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A proof of the theorem of supports

by

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Dedicated to Professor Jan Mikusiński on the occasion of his seventieth birthday

Abstract. A proof of Lions' Theorem of Supports is obtained by reducing the theorem to an analogue of the Titchmarsh theorem, and then applying the Titchmarsh theorem in a form due to Mikusiński. An advantage of the proof is that analytic functions of several complex variables enter only at the end and in a particularly simple way.

§ **0. Preface.** The one-dimensional version of Lions' theorem of supports for distributions with compact support (Theorem I below) follows easily from the Titchmarsh theorem on convolution products. In more than one dimension the proof of the theorem of supports is more sophisticated. It is the case of more than one dimension that we shall consider. In the present proof we show first that Lions' theorem in several variables is equivalent to the formulation given below in Theorem I'', which looks very much like the one-dimensional Titchmarsh theorem. Finally, in the last section, Theorem I'' is shown to follow from Mikusiński's proof of the one-dimensional Titchmarsh theorem.

An advantage of this proof of Lions' theorem is that the theory of analytic functions of several complex variables appears only at the very end of the proof an in a particularly simple form. Only two properties of analytic functions of several complex variables are used; the first is the property that the product of two entire functions of several complex variables vanishes identically only if one of the two factors vanishes identically. The second property is the uniqueness of the Laplace transform of a function of several variables.

In reducing Lions' theorem to the equivalent form given in Theorem I'' two properties of convex sets are assumed. The first property is that in a finite-dimensional space the convex hull of a compact set is compact. The second property (also used only in a finite-dimensional space, but true more broadly) is that a convex compact set is equal to the intersection of the closed half-spaces which contain the set.

Besides the properties mentioned above we use only elementary properties of distributions.

§ 1. **Introduction.** Let $\mathscr{E}'(\boldsymbol{R}^N)$ be the space of distributions with compact support, and $\mathscr{D}(\boldsymbol{R}^N)$ the space of infinitely differentiable functions with compact support. The theorem we shall prove is due to Lions [1].

We shall denote the convex hull of $K \subset \boldsymbol{R}^N$ by $[K]$. If $K \subset \boldsymbol{R}^N$ is compact then $[K]$ is compact. We shall denote convolution by $*$.

THEOREM I (Lions). *If S and T are in $\mathscr{E}'(\boldsymbol{R}^N)$ then*

$$[\operatorname{Supp} S * T] = [\operatorname{Supp} S] + [\operatorname{Supp} T].$$

This theorem is closely related to a theorem of Titchmarsh.

THEOREM II (Titchmarsh). *If f and g are continuous on $[0, T]$ and $(f*g)(t) = \int_0^t f(t-u)g(u)\,du = 0$ for all $t \in [0, T]$ then there exist T_1 and T_2 such that*

$$f(t) \equiv 0, \quad t \in [0, T_1],$$
$$g(t) \equiv 0, \quad t \in [0, T_2],$$

and $T_1 + T_2 \geqslant T$.

Indeed, in one dimension Theorem I follows from II. It might be hoped that once Theorem I is reduced to a standard form (Theorem I″ below), it would follow from the theorem of Titchmarsh by taking partial Laplace transforms. This does not seem to be the case; however, in his proof of the Titchmarsh Theorem [2] Mikusiński proves more than is stated in the Titchmarsh Theorem, namely Theorem III. We point out that the theorem of supports (in the standard form) follows directly from the theorem of Mikusiński by taking partial Laplace transforms. We first show that Theorem I is equivalent to Theorem I′ below.

§ 2. **Regularization.** The *closed ball with center at x and radius $\varepsilon > 0$* is the set $B_\varepsilon(x) = \{y|\ y \in \boldsymbol{R}^N, \|y-x\| \leqslant \varepsilon\}$. For $B_\varepsilon(0)$ we simply write B_ε. An *approximate identity* is a set of functions $\{\varphi_\varepsilon|\ \varepsilon > 0\}$ with the properties

(i) $\varphi_\varepsilon \in \mathscr{D}(B_{\varrho(\varepsilon)})$ where $\varrho(\varepsilon) \to 0$ as $\varepsilon \to 0$,

(ii) $\varphi_\varepsilon(x) \geqslant 0$ for all $x \in \boldsymbol{R}^N$, $\varepsilon > 0$,

(iii) $\int_{\boldsymbol{R}^N} \varphi_\varepsilon(x)\,dx = 1$ for all $\varepsilon > 0$.

If φ_ε is an approximate identity then $\check{\varphi}_\varepsilon(x) = \varphi_\varepsilon(-x)$ is also an approximate identity.

We state the following two well-known lemmas but omit the proofs.

LEMMA 1. *If φ and θ are in $\mathscr{D}(\boldsymbol{R}^N)$ then*

$$\operatorname{Supp} \varphi * \theta \subset \operatorname{Supp} \varphi + \operatorname{Supp} \theta.$$

LEMMA 2. *If φ_ε is an approximate identity and $\theta \in \mathscr{D}$ then $\varphi_\varepsilon * \theta \to \theta$ with convergence in $\mathscr{D}$ as $\varepsilon \to 0$.*

Lemma 2 justifies the name "approximate identity".

A *regularization* of $T \in \mathscr{D}'$ is a set of functions $\{T_\varepsilon = T * \varphi_\varepsilon|\ \varepsilon > 0\}$ where φ_ε is an approximate identity. For any $T \in \mathscr{D}'(\boldsymbol{R}^N)$, T_ε lies in $C^\infty(\boldsymbol{R}^N)$ and the support of T_ε is close to the support of T in a sense made precise by the following lemma.

LEMMA 3. *Let T_ε be a regularization of $T \in \mathscr{E}'$. For $\varepsilon > 0$*

(1) $$\operatorname{Supp} T_\varepsilon \subset \operatorname{Supp} T + B_{\varrho(\varepsilon)},$$

(2) $$\operatorname{Supp} T \subset \operatorname{Supp} T_\varepsilon + B_{\delta(\varepsilon)},$$

where $\varrho(\varepsilon)$ is given by (i) *above and $\varrho(\varepsilon)$ and $\delta(\varepsilon)$ tend to zero as ε tends to zero.*

Proof. (1). We shall show that if $\operatorname{distance}(x, \operatorname{Supp} T) > \varrho(\varepsilon)$ then $x \notin \operatorname{Supp} T_\varepsilon$. Suppose the distance from x to $\operatorname{Supp} T$ is $\varrho(\varepsilon) + a$ where $a > 0$. Then for $\theta \in \mathscr{D}(B_{a/2}(x))$ we have

$$\langle T_\varepsilon, \theta\rangle = \langle T * \varphi_\varepsilon, \theta\rangle = \langle T, \check{\varphi}_\varepsilon * \theta\rangle$$

and since $\operatorname{Supp} \check{\varphi}_\varepsilon * \theta \subset B_{\varrho(\varepsilon)+a/2}(x)$, which is disjoint from $\operatorname{Supp} T$, $T = 0$ on a neighborhood of x. Thus $x \notin \operatorname{Supp} T$.

(2). If (2) does not hold, there is a $\delta > 0$, and two sequences, $\varepsilon_n \to 0$, and $x_n \in \operatorname{Supp} T$, such that

$$\operatorname{distance}(x_n, \operatorname{Supp} T_{\varepsilon_n}) > \delta \quad \text{for all } n.$$

By compactness there is a subsequence of x_n convergent to some $x \in \operatorname{Supp} T$, and for an infinite number of $\varepsilon_n > 0$, $\operatorname{distance}(x, \operatorname{Supp} T_{\varepsilon_n}) > \delta/2$.

If $\theta \in \mathscr{D}(B_{\delta/4}(x))$ then, since $\check{\varphi}_\varepsilon$ is an approximate identity,

$$\langle T, \theta\rangle = \lim_{n\to\infty} \langle T, \check{\varphi}_{\varepsilon_n} * \theta\rangle = \lim_{n\to\infty} \langle T_{\varepsilon_n}, \theta\rangle.$$

But since $\operatorname{Supp} \theta$ is disjoint from $\operatorname{Supp} T_{\varepsilon_n}$ for an infinite number of n it follows that $\langle T, \theta\rangle = 0$. Thus $T = 0$ on a neighborhood of x contradicting the fact that $x \in \operatorname{Supp} T$.

Inclusions (1) and (2) imply that it is sufficient to prove the theorem of supports for $T = \varphi$, $S = \theta$, with φ and θ in $\mathscr{D}$. In fact,

THEOREM. *The inclusions*

(1′) $$[\operatorname{Supp} S * T] \subset [\operatorname{Supp} T] + [\operatorname{Supp} S],$$

(2′) $$[\operatorname{Supp} S * T] \supset [\operatorname{Supp} T] + [\operatorname{Supp} S]$$

hold for all T and S in $\mathscr{E}'(\boldsymbol{R}^N)$ if they hold for all regularizations T_ε and S_ε of such distributions.

Proof. (1′). For each $\varepsilon > 0$ it follows from (2) that

$$[\operatorname{Supp} S * T] \subset [\operatorname{Supp} S_\varepsilon * T_\varepsilon] + B_{\delta(\varepsilon)}.$$

Applying (1′) to $S_\varepsilon * T_\varepsilon$ and using the inclusion (1) gives

$$[\operatorname{Supp} S * T] \subset [\operatorname{Supp} S] + [\operatorname{Supp} T] + B_{2\varrho(\varepsilon)+\delta(\varepsilon)}$$

for all $\varepsilon > 0$. Taking the intersection of the sets on the right yields the results.

(2′). A similar application of (1) and (2) gives (2′).

§ 3. **Reduction to standard form.** Since the inclusion one way follows from Lemma 1 we need only prove the following.

THEOREM I′. *If* φ *and* θ *are in* $\mathscr{D}(\boldsymbol{R}^N)$ *then*

(3) $$[\operatorname{Supp} \varphi * \theta] \supset [\operatorname{Supp} \varphi] + [\operatorname{Supp} \theta].$$

Suppose $\zeta \in \boldsymbol{R}^N$, $|\zeta| = 1$, $T \in \boldsymbol{R}$. A *closed half-space* is a set $H_\zeta(T) = \{x|\ \langle x, \zeta\rangle \geqslant T\}$. If $K \subset \boldsymbol{R}^N$ is compact then $[K]$ is also compact and

$$[K] = \bigcap_{H_\zeta(T) \supset K} H_\zeta(T).$$

LEMMA 3. *To prove the inclusion* (3) *it is sufficient to show that*

(4) $$H_\zeta(T) \supset \operatorname{Supp} \varphi * \theta$$

implies there exist T_1 *and* T_2 *with* $T_1 + T_2 \geqslant T$ *and*

(5) $$H_\zeta(T_1) \supset \operatorname{Supp} \varphi,$$

(6) $$H_\zeta(T_2) \supset \operatorname{Supp} \theta.$$

Proof. If a closed half-space contains K it also contains $[K]$. Thus for $x \in [\operatorname{Supp} \varphi]$ and $y \in [\operatorname{Supp} \theta]$

$$\langle x+y, \zeta\rangle = \langle x, \zeta\rangle + \langle y, \zeta\rangle \geqslant T_1 + T_2 \geqslant T.$$

Therefore $H_\zeta(T) \supset [\operatorname{Supp} \varphi] + [\operatorname{Supp} \theta]$. Since $[\operatorname{Supp} \varphi * \theta]$ is the intersection of such half-spaces the assumption of the lemma gives $[\operatorname{Supp} \varphi * \theta] \supset [\operatorname{Supp} \varphi] + [\operatorname{Supp} \theta]$. Thus the proof is complete.

For $\varphi \in \mathscr{D}$, $\sigma \in \boldsymbol{R}^N$ define φ_σ by the equation

$$\varphi_\sigma(t) = \varphi(t - \sigma), \quad t \in \boldsymbol{R}^N.$$

It then follows that for $\varphi, \theta \in \mathscr{D}$, $\sigma, \tau \in \boldsymbol{R}^N$

(7) $$\varphi_\sigma * \theta_\tau = (\varphi * \theta)_{\sigma+\tau}.$$

For any $\varphi \in \mathscr{D}$ the linear function $\langle x, \zeta\rangle$ has a minimum, say T_1, on $\operatorname{Supp} \varphi$. Thus for any $\varphi, \theta \in \mathscr{D}$ and $\zeta \in \boldsymbol{R}^N$, $|\zeta| = 1$ there exist three numbers T, T_1, and T_2 satisfying (4), (5) and (6). It is only necessary to check that $T_1 + T_2 \geqslant T$.

Since for any $\alpha \in \boldsymbol{R}$

$$\alpha\zeta + H_\zeta(T) = H_\zeta(T+\alpha)$$

and

$$\alpha\zeta + \operatorname{Supp} \varphi = \operatorname{Supp} \varphi_{\alpha\zeta}$$

in view of (7) *it is sufficient to verify Lemma* 3 *for* T, T_1, $T_2 > 0$.

For $\varphi \in \mathscr{D}$, $\mathscr{R}$ a rotation about the origin in $\boldsymbol{R}^N$, define $\varphi_{\mathscr{R}}$ by the equation

$$\varphi_{\mathscr{R}}(t) = \varphi(\mathscr{R}^{-1} t), \quad t \in \boldsymbol{R}^N.$$

It then follows that for $\varphi, \theta \in \mathscr{D}(\boldsymbol{R}^N)$,

$$\varphi_{\mathscr{R}} * \theta_{\mathscr{R}} = (\varphi * \theta)_{\mathscr{R}}.$$

Since

$$\mathscr{R} H_\zeta(T) = H_{\mathscr{R}\zeta}(T)$$

and

$$\mathscr{R} \operatorname{Supp} \varphi = \operatorname{Supp} \varphi_{\mathscr{R}}, \quad \varphi \in \mathscr{D},$$

it is sufficient to verify Lemma 3 *for* T, T_1, $T_2 \geqslant 0$ *and* $\zeta = (1, 0, \ldots, 0)$.

Thus we need only prove

THEOREM I″. *If* φ *and* θ *are in* $\mathscr{D}(\boldsymbol{R}^N)$, *and* $\operatorname{Supp} \varphi$ *and* $\operatorname{Supp} \theta$ *are contained in* $\{t|\ t_1 \geqslant 0\}$, *then*

$$(\varphi * \theta)(t) = 0 \quad \textit{for } 0 \leqslant t_1 \leqslant T$$

implies there exist T_1 *and* T_2 *such that*

$$\varphi(t) = 0 \textit{ for } 0 \leqslant t_1 \leqslant T_1, \quad \theta(t) = 0 \textit{ for } 0 \leqslant t_1 \leqslant T_2,$$

where $T_1 + T_2 \geqslant T$.

§ 4. **Proof of the Theorem.** The proof of Theorem I″ will use partial Laplace transforms. Denote a point in $\boldsymbol{R}^{N-1}$ by $t' = (t_2, t_3, \ldots, t_N)$, thus $t = (t_1, t') \in \boldsymbol{R}^N$. For $z = (z_1, \ldots, z_{N-1}) \in \boldsymbol{C}^{N-1}$ let

$$\langle z, t'\rangle = z_1 t_2 + \ldots + z_{N-1} t_N.$$

The partial Laplace transform of $\varphi \in \mathscr{S}(\boldsymbol{R}^N)$ is

$$\varphi_z(t_1) = \int_{\boldsymbol{R}^{N-1}} e^{-\langle z, t'\rangle} \varphi(t)\, dt'.$$

In [2, Appendix] Mikusiński proves the Titchmarsh theorem in the following form.

THEOREM III (Mikusiński). *If* f *and* $g \in C[0, T]$ *and* $(f * g)(t) = 0$ *for* $0 \leqslant t \leqslant T$, *then we have*

$$f(\sigma) g(\tau) = 0 \quad \textit{for all } \sigma \geqslant 0,\ \tau \geqslant 0,\ \sigma + \tau \leqslant T.$$

The Titchmarsh Theorem follows immediately by letting T_1 be the largest value such that f vanishes identically on $[0, T_1]$. Then g must vanish on $[0, T-T_1]$.

Theorem I″ also follows from Mikusiński's Theorem. Indeed, taking partial Laplace transforms we have that, for each $z \in C^{N-1}$, the functions $\varphi_z(t_1)$ and $\theta_z(t_1)$ satisfy the assumption $(\varphi_z * \theta_z)(t_1) = 0$ for $0 \leqslant t_1 \leqslant T$. Thus for each $z \in C^{N-1}$, by Mikusiński's Theorem,

$$\varphi_z(\sigma)\theta_z(\tau) = 0 \quad \text{for all } \sigma \geqslant 0,\ \tau \geqslant 0,\ \sigma+\tau \leqslant T.$$

Let T_1 be the largest value such that $\varphi_z(\sigma)$ is zero for all $\sigma \leqslant T_1$ and all $z \in C^{N-1}$. If $T_1 < T$ there is a σ_1 larger than T_1 and arbitrarily close to T_1, and a $z' \in C^{N-1}$ such that $\varphi_{z'}(\sigma_1) \neq 0$. For all $\tau_1 \leqslant T-\sigma_1$ we have

$$\varphi_z(\sigma_1)\theta_z(\tau_1) = 0 \quad \text{for all } z.$$

Since the product of two entire functions of z is zero only if one of them is zero, it follows that $\theta_z(\tau_1) = 0$ for all z. Since $\sigma_1 > T_1$ can be taken arbitrarily close to $T_2 = T-T_1$,

$$\varphi_z(t_1) = 0, \quad 0 \leqslant t_1 \leqslant T_1,\ z \in C^{N-1},$$
$$\theta_z(t_1) = 0, \quad 0 \leqslant t_1 \leqslant T_2,\ z \in C^{N-1},$$

and $T_1 + T_2 = T$. By the uniqueness of the Laplace transform we have the result of Theorem I″.

References

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For a Banach space isomorphic to its square the Radon–Nikodým property and the Krein–Milman property are equivalent

by

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Abstract. We prove the result announced in the title.

0. Introduction. For the definition of the Radon–Nikodým property and the Krein–Milman property (abbreviated RNP and KMP) we refer to [5]. In the past ten years some effort has been made to prove or disprove the conjecture that these two notions are equivalent. For some classes of Banach spaces it has been shown that they are equivalent (see [7] and [3]).

There is an intermediate notion between RNP and KMP, namely the "Integral Representation Property" (abbr. IRP) [13]: A Banach space X (which we suppose to be separable to avoid measure-theoretic complications, in which we are not interested here) has IRP if for every bounded, closed, convex set $C \subseteq X$ and every $x \in C$ there is a probability measure μ on the extreme points of C such that x is the barycenter of μ.

A theorem due to Edgar shows that RNP $\Rightarrow$ IRP ([5], p. 145) and it is easy to see that IRP $\Rightarrow$ KMP. The converse implications are open.

E. Thomas [13] has shown that a Fréchet space X has RNP iff X^N has IRP. In the context of Banach spaces this may be formulated as follows: A Banach space X has RNP iff $l^2(X)$ (or any other appropriate space of sequences in X) has IRP.

It was observed in [11] that if X is isomorphic to its square then $l^2(X)$ has IRP iff X has IRP, i.e. in this case IRP and RNP are equivalent.

In the present paper we obtain — inspired by the argument of E. Thomas but using quite a different reasoning and an observation due to H. P. Rosenthal — analogous results for KMP in place of IRP.

The aim of the introductory Section 1 is to establish Corollary 1.3; this result — or at least some variant of it — seems to be known to people working in the field and its content can be derived from a construction of J. Bourgain [1]. However, we have preferred to use an approach due to C. Stegall [12]. This approach seems at the same time elementary and powerful; it also shows that the pathologies arising in the absence of RNP need not to be "constructed" but are already contained in any nonrepresentable operator from $L^1[0, 1]$ to X.