasing sequence Σ_n of finite partitions of A such that for every decreasing sequence A^n of atoms of Σ_n we have

$$\bigcap_{n=1}^{\infty} \mathscr{L}_{A^n}^T = \emptyset.$$

Let d_n be the number of atoms of Σ_n and find $\alpha_n > 0$ which tend sufficiently fast to zero so that

$$\sum_{n=1}^{\infty} d_n \alpha_n^2 < \infty.$$

Let D be the denumerable set consisting of the pair $(0, 1)$ and the pairs $\{(n, j)\colon n \geqslant 1, 1 \leqslant j \leqslant d_n\}$ and index $l^2(X)$ by D, i.e.

$$l^2(D; X) = \{(x_{n,j})_{(n,j)\in D}\colon \|(x_{n,j})_{(n,j)\in D}\|^2 = \sum_D \|x_{n,j}\|^2 < \infty\}.$$

Define the operator

$$S\colon L^1(\Omega, \Sigma, P) \to l^2(D; X)$$

coordinatewise by

$$S_{(0,1)} = T, \qquad S_{(n,j)} = \alpha_n \cdot T \circ R_{A_j^n}, \quad n \geqslant 1,\ 1 \leqslant j \leqslant d_n.$$

Recall that for $A \in \Sigma$, R_A is the restriction to A. It is easily checked that, by the choice of α_n, the operator S is well defined. S has the property that for every decreasing sequence (A^n) of atoms of Σ_n

$$\bigcap_{n=1}^{\infty} \mathscr{L}^S_{A^n} = \emptyset$$

because the coordinate $(0, 1)$ of S has this property.

We shall show that S satisfies condition $(*)$ of Proposition 2.1. Indeed, fix (n, j) in D and let $(n+1, j(1)), \ldots, (n+1, j(q))$ be the indices of the successors of A_j^n. Let $(x^i)_{i=1}^{\infty}$ be a Cauchy sequence in $\mathscr{L}^S_{A_j^n}$. Find

$$f^i = f^i \chi_{A_j^n} \geqslant 0, \quad \int f^i\, dP = 1 \quad \text{with} \quad Sf^i = x^i.$$

For every $i \in N$ we may write f^i as a convex combination

$$f^i = \sum_{p=1}^{q} \lambda_p^i f_p^i, \qquad \sum_{p=1}^{q} \lambda_p^i = 1$$

where

$$f_p^i = f_p^i \chi_{A_{j(p)}^{n+1}} \geqslant 0, \quad \int f_p^i\, dP = 1.$$

By passing to a subsequence we may assume that $(\lambda_p^i)_{i=1}^{\infty}$ converges to λ_p. Clearly $\sum \lambda_p = 1$. Fixing for $1 \leqslant p \leqslant q$ our attention on the coordinate $(n+1, j(p))$ of S we see that

$$S_{(n+1,j(p))} f^i = \alpha_{n+1} T(\lambda_p^i f_p^i).$$

From the assumption that $(x^i)_{i=1}^{\infty}$ is a Cauchy sequence we infer that

$$(T(\lambda_p^i f_p^i))_{i=1}^{\infty}$$

converges in X; we may deduce that

$$(S(\lambda_p^i f_p^i))_{i=1}^{\infty}$$

converges coordinatewise in $l^2(D; X)$ and — again by the choice of the α_n — this sequence converges with respect to the norm of $l^2(D; X)$. Hence

$$\lim_{i\to\infty} x^i = \sum_{p=1}^{q} \lambda_p (\lim_{i\to\infty} Sf_p^i) \in \mathrm{co}\Big(\bigcup_{p=1}^{q} \mathscr{L}^S_{A_{j(p)}^{n+1}}\Big).$$

Hence S fulfills condition $(*)$ of 2.1, which finishes the proof of the theorem. ∎

3. 3/4-embeddings and the Krein–Milman property.

DEFINITION 3.1 [6]. X *semi-embeds* into Y if there is an injective continuous operator $j\colon X \to Y$ such that the image of the unit ball under j is closed in Y.

Bourgain and Rosenthal [2] showed that a separable Banach space X that semi-embeds into a Banach space Y which has RNP, has already RNP (see also [12]).

For the purpose of KMP we need a stronger notion.

DEFINITION 3.2. A Banach space X *3/4-embeds* into a Banach space Y if there is an injective operator $j\colon X \to Y$ which maps closed, bounded, convex sets onto closed, bounded, convex sets.

PROPOSITION 3.3. *If Y has KMP and X 3/4-embeds into Y, then X also has KMP.*

Proof (trivial). If X fails KMP then [5] there is a closed bounded convex set $C \subseteq X$ which has no extreme point. If $j\colon X \to Y$ is a 3/4-embedding then $j(C)$ is closed, bounded and convex and has no extreme points, which gives a contradiction. ∎

COROLLARY 3.4. *If $l^2(X)$ 3/4-embeds into X then X has KMP iff X has RNP.* ∎

PROPOSITION 3.5. *If $X \times X$ 3/4-embeds into X then $l^2(X)$ 3/4-embeds into X.*

Remark 3.6. Before proving 3.5 we shall deduce the corollary, which is the main result of the paper.

COROLLARY 3.7. *If $X \times X$ 3/4-embeds into X (in particular, if X is isomorphic to $X \times X$) then X has KMP iff X has RNP.* ∎

Proof of Proposition 3.5. The idea of the proof is very simple and consists in "repeatedly splitting X as $X \times X$". However, it turns out that it is rather laborious to write up the details.

Let $l^2(X)$ this time be indexed by the natural numbers N; for $n \in N$ let X^n be the subspace of $l^2(X)$ such that the coordinates greater than n are zero and for $1 \leqslant k \leqslant l \leqslant n$ let $X^n_{[k,l]}$ be the subspace of X^n such that all coordinates outside the interval $[k, l]$ equal zero. X^n (resp. $X^n_{[k,l]}$) may be identified in an obvious way with the n-fold (resp. $l-k+1$-fold) product $X \times \ldots \times X$; we shall freely use these identifications in the sequel.

By hypothesis there is a 3/4-embedding $i\colon X\times X\to X$; we may suppose $\|i\| \leqslant 1/2$. Let

$$E_1\colon X^2 \to X$$

be simply i and, for $n \geqslant 2$, define

$$E_n\colon X^{n+1} \to X^n$$

to be the identity on $X^{n+1}_{[1,n-1]}$ with values in $X^n_{[1,n-1]}$ and to be i on $X^{n+1}_{[n,n+1]}$ with values in $X^n_{[n,n]}$.

(i) *For $n\in N$, E_n is a 3/4-embedding from X^{n+1} into X^n.* E_n is clearly injective. As regards the second property of 3/4-embeddings, note that a continuous, injective $i\colon Y\to Z$ maps closed, convex, bounded sets into closed sets iff for every bounded sequence $(y_j)_{j=1}^\infty$ such that $(i(y_j))_{j=1}^\infty$ converges we may find convex combinations $\tilde{y}_k$ of $(y_i)_{i=k}^\infty$ which converge in Y.

In the special case of E_n, if $(x_j)_{j=1}^\infty$ is bounded in X^{n+1} and such that $(E_n(x_j))_{j=1}^\infty$ converges, it is clear that the first $n-1$ coordinates of x_j converge. Hence E_n maps closed, bounded, convex sets to closed sets, as i does so by assumption.

(ii) *For $n \geqslant 1$ the map*

$$E_1\circ\ldots\circ E_n\colon X^{n+1}\to X$$

is a 3/4-embedding. Indeed, the composition of 3/4-embeddings is a 3/4-embedding.

(iii) $$\|E_1\circ\ldots\circ E_n|_{X^{n+1}_{[n,n+1]}}\| \leqslant 2^{-n}.$$

This follows from the assumption $\|i\| \leqslant 1/2$.

Let P_n be the natural projection from $l^2(X)$ onto X^n and put $j_n = E_1\circ\ldots\circ E_n\circ P_{n+1}$.

(iv) $$\|j_{n+1}-j_n\| \leqslant 2^{-n+1} \quad \textit{for } n\geqslant 1.$$

j_{n+1} and j_n coincide on the space X^n and are both equal to zero on the range of $I-P_{n+2}$. Using (iii) on the spaces $X^{n+1}_{[n+1,n+1]}$ and $X^{n+2}_{[n+2,n+2]}$ we obtain (iv). Hence the sequence $j_n\colon l^2(X)\to X$ converges to an operator j. We shall show that j is a 3/4-embedding.

(v) *j is injective.* Let $x=(x_n)_{n=1}^\infty$ be an element of $l^2(X)$, $\|x\|=1$ and choose k large enough that $P_k(x)\neq 0$. Then

$$j(x) = j\big(P_k(x)+(I-P_k)(x)\big) = j_k\big(P_k(x)\big)+\lim_{n\to\infty} j_n\big((I-P_k)(x)\big).$$

A glance at the definition of j_k shows that the sequence $\{j_n\big((I-P_k)(x)\big)\}_{n=k}^\infty$ lies in the closed, convex, bounded set

$$j_k\circ(I-P_k)\big(\text{unit ball } (l^2(X))\big) = E_1\circ\ldots\circ E_k\big(\text{unit ball } (X^{k+1}_{[k+1,k+1]})\big).$$

But $-j_k\big(P_k(x)\big)$ is a nonzero element of

$$E_1\circ\ldots E_k\big(\text{unit ball } (X^{k+1}_{[1,k]})\big)$$

and hence does not belong to the above set. Thus $j(x)\neq 0$.

(vi) *j maps closed, bounded, convex sets onto closed sets.* Let $(x^i)_{i=1}^\infty$ be a bounded sequence in $l^2(X)$ such that $(j(x^i))_{i=1}^\infty$ converges in X. We have to show that there are convex combinations $(\tilde{x}^i)_{i=1}^\infty$ of $(x^j)_{j=i}^\infty$ converging in $l^2(X)$. Note that

$$j_1\big(\text{unit ball } (l^2(X))\big) \supseteq j_2\big(\text{unit ball } (l^2(X))\big) \supseteq \ldots \supseteq j\big(\text{unit ball}(l^2(X))\big).$$

Hence for every $n\in N$ the sequence $\big((E_1\circ\ldots\circ E_n)^{-1}\circ j(x^i)\big)_{i=1}^\infty$ is bounded and therefore by (ii) we may find convex combinations converging in X^{n+1}. By a diagonal procedure, we may find convex combinations $(\tilde{x}^i)_{i=1}^\infty$ such that, for every $n\in N$, we have this convergence. But this means that $(\tilde{x}^i)_{i=1}^\infty$ converges coordinatewise in $l^2(X)$. Hence – by passing once again to convex combinations if necessary – $(\tilde{x}^i)_{i=1}^\infty$ converges in $l^2(X)$.

Summing up, we have constructed a 3/4-embedding $j\colon l^2(X)\to X$, thus finishing the proof of 3.5. ∎

4. Problems and remarks. Theorem 2.3 shows that RNP and KMP are equivalent if we pass from X to the larger space $l^2(X)$. It is natural to ask whether one can do any better, e.g. by passing to a smaller space. For example one may ask:

QUESTION 4.1. If $X\times X$ has KMP, does this imply that X has RNP?

In the other direction one may ask:

QUESTION 4.2. If X has KMP, does this imply that $X\times X$ has KMP?

We do not even know the answer to the following question:(¹)

QUESTION 4.3. If X has KMP, does this imply that $X\times l^2$ has KMP?

However, we have a very modest positive result in this direction:

PROPOSITION 4.4. *If X has KMP then $X\times \mathbf{R}$ has KMP.*

Proof (elementary geometry). Let $B\subseteq X\times \mathbf{R}$ be closed, bounded, convex and suppose that B has no extreme point. We may assume $0\in B$.

Let $B_1 = B\cap (X\times\{0\})$ which, by assumption, has an extreme point $x_0 = \{\xi_0, 0)$. As x_0 is not extremal in B there is a straight line in B through x_0; let x_1 and x_2 be the endpoints of this line, i.e. we may find $0<\lambda<1$ and

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Added in proof. It has been shown by M. Talagrand (personal communication) that if X has KMP and X is reflexive, then $X\times Y$ has KMP, thus solving Question 4.3 in the positive and making Proposition 4.4 obsolete.

$x_1, x_2 \in B$ such that

$$x_0 = \lambda x_1 + (1-\lambda) x_2$$

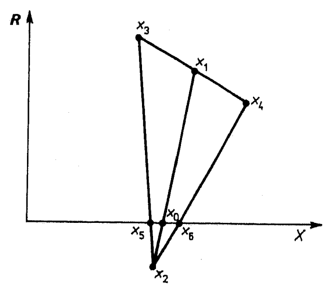
and such that for $\varepsilon > 0$

$$x_i + \varepsilon (x_i - x_0) \notin B, \quad i = 1, 2.$$

Clearly $x_1 = (\xi_1, t_1)$, $x_2 = (\xi_2, t_2)$ with $t_i \neq 0$ and we may assume $t_1 > 0$. As x_1 is not extremal in B we may find $x_3, x_4 \in B$ such that

$$x_1 = \tfrac{1}{2}(x_3 + x_4) = \tfrac{1}{2}\big((\xi_3, t_3) + (\xi_4, t_4)\big).$$

By choosing x_3 and x_4 close enough to x_1 we may assume $t_3, t_4 > 0$.



Define

$$x_5 = \big((-t_2)^{-1} + t_3^{-1}\big)\big(t_3^{-1} x_3 + (-t_2)^{-1} x_2\big),$$

and

$$x_6 = \big((-t_2)^{-1} + t_4^{-1}\big)\big(t_4^{-1} x_4 + (-t_2)^{-1} x_2\big).$$

Then $x_5, x_6 \in B_1$ and x_0 is a convex combination of x_5 and x_6, which gives the contradiction. ∎

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