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Note on differentiation of integrals and the halo conjecture

by

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Abstract. In this paper several results on differentiation of integrals are obtained from restricted weak type estimates of the maximal operator associated to certain differentiation bases in $\boldsymbol{R}^n$. The only tool used is a simple lemma in measure theory due to E. Stein and N. Weiss which explains how functions add up in weak-L^1 (Lemma 5). In the process, we construct for each index $m \geqslant 0$ a quasi-Banach function space which plays with respect to $L(\log^+ L)^m$ the same role as the Lorentz class $L(p, 1)$ does with respect to L^p, $1 < p < \infty$ (see Theorems 2 and 3). We follow here some ideas originated in Taibleson–Weiss [7].

The same methods are used to exhibit a weak type estimate for the maximal operator on the partial sums of Fourier series and, as a consequence, a.e. convergence a little bit beyond $L\log^+ L\log^+\log^+ L$.

1. Introduction and statement of results. Let $\mathscr{B}$ be a differentiation basis in $\boldsymbol{R}^n$ and $\Phi(u)$ its halo function; that is, $\Phi(u) = u$ if $0 < u \leqslant 1$ and, if $u > 1$,

$$\Phi(u) = \sup\{|A|^{-1}|\{x\colon (M\chi_A)(x) > 1/u\}|\colon A \text{ a msble. subset, } |A| > 0\},$$

where $M = M_{\mathscr{B}}$ denotes the maximal operator associated to $\mathscr{B}$, $|A|$ the Lebesgue measure of the subset A and χ_A its characteristic function.

In the present work we give partial answers to the following question: Assuming certain knowledge on the growth of $\Phi(u)$ at infinity, what can be said about differentiation properties of the basis $\mathscr{B}$?. (For an introduction to the subject, including some basic definitions, the reader is referred to de Guzmán [1].) We will state now a first result in this direction.

THEOREM 1. *Suppose that $\Phi(u) \leqslant c_0 u(1+\log^+ u)^m$ for some non-negative constants m and c_0. Then $\mathscr{B}$ differentiates any function which is locally in $L(\log^+ L)^m \log^+\log^+ L$.*

$L(\log^+ L)^m \log^+\log^+ L$ is not, however, the appropriate class to fully exploit the information given by such a behavior of the halo function and our next step will be to introduce more adequate classes to deal with this kind of problem. We will also show that, at least in one case, our results are best possible (see Theorem 3).

Notation. Let $\varphi_m(s)$ denote the function $s(1+\log^+ 1/s)^m$, $s \geq 0$. Given a function $f(x)$, we set

$$\|f\|_{\varphi_m} = \int_0^\infty \varphi_m(\lambda_f(t))\,dt,$$

where $\lambda_f(t)$ stands for the distribution function of f; that is,

$$\lambda_f(t) = |\{x \in \boldsymbol{R}^n\colon\ |f(x)| > t\}|.$$

We define B_m as the collection of all measurable functions $f(x)$ for which

$$\|f\|_{B_m} = \int_0^\infty \varphi_m(\lambda_f(t))\left[1+\log\left(\|f\|_{\varphi_m}/t\varphi_m(\lambda_f(t))\right)\right] dt$$

is finite.

Theorem 1 will be obtained as an easy consequence of the following

THEOREM 2. *If $\Phi(u) \leq c_0 u(1+\log^+ u)^m$, there exists a constant C such that for any measurable subset E and for any function f in the class B_m we have*

$$|\{x \in E\colon (Mf)(x) > t\}| \leq \frac{C}{t}\max(c_0, |E|)\|f\|_{B_m}.$$

Remarks. (a): For each $m \geq 0$, B_m has a structure of quasi-Banach space, the quasi-norm given by $\|\cdot\|_{B_m}$. Homogeneity and completeness can easily be proved. To show the inequality

$$\|f+g\|_{B_m} \leq K(\|f\|_{B_m}+\|g\|_{B_m}) \tag{1.1}$$

for some absolute constant K, it suffices to point out that from the definition of $\|\cdot\|_{B_m}$ and the monotonicity of φ_m we have

$$\begin{aligned}\|f\|_{B_m} &\sim \sum_{k=-\infty}^{\infty} 2^k\varphi_m(\lambda_f(2^k))\left[1+\log\frac{\sum_{j=-\infty}^{\infty} 2^j\varphi_m(\lambda_f(2^j))}{2^k\varphi_m(\lambda_f(2^k))}\right] \\ &= N(\{2^k\varphi_m(\lambda_f(2^k))\}),\end{aligned} \tag{1.2}$$

where, given a numerical sequence $c = \{c_k\}$, we write

$$N(c) = \sum_k |c_k|\left[1+\sum_j |c_j|/|c_k|\right].\ (^1)$$

(1) "$A \sim B$" means as usual that there are constants $0 < c, C < \infty$ such that $cB \leq A \leq CB$.

Now, (1.1) follows from the facts that $\lambda_{f+g}(2s) \leq \lambda_f(s)+\lambda_g(s)$, that $\varphi_m(s)$ is subadditive and that for two sequences $c = \{c_k\}$, $d = \{d_k\}$ we have

$$N(\{c_k+d_k\}) \leq (1+\log 2)[N(c)+N(d)]$$

(see Taibleson–Weiss [7], p. 102).

(b): For the case $m = 0$, Theorem 2 has the following refinement:

THEOREM 3. *The following two statements are equivalent*:

(i) $\Phi(u) \leq c_0 u$ (*i.e., M is of restricted weak type* (1, 1));

(ii) *there exists a constant C such that*

$$|\{x \in \boldsymbol{R}^n\colon (Mf)(x) > t\}| \leq \frac{C}{t}\|f\|_{B_0} = \frac{C}{t}\int_0^\infty \lambda_f(t)\left[1+\log\frac{\|f\|_1}{t\lambda_f(t)}\right]dt.$$

The class B_0 has been studied in [5] in a different context. It is shown there that B_0 is the rearrangement-invariant hull of certain "block spaces" introduced by M. Taibleson and G. Weiss in [7] in connection with almost everywhere convergence of Fourier series. As it was pointed out in [5], B_0 is close to L^1 and contains some elements whose Fourier series do not converge almost everywhere. In Section 3, however, we prove that this property about a.e. convergence holds for functions in the slightly smaller class B_1.

(c): The "halo conjecture" states that if the basis $\mathscr{B}$ is invariant by homotheties then $\mathscr{B}$ differentiates the class L_Φ of functions h such that $\int\Phi(|h|) < \infty$ (see de Guzmán [1], Chapter VIII). The halo conjecture has been positively solved only for the case $\Phi(u) \sim u$. In fact, it is known that $\Phi(u) \sim u$ if and only if M is of weak type (1, 1).

Let us see how close our results are from the halo conjecture. Theorem 2 and the fact that any function in B_m can be approximated (in the topology of B_m) by continuous functions show that $\mathscr{B}$ differentiates the class B_m.

If the conjecture were true we would have, for $\Phi(u) \sim u(1+\log^+ u)^m$ and $\mathscr{B}$ invariant under homotheties, that $\mathscr{B}$ differentiates $L(\log^+ L)^m$. One can easily prove that any function which is locally in B_m lies then locally in $L(\log^+ L)^m$. As a matter of fact, on a subset X of finite measure (say $|X| < 1$), $\|\ \|_{\varphi_m}$ represents an equivalent norm in $L(\log^+ L)^m(X)$ and, since $\|\ \|_{\varphi_m} \leq \|\ \|_{B_m}$, we see that $B_m(X)$ is continuously imbedded in $L(\log^+ L)^m(X)$. However, the former class is not far from the latter since it contains any function in the class $L(\log^+ L)^m\log^+\log^+ L(X)$. Thus, any function which is locally in $L(\log^+ L)^m\log^+\log^+ L(\boldsymbol{R}^n)$ lies (locally) in $B_m(\boldsymbol{R}^n)$ (see Section 4).

Observe that our results hold for a general basis without further assumptions.

2. Proof of Theorems 2 and 3. For $u \geq 0$ set $\Phi_0(u) = u(1+\log^+ u)^m$. Thus, $\Phi(u) \leq c_0\Phi_0(u)$.

LEMMA 4. (iii) *Φ_0 is submultiplicative; i.e.,* $\Phi_0(u_1 u_2) \leqslant \Phi_0(u_1)\Phi_0(u_2)$.

(iv) $\Phi_0(1/\varphi_m(s)) \leqslant 1/s$.

Proof. (iii) is trivial. For (iv) we notice that $\varphi_m(s) \geqslant s$ and, therefore,

$$\Phi_0(1/\varphi_m(s)) = \frac{1}{\varphi_m(s)}\left(1+\log^+\frac{1}{\varphi_m(s)}\right)^m \leqslant \frac{1}{s(1+\log^+(1/s))^m}\left(1+\log^+\frac{1}{s}\right)^m = \frac{1}{s}. \blacksquare$$

Now, for any set A we have from the definition of Φ and Lemma 4

$$\left|\left\{x\in \boldsymbol{R}^n:\ \left(M\frac{1}{\varphi_m(|A|)}\chi_A\right)(x) > t\right\}\right| = |\{x\in \boldsymbol{R}^n:\ (M\chi_A)(x) > \varphi_m(|A|)\,t\}|$$
$$\leqslant \Phi(1/t\varphi_m(|A|))|A| \leqslant c_0\,\Phi_0(1/t)\,\Phi_0(1/\varphi_m(|A|))|A| \leqslant c_0\,\Phi_0(1/t).$$

Fix the set E. We may assume that $|E| < \infty$. Then,

$$\left|\left\{x\in E:\ \left(M\frac{1}{\varphi_m(|A|)}\chi_A\right)(x) > t\right\}\right| \leqslant \begin{cases} c_0\,\Phi_0(1/t) = c_0(1/t) & \text{if } t \geqslant 1,\\ |E| \leqslant |E|/t & \text{if } t < 1.\end{cases}$$

Therefore,

$$\left|\left\{x\in E:\ \left(M\frac{1}{\varphi_m(|A|)}\chi_A\right)(x) > t\right\}\right| \leqslant \frac{\hat{c}}{t}, \quad \text{where} \quad \hat{c} = \max(c_0, |E|). \tag{2.1}$$

We will need the following known result (see [6] and [7], p. 97).

LEMMA 5 (E. Stein–N. Weiss). *Let g_k be a sequence of functions satisfying $|\{x\in E:\ |g_k(x)| > t\}| \leqslant K/t$, for some constant K independent of the index k, and let $c = \{c_k\}$. Then,*

$$|\{x\in E:\ |\sum c_k g_k(x)| > t\}| \leqslant \frac{6K}{t}N(c).$$

Let f be an element of B_m. For $k = 0, \pm 1, \pm 2, \ldots$ we define $A_k = \{x:\ 2^k < |f(x)| \leqslant 2^{k+1}\}$. Thus,

$$|f(\cdot)| \leqslant \sum_{k=-\infty}^{\infty} 2^{k+1}\chi_{A_k} = \sum_{k=-\infty}^{\infty} 2^{k+1}\varphi_m(|A_k|)\left(\frac{1}{\varphi_m(|A_k|)}\chi_{A_k}\right).$$

Using this and the sublinearity of the operator M, we obtain from (2.1), Lemma 5 and (1.2)

$$|\{x\in E:\ (Mf)(x) > t\}| \leqslant \frac{6\hat{c}}{t}N(\{2^{k+1}\varphi_m(|A_k|)\})$$
$$\leqslant \frac{12\hat{c}}{t}N(\{2^k\varphi_m(\lambda_f(2^k))\}) \leqslant c'\frac{\hat{c}}{t}\|f\|_{B_m}. \blacksquare$$

The proof of (i) $\Rightarrow$ (ii) in Theorem 3 is similar, with the only difference that the estimate (2.1) holds for $\hat{c} = c_0$ and $E = \boldsymbol{R}^n$.

In order to prove that (ii) implies (i) we pick an arbitrary subset A and we let $f = \chi_A$. Then, assuming (ii) is true, we have

$$|\{x\in \boldsymbol{R}^n:\ (Mf)(x) > t\}| \leqslant \frac{C}{t}\|f\|_{B_0} = \frac{C}{t}\int_0^1 |A|\left(1+\log\frac{1}{r}\right)dr = \frac{C}{t}|A|\,2,$$

and (i) holds with $c_0 = 2C$. $\blacksquare$

3. An application to a.e. convergence of Fourier series. The setting in this section could be any measure space, (X, μ), of finite measure but, for simplicity, we will asume that X is a subset of $\boldsymbol{R}^n$, $\mu = |\cdot|$ (the Lebesgue measure) and $|X| = 1$.

A measurable function $s(x)$ is said to be *special* if it is of the form $s(x) = i(x)\chi_E(x)$ for some subset E of X, and with $i(x)$ satisfying $1/2 < |i(x)| \leqslant 1$.

THEOREM 6. *Let T be a sublinear operator mapping functions in, say, $L^1(X)$ into measurable functions. Assume that, for $1 < p \leqslant 2$, T satisfies*

$$|\{x\in X:\ |(Ts)(x)| > t\}|^{1/p} \leqslant \frac{C}{(p-1)^m}\frac{1}{t}\|s\|_p, \tag{3.1}$$

where $m \geqslant 0$ and C is independent of p and the special function s. Then, for any function $f\in B_m(X)$ we have

$$|\{x\in X:\ |(Tf)(x)| > t\}| \leqslant \frac{C'}{t}\|f\|_{B_m} \tag{3.2}$$

(C' independent of f) and, therefore, T is bounded from B_m into weak-L^1.

An example of an operator T satisfying (3.1) with $m = 1$ is given by the maximal operator on the partial sums of Fourier series. For the ring of integers in a local field (which includes the Walsh–Paley or dyadic group) the proof can be seen in Hunt–Taibleson [3]. For the torus one can adapt Hunt's basic result in [2] about characteristic functions of sets, $f = \chi_E$, observing that the only property of such functions he uses is $|E| = \|f\|_p^p$, for $1 \leqslant p < \infty$. One can realize that the weaker assumption $\|f\|_1 \sim \|f\|_p^p$, $1 \leqslant p < \infty$, is enough and that this is clearly true for special functions.

COROLLARY 7. *Let X be the one-dimensional torus or the ring of integers in a local field. Then the Fourier series of any function in $B_1(X)$ converges almost everywhere.*

This represents a double improvement with respect to the well-known results in [3] and [4]. If $\psi(t) = t\log^+ t\log^+\log^+ t$, $t > 0$, we will see in the next section that the Orlicz class L_ψ is properly included in B_1. So, convergence holds for a larger class. On the other side, it was shown in [3] that if $J(f)=\int\psi(|f|)$ and $J(f)<1/2$, then there exist two absolute constants c_1, c_2 for which

$$|\{x\in X:\ (Tf)(x)>c_1 J(f)^{1/5}\}|\leqslant c_2 J(f)^{1/5},$$

while here we have the better estimate (3.2). The reader may compare this Theorem with Theorem (4.41) in [8], Vol. II, p. 119, and Theorem (3.1) in [1], p. 184.

Proof of the theorem. Let $s=i\chi_E$ be a special function. Take $p=1+[1+\log(1/|E|)]^{-1}$. Then $1<p\leqslant 2$ and, by (3.1) and the fact that $|X|=1$,

$$\left|\{x\in X:\ |(Ts)(x)|>t\}\right|\leqslant\frac{C}{t}\frac{1}{(p-1)^m}|E|^{1/p}$$

$$=\frac{C}{t}|E|\left(1+\log\frac{1}{|E|}\right)^m|E|^{-(1-1/p)}\leqslant\frac{eC}{t}\varphi_m(|E|);$$

the last inequality follows since $|E|\leqslant 1$ and

$$|E|^{-(1-1/p)}=\left(\frac{1}{|E|}\right)^{\frac{1}{2+\log 1/|E|}}\leqslant\left(\frac{1}{|E|}\right)^{\frac{1}{\log 1/|E|}}=e.$$

Thus,

$$\left|\left\{x:\ \left|\left(T\frac{1}{\varphi_m(|E|)}s\right)(x)\right|>t\right\}\right|\leqslant\frac{C'}{t}, \tag{3.3}$$

for every special function.

Now, let $f\in B_m$. For $k=0,\pm1,\pm2,\ldots$ we set $E_k=\{x\in X:\ 2^k<|f(x)|\leqslant 2^{k+1}\}$ and $f_k=f\chi_{E_k}$. Then

$$f=\sum_{-\infty}^{\infty}f_k=\sum_{-\infty}^{\infty}2^{k+1}\varphi_m(|E_k|)\frac{1}{2^{k+1}\varphi_m(|E_k|)}f_k.$$

Since each $s_k=2^{-(k+1)}f_k$ is a special function supported on E_k, we obtain from (3.3) and Lemma 5

$$\left|\{x\in X:\ |(Tf)(x)|>t\}\right|\leqslant\frac{6C'}{t}N\left(\{2^{k+1}\varphi_m(|E_k|)\}\right)$$

$$\leqslant\frac{12C'}{t}N\left(\{2^k\varphi_m(\lambda_f(2^k))\}\right)\leqslant\frac{C''}{t}\|f\|_{B_m}. \blacksquare$$

4. Final remark. In this section we show that if $f(x)$ is locally in $Y_m=L(\log^+L)^m\log^+\log^+L$ then it is locally in B_m. For that we may assume that $f\in Y_m$ and f has compact support, A. Observe that $f\in Y_m$ implies

$$\int_{e^2}^{\infty}\lambda_f(t)(\log t)^m\log\log t\,dt<\infty. \tag{4.1}$$

We may also assume $\|f\|_{\varphi_m}\leqslant 1$. Thus,

$$\|f\|_{B_m}\leqslant\Big(\int_{E_1}+\int_{E_2}+\int_{E_3}\Big)\varphi_m(\lambda_f(t))\left[1+\log^+\left(1/t\varphi_m(\lambda_f(t))\right)\right]dt=I_1+I_2+I_3$$

where $E_1=[0,e^2]$, $E_2=\{t>e^2:\ \lambda_f(t)<1/t(\log t)^{m+2}\}$ and $E_3=(e^2,\infty)-E_2$. Clearly,

$$I_1\leqslant\int_0^{e^2}\varphi_m(|A|)\left[1+\log(1/t\varphi_m(|A|))\right]dt\leqslant 2e^2\varphi_m(|A|)<\infty.$$

Also

$$I_2\leqslant\int_{e^2}^{\infty}\varphi_m\big(t^{-1}(\log t)^{-(m+2)}\big)\left[1+\log^+\frac{1}{t\varphi_m\big(t^{-1}(\log t)^{-(m+2)}\big)}\right]dt$$

$$\leqslant C_m\int_{e^2}^{\infty}\frac{\log\log t}{t(\log t)^2}dt<\infty,$$

and

$$I_3\leqslant\int_{e^2}^{\infty}\lambda_f(t)\left[1+\log\big(t(\log t)^{m+2}\big)\right]^m\left[1+\log(\log t)^{m+2}\right]dt$$

$$\leqslant C'_m\int_{e^2}^{\infty}\lambda_f(t)(1+\log t)^m(1+\log\log t)<\infty,\quad\text{(by (4.1))}.$$

Therefore, $f\in B_m$.

Now, if we consider a sequence $\{I_k\}$ of disjoint subintervals of $A=[0,1]$ and a numerical sequence $c=\{c_k\}$ it is not hard to see that the function $f=\sum c_k[\varphi_m(|I_k|)]^{-1}\chi_{I_k}$ belongs to $B_m(A)$ provided that $N(c)<\infty$. Given $\varepsilon>0$, set $G_m^\varepsilon(u)=u(\log^+u)^m(\log^+\log^+u)^\varepsilon$; for $k=1,2,\ldots$, let $c_k=k^{-(1+\varepsilon)}$ and choose I_k so that $|I_k|=c_k2^{-mk}2^{-2^k}$. Then

$$\int_A G_m^\varepsilon(|f(x)|)\,dx=\sum|I_k|\,G_m^\varepsilon\big(c_k[\varphi_m(|I_k|)]^{-1}\big)$$

$$\sim\sum k^{-(1+\varepsilon)}2^{-mk}2^{-2^k}G_m^\varepsilon(2^{2^k})\sim\sum k^{-1}=\infty.$$

This example shows that B_m is not contained in any of the spaces $L(\log^+L)^m(\log^+\log^+L)^\varepsilon$, $\varepsilon>0$, and, in particular, that $Y_m(A)$ is a proper subclass of $B_m(A)$ as mentioned in Section 3.

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On the capacity of a continuum with a non-dense orbit under a hyperbolic toral automorphism

by

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Abstract. In this paper we compute an upper and lower estimation for the capacity of a continuum (connected compact set) lying in the torus $T^n = R^n/Z^n$ whose orbit under a hyperbolic toral automorphism is not dense in T^n. Also estimations of capacity in Pesin's sense are considered.

Introduction. The main results.

1. First we define capacity. Let (X, ϱ) be a compact metric space and let A be any subset in X. Cover it with finitely many balls $\{B(x_i, r_i)\}_{i=1}^k$ with centres in A of radii $r_i \leqslant \varepsilon$. By $I(A, \varepsilon)$ denote the minimal possible k. The number

$$C_A = \limsup_{\varepsilon \to 0} \frac{\log I(A, \varepsilon)}{-\log \varepsilon}$$

is called the *capacity* of the set A. Observe that $\dim_H A \leqslant C_A$, where $\dim_H$ is the Hausdorff dimension and that

(1) *if* $\varepsilon_i \searrow 0$, $\limsup_{i \to \infty} (\varepsilon_i/\varepsilon_{i+1}) < +\infty$, *then*

$$C_A = \limsup_{i \to \infty} \frac{\log I(A, \varepsilon_i)}{-\log \varepsilon_i}.$$

2. Denote by $\pi\colon R^n \to R^n/Z^n$ the standard covering projection. A hyperbolic toral automorphism is a map $f\colon T^n \to T^n$ which has a linear lift $\tilde{f}\colon R^n \to R^n$ without eigenvalues of modulus 1. It is clear that there exists a minimal number $r \geqslant 1$ such that either the eigenvalues of $\tilde{f}^r$ are real and positive or they are not roots of real numbers. By $\bar{f}$ we denote $\tilde{f}^r$. We define

$$E_\lambda = \bigcup_{j=0}^{\infty} (\bar{f} - \lambda \mathrm{id})^{-j}(0)$$

if an eigenvalue λ of $\bar{f}$ is real and

$$E_\lambda = \Big(\bigcup_{j=0}^{\infty} (\hat{f} - \lambda \mathrm{id})^{-j}(0) \cup \bigcup_{j=0}^{\infty} (\hat{f} - \bar{\lambda} \mathrm{id})^{-j}(0)\Big) \cap R^n,$$