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Unique continuation for Schrödinger equations in dimensions three and four

by

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Abstract. The unique continuation property for solutions of the Schrödinger equations (or inequalities) in dimensions three and four is proved. The L_p-conditions assumed for the strong uniqueness are shown to be optimal. The main result follows from the Carleman type inequalities, which are obtained from L_p-estimates of integral operators.

1. Introduction. The study of the unique continuation property of solutions to differential equations with nonanalytic coefficients began with a paper by Carleman [4] (in 2 dimensions), in which he introduced the important method of weighted inequalities. Then the unique continuation theorem was proved for the second order elliptic equations under the assumption that the coefficients of the lower order terms are of class L_∞. The case where the leading part is the Laplacian was considered by Müller [12] and Heinz [8] while Aronszajn [2] and Cordes [5] studied equations whose leading part had variable coefficients. The sufficiency of the Lipschitz continuity was shown by Hörmander [9] and in a stronger form by Aronszajn, Krzywicki and Szarski [3]. An example given by Pliś [13] shows that it is almost a necessary condition.

There are partial results on the unique continuation for the equations with unbounded lower order coefficients. Sufficient conditions are given e.g. by Schechter and Simon [14], Amrein, Berthier and Georgescu [1] and Hörmander [10]. Our result on the strong uniqueness concerns rather special cases but the L_p-conditions on the coefficients are sharp.

In this paper we consider differential inequalities of the form

$$|\Delta u(x)| \leqslant a(x)\,|u(x)| \tag{1.1}$$

where the function $a(x)$ is assumed to be of class $L_{\sigma,\mathrm{loc}}$, u is a function on a connected subset Ω of $\boldsymbol{R}^n$ $(n \geqslant 3)$, and Δ is the Laplacian. Without loss of generality one may replace inequality (1.1) by the Schrödinger equation $\Delta u + Vu = 0$ with $|V(x)| \leqslant a(x)$.

We prove the strong uniqueness for solutions of (1.1) under the assumption $\sigma \geqslant n/2$ for $n = 3$ or 4. This improves a result of [1], where it is assumed that $\sigma > n/2$. We also give some examples that show the theorem does not hold for $\sigma < n/2$ $(n \geqslant 3)$.

We use the Carleman type inequalities of the form

$$(1.2)\qquad \Big(\int \big||x|^{-k} v(x)\big|^q dx\Big)^{1/q} \leqslant C\Big(\int \big||x|^{-k} \Delta v(x)\big|^p dx\Big)^{1/p}$$

where v is a function of class $C_0^\infty(\boldsymbol{R}^n - \{0\})$ and $k = 1, 2, \ldots$ The uniqueness theorems are derived from (1.2) in a routine manner (Lemma 1.1).

DEFINITION 1.1. A function u is said to have *zero of infinite order in L_q-sense* at $x_0 \in \Omega$ if for any positive integer k the functions $|x-x_0|^{-k} u(x)$ are of class $L_{q,\mathrm{loc}}$. If $q < \infty$ can be arbitrary we say that u has *zero of infinite order.*

One says that (1.1) has the *strong unique continuation property* if every function u satisfying (1.1) and having zero of infinite order must vanish identically. Inequality (1.1) has the *unique continuation property* if every function u satisfying (1.1) and equal to zero on some open set is identically zero on Ω.

By saying that a function u satisfies (1.1) we mean here that $u \in L_{q,\mathrm{loc}}$ for any $q < \infty$, Δu (distributionally) is a locally integrable function and inequality (1.1) is fulfilled almost everywhere in Ω.

The following lemma is proved in Section 6.

LEMMA 1.1. *Let $a \in L_{\sigma,\mathrm{loc}}$, $1 \leqslant \sigma \leqslant \infty$. Inequality* (1.1) *has the strong unique continuation property provided there exist real numbers p, q, C, R such that $1 \leqslant p, q < \infty$, $1/\sigma = 1/p - 1/q$, $R > 0$ and inequality* (1.2) *holds for all $v \in C_0^\infty(B_R - \{0\})$ where $B_R = \{x\colon |x| < R\}$.*

We prove (1.2) with $1/p - 1/q = 2/n$ $(n = 3, 4)$. The proof is based on L_p-estimates of some majorizing integral operators given in Section 3. (The case $n = 4$ was presented in [15]). The case $n > 4$ cannot be treated in the same way because of the lack of an appropriate majorization. The discussion of this case will be given elsewhere.

Remark 1.1. By a homothetic change of variable in (1.2) we conclude that if inequality (1.2) holds for all $v \in C_0^\infty(B_{R_0} - \{0\})$ with a constant $C = C_0$ then (1.2) holds for all $v \in C_0^\infty(B_R - \{0\})$ with the constant $C = C_0 (R/R_0)^{2-n(1/p-1/q)}$. Hence it follows that inequality (1.2) cannot be proved if $1/p - 1/q > n/2$. Moreover, if $1/p - 1/q < n/2$ then the constant C may be taken arbitrarily small.

Remark 1.2. The condition that u has zero of infinite order in the L_q-sense at $x_0 \in \Omega$ is equivalent to the following: for any positive integer k

$$(1.3)\qquad \int\limits_{|x-x_0|<\varepsilon} |u(x)|^q dx = O(\varepsilon^k) \quad \text{as } \varepsilon \to 0.$$

2. Estimates of Taylor remainders. Now we prove some estimates of Taylor remainders which give us estimates of the kernels of the integral operators involved in our considerations.

We introduce the following operators in the complex domain:

$$(2.1)\qquad J_1 f(z) = \operatorname{Re} 2\pi^{-1} \int_0^1 (1-t^2)^{-1/2} f(\operatorname{Re} z + it \operatorname{Im} z)\, dt,$$

$$(2.2)\qquad J_2 f(z) = \operatorname{Re} \int_0^1 f(\operatorname{Re} z + it \operatorname{Im} z)\, dt,$$

which allow us to reduce the problem to an investigation of the complex geometric series.

LEMMA 2.1. (i) *$J_m \operatorname{Re}(z^l)$ $(m = 1, 2)$ are homogeneous polynomials of degree l in the variable z $(l = 0, 1, \ldots)$.*

(ii) *If $f(z)$ is an analytic function with real values for real arguments then*

$$(2.3)\qquad J_2\big(f'(z)\big) = \operatorname{Im} f(z)/\operatorname{Im} z \qquad (\operatorname{Im} z \neq 0).$$

(iii) *The following equalities hold:*

$$(2.4)\qquad J_1\big((1-z)^{-1}\big) = \operatorname{sign}(1-\operatorname{Re} z)\, |1-z|^{-1},$$

$$(2.5)\qquad J_2\big((1-z)^{-2}\big) = |1-z|^{-2}.$$

(iv) *The following inequality is true:*

$$(2.6)\qquad J_1(|1-z|^{-1}) \leqslant 2\left[\log\left(1+\left|\frac{\operatorname{Im} z}{1-\operatorname{Re} z}\right|\right)+1\right]|1-z|^{-1}.$$

Proof. It is easy to check (i) and (ii), then (2.5) follows from (ii). We get (2.4) substituting $\alpha = \operatorname{Im} z/(1-\operatorname{Re} z)$ in the formula

$$2\pi^{-1} \int_0^1 (1-t^2)^{-1/2} (1+\alpha^2 t^2)^{-1}\, dt = (1+\alpha^2)^{-1/2}$$

which can be obtained by means of the well-known methods of integration.

(iv): The following inequality is obtained by means of integration by parts and by applying the estimate $\arcsin t \leqslant \pi t/2$:

$$2\pi^{-1} \int_0^1 (1-t^2)^{-1/2} (1+\alpha^2 t^2)^{-1/2}\, dt \leqslant (1+\alpha^2)^{-1/2} + \int_0^1 \alpha^2 t^2 (1+\alpha^2 t^2)^{-3/2}\, dt.$$

For the last term we have

$$\int_0^1 |\alpha|\, t (1+\alpha^2 t^2)^{-1}\, dt = \tfrac{1}{2} |\alpha|^{-1} \log(1+\alpha^2)$$
$$\leqslant (1+\alpha^2)^{-1/2} + 2\log(1+|\alpha|)(1+\alpha^2)^{-1/2}.$$

Putting $\alpha = \operatorname{Im} z/(1-\operatorname{Re} z)$ in these inequalities we get (iv).

One can observe that the operator J_1 can be expressed as a composition of the operator of fractional integration of order $\frac{1}{2}$ along the imaginary axis and some elementary operators.

LEMMA 2.2. *Let $\Phi_{m,k}$ be the k-th Taylor polynomial at the origin of the function* $1/|1-z|^m$ $(m = 1, 2,\ k = 0, 1, \ldots)$. *Then*

$$\left||1-z|^{-1}-\Phi_{1,k}(z)\right| \leqslant 2\left[2+\log\left(1+\left|\frac{\operatorname{Im} z}{1-\operatorname{Re} z}\right|\right)\right]\frac{|z|^{k+1}}{|1-z|}, \tag{2.7}$$

$$\left||1-z|^{-2}-\Phi_{2,k}(z)\right| \leqslant |z|^{k+2}|1-z|^{-1}|\operatorname{Im} z|^{-1}. \tag{2.8}$$

Proof. By Lemma 2.1 (i)–(iii)

$$\Phi_{1,k}(z) = J_1(1+\ldots+z^k), \qquad \Phi_{2,k}(z) = J_2\left(\frac{d}{dz}(1+\ldots+z^{k+1})\right).$$

If $\operatorname{Re} z < 1$ we have

$$|1-z|^{-1}-\Phi_{1,k}(z) = J_1\left(z^{k+1}(1-z)^{-1}\right)$$

and for $\operatorname{Re} z > 1$

$$|1-z|^{-1}-\Phi_{1,k}(z) = J_1\left(z^{k+1}(1-z)^{-1}\right)+2|1-z|^{-1}.$$

Then (2.7) is a consequence of (2.6) and the obvious properties of the operator J_1.

Estimate (2.8) follows immediately from the identity

$$|1-z|^{-2}-\Phi_{2,k}(z) = (\operatorname{Im} z)^{-1}\operatorname{Im}\left(z^{k+2}(1-z)^{-1}\right),$$

which results from (2.5) and (2.3).

LEMMA 2.3. *Let $h_{m,k}$ be the kth Taylor polynomial at the origin of the function $x \mapsto |x-y|^{-m}$ in $\boldsymbol{R}^{m+2}$ $(y \neq 0)$. Then*

(i) *$h_{m,k}(x, y)$ is harmonic in the variable y,*

(ii) *$h_{m,k}(x, y) = |y|^{-m}\Phi_{m,k}(z)$, where $\operatorname{Re} z = xy|y|^{-2}$, $\operatorname{Im} z = \left(x^2y^2-(xy)^2\right)^{1/2}|y|^{-2}$, and $\Phi_{m,k}$ are the polynomials introduced in Lemma 2.2.*

(iii) $\left||x-y|^{-1}-h_{1,k}(x, y)\right|$

$$\leqslant 2\left[\log\left(1+\frac{|x^2y^2-(xy)^2|^{1/2}}{|y^2-xy|}\right)+2\right]\left(\frac{|x|}{|y|}\right)^{k+1}|x-y|^{-1},$$

$$\left||x-y|^{-2}-h_{2,k}(x, y)\right| \leqslant \frac{|x|}{|x-y|\left(x^2y^2-(xy)^2\right)^{1/2}}\left(\frac{|x|}{|y|}\right)^{k+1}.$$

Proof. (i) is a consequence of the harmonicity of the derivatives of $|x-y|^{-m}$.

For any $\lambda > 0$

$$h_{m,k}(\lambda x, \lambda y) = \lambda^{-m}h_{m,k}(x, y), \tag{2.9}$$

and for any orthogonal matrix A

$$h_{m,k}(Ax, Ay) = h_{m,k}(x, y). \tag{2.10}$$

There exists an orthogonal matrix A such that $Ay = |y|(1, 0, \theta)$ and $Ax = |y|(\operatorname{Re} z, \operatorname{Im} z, \theta)$ (θ is the origin in $\boldsymbol{R}^m$). Now by (2.9) and (2.10) and the obvious equality

$$\Phi_{m,k}(z) = h_{m,k}\left((\operatorname{Re} z, \operatorname{Im} z, \theta), (1, 0, \theta)\right)$$

we get (ii).

Substituting the identities $|z-1| = |x-y|/|y|$, $|z| = |x|/|y|$ and $\Phi_{m,k}(z) = |y|^m h_{m,k}(x, y)$ into (2.7) and (2.8) we obtain (iii).

3. L_p-estimates of the majorizing operators. We are interested in L_p-estimates of the integral operators M_1, M_2 in $\boldsymbol{R}^n$ given formally by

$$M_1 f(x) = \int k\left(\frac{|x^2y^2-(xy)^2|^{1/2}}{|y^2-xy|}\right)|x-y|^{-n/\sigma}f(y)\,dy \tag{3.1}$$

where $k(t) = 2\log(1+t)+4$,

$$M_2 f(x) = \int |x-y|^{-n/\sigma_1}\left[|x|^{-1}\left(x^2y^2-(xy)^2\right)^{1/2}\right]^{-n/\sigma_2}f(y)\,dy \tag{3.2}$$

where $1 < \sigma$, $\sigma_1 < \infty$ and $1 < \sigma_2 \leqslant \infty$, $n \geqslant 3$.

Remark that $|x|^{-1}\left(x^2y^2-(xy)^2\right)^{1/2}$ is the distance from the point y to the straight line spanned by x and the origin.

First we prove two technical lemmas. If F is a function on the interval $[0, 2]$ we denote by $\|F|S^{n-1}\|_\sigma$ the $L_\sigma(S^{n-1})$-norm of the function $\xi \mapsto F(|\xi_0-\xi|)$ (this norm does not depend on ξ_0). We shall use the following formula $(n = 2, 3, \ldots)$

$$\int_{S^{n-1}} F(|\xi_0-\xi|)\,d\omega_\xi = c_{n-1}\int_0^2 F(t)\,t^{n-2}\left(1-\tfrac{1}{4}t^2\right)^{(n-3)/2}dt, \tag{3.3}$$

where $d\omega$ is the surface element of S^{n-1} and c_n is the surface measure of the unit sphere S^{n-1}, $\xi_0 \in S^{n-1}$.

This formula can easily be derived e.g. from the integral formula for a plane wave function over the unit sphere ([11], Ch.I).

LEMMA 3.1. *Let $k(t) \geqslant 0$ be an increasing function on $\boldsymbol{R}_+$, $n \geqslant 3$ and $\sigma > 1$. Put*

$$k_{r,\varrho}(t) = k\left(2t|t^2+\nu|^{-1}\right)\left((r-\varrho)^2+r\varrho t^2\right)^{-n/2\sigma}$$

where $t, r, \varrho > 0$ and $\nu = 2(\varrho-r)/r$. There exists a constant $C = C(\sigma, n)$ such that for any $\alpha, \beta \geqslant 0$, $\alpha+\beta = n-1$

$$\|k_{r,\varrho}|S^{n-1}\|_\sigma^\sigma \leqslant C\varkappa_\sigma^\sigma|r-\varrho|^{-1}r^{-\alpha}\varrho^{-\beta} \tag{3.4}$$

where

$$\varkappa_\sigma = \left(\int_0^\infty k^\sigma(t)(t^2+1)^{-1}dt\right)^{1/\sigma}.$$

Proof. By (3.3)

$$\| k_{r,\varrho} | S^{n-1} \|_\sigma^\sigma \leqslant c_{n-1} \int_0^2 k_{r,\varrho}^\sigma t^{n-2}\, dt.$$

Put

$$k_v(t) = k^\sigma\big(2t\,|t^2 - |v||^{-1}\big)(v^2+t^2)^{-1}.$$

Denote by I_1, I_2 the integrals of the function $k_v(t)$ over the intervals $0 \leqslant t \leqslant (\frac{1}{2}|v|)^{1/2}$ and $(\frac{1}{2}|v|)^{1/2} \leqslant t \leqslant 2$, respectively.

If $t \leqslant (\frac{1}{2}|v|)^{1/2}$ then $t\,|t^2-|v||^{-1} \leqslant 2t\,|v|^{-1}$. By the change of the variable $t = s|v|$ and the last estimate we obtain

$$I_1 \leqslant |v|^{-1} \int_0^\infty k^\sigma(4s)(1+s^2)^{-1}\, ds.$$

If $(\frac{1}{2}|v|)^{1/2} \leqslant t \leqslant 2$ then $(v^2+t^2)^{-1} \leqslant 2|v|^{-1}$. Since $t\,|t^2-|v||^{-1} \leqslant |t-|v|^{1/2}|^{-1}$ then, we have

$$I_2 \leqslant 2|v|^{-1} \int_{-2}^2 k^\sigma(2|\tau|^{-1})\, d\tau \leqslant C\varkappa_\sigma^\sigma |v|^{-1}.$$

Adding I_1 and I_2 we get

$$\int_0^2 k_v(t)\, dt \leqslant C\varkappa_\sigma^\sigma |v|^{-1}. \tag{3.5}$$

If $\frac{1}{2} \leqslant \varrho/r \leqslant 2$ then $|v| \leqslant 2\sqrt{2}\,|r-\varrho|\,(r\varrho)^{-1/2}$. Therefore

$$t^{n-2}\big((r-\varrho)^2 + r\varrho t^2\big)^{-n/2} \leqslant 8\,(r\varrho)^{-n/2}(v^2+t^2)^{-1}.$$

From this and (3.5) we obtain

$$\int_0^2 k_{r,\varrho}^\sigma(t)\, t^{n-2}\, dt \leqslant C\varkappa_\sigma^\sigma r^{(2-n)/2} \varrho^{-n/2} |r-\varrho|^{-1}$$

and (3.4) follows from $r^{(2-n)/2}\varrho^{-n/2} \leqslant Cr^{-\alpha}\varrho^{-\beta}$.

If $\varrho/r < \frac{1}{2}$ or $\varrho/r > 2$ then $r^\alpha \varrho^\beta \leqslant (2|r-\varrho|)^{n-1}$ and it is enough to prove that

$$\int_0^2 k_{r,\varrho}^\sigma(t)\, t^{n-2}\, dt \leqslant C\varkappa_\sigma^\sigma |r-\varrho|^{-n},$$

which immediately results from the inequality

$$k_{r,\varrho}(t) \leqslant |r-\varrho|^{-n/\sigma} k\big(2\,|t-|v|^{1/2}|^{-1}\big).$$

Lemma 3.2. *Let*

$$k_{r,\varrho}(t) = \big((r-\varrho)^2 + r\varrho t^2\big)^{-n/2\sigma_1}\big(\varrho t(4-t^2)^{1/2}\big)^{-n/\sigma_2}$$

where $0 < t < 2$, $1 < \sigma_1 < \infty$, $1 < \sigma_2 \leqslant \infty$, $n \geqslant 3$. *If* $1/\sigma = 1/\sigma_1 + 1/\sigma_2 < 1$, $\sigma_1/\sigma_2 < n-1$, *then there exists a real constant* $C = C(n, \sigma_1, \sigma_2)$ *such that for any* α, β *with* $\alpha+\beta = n-1$, $\alpha \geqslant 0$, $\beta \geqslant n\sigma/\sigma_2$ *we have*

$$\| k_{r,\varrho} | S^{n-1} \|_\sigma^\sigma \leqslant C\,|r-\varrho|^{-1} r^{-\alpha} \varrho^{-\beta}. \tag{3.6}$$

Proof. We begin with an auxiliary fact that for any $l > 1$ there exists a constant $C(l)$ such that

$$\int_0^1 t^{l-2}\big(|r-\varrho|^2 + r\varrho t^2\big)^{-l/2}\, dt \leqslant C(l)\,|r-\varrho|^{-1} r^{-\alpha_1} \varrho^{-\beta_1} \tag{3.7}$$

for any $\alpha_1, \beta_1 \geqslant 0$ with $\alpha_1 + \beta_1 = l-1$.

Substituting $t = s|r-\varrho|\,(r\varrho)^{-1/2}$ one can estimate the left-hand side of (3.7) by $|r-\varrho|^{-1}(r\varrho)^{1-l}$, the estimate by $|r-\varrho|^{-l}$ is obvious. (3.7) follows by applying the first estimate if $\frac{1}{2} < \varrho/r < 2$ and the second otherwise.

In view of (3.3) we have to estimate the integral of the function

$$k_{r,\varrho}(t)\, t^{n-2}(4-t^2)^{(n-3)/2}$$

along the interval $[0, 2]$. The integral from 0 to 1 may be estimated directly from (3.7), and that from 1 to 2 can be reduced to the first one by a suitable estimation.

The L_p-estimates of the invariant integral operators on the unit sphere will be derived from the following variant of the Young inequality ([6], Th. 9.5.1).

Let K be a symmetric function on a set $X \times X$ and let $\mathscr{K}$ be the integral operator corresponding to the kernel K and some measure μ on X. If $1 \leqslant p, q, \sigma \leqslant \infty$ and $1/p - 1/q = 1 - 1/\sigma$ then

$$\|\mathscr{K} f\|_q / \|f\|_p \leqslant \operatorname*{ess\,sup}_\eta \Big(\int_X |K(\xi, \eta)|^\sigma\, d\mu_\xi\Big)^{1/\sigma}. \tag{3.8}$$

We shall also make use of the Hardy–Littlewood inequality ([7]):

$$\int_0^\infty\!\!\int \frac{\varphi(\varrho)\,\psi(r)\, d\varrho\, dr}{\varrho^\gamma r^{-\gamma} |r-\varrho|^\lambda} \leqslant C\,\|\varphi\|_p \|\psi\|_s$$

where $\lambda = 2 - 1/p - 1/s$, $1/p + 1/s > 1$, $1 < p, s < \infty$ and $-1 + 1/s < \gamma < 1 - 1/p$. The constant C depends only on p, s and γ.

Theorem 3.1. *Let* M_1, M_2 *be integral operators in* $\mathbf{R}^n$ $(n \geqslant 3)$ *given by* (3.1) *and* (3.2).

If $1 < p < \sigma/(\sigma-1)$, $1/q = 1/p - (\sigma-1)/\sigma$ *then there exists a constant* $C = C(p, q, n)$ *such that*

$$\|M_1 f\|_{L_q(\mathbf{R}^n)} \leqslant C\,\|f\|_{L_p(\mathbf{R}^n)}. \tag{3.9}$$

If σ_1 and σ_2 satisfy the assumptions of Lemma 3.2 and

$$1-\frac{1}{\sigma_1}-\frac{1}{\sigma_2}<\frac{1}{p}<1-\frac{1}{\sigma_2},\qquad \frac{1}{q}=\frac{1}{p}-1+\frac{1}{\sigma_1}+\frac{1}{\sigma_2},$$

then there exists a constant $C=C(p,q,n,\sigma_1,\sigma_2)$ such that

$$\|M_2 f\|_{L_q(\mathbf{R}^n)}\leqslant C\|f\|_{L_p(\mathbf{R}^n)}. \tag{3.10}$$

Proof. Denote by $K_1(x,y)$ the kernel of the operator M_1. Then

$$K_1(r\xi,\varrho\eta)=k\left(\frac{|\xi-\eta|\,|\xi+\eta|}{\left||\xi-\eta|^2+\nu\right|}\right)\left((r-\varrho)^2+r\varrho|\xi-\eta|^2\right)^{-n/2\sigma}$$

and $K_1(r\xi,\varrho\eta)\leqslant k_{r,\varrho}(|\xi-\eta|)$ where $k_{r,\varrho}$ is given in Lemma 3.1.

Let $f\in L_p(\mathbf{R}^n)$, $g\in L_{q'}(\mathbf{R}^n)$ $(1/q'+1/q=1)$, and put

$$\varphi(\varrho)=\Big(\int_{S^{n-1}}|f(\varrho\eta)|^p\,d\omega_\eta\Big)^{1/p},\qquad \psi(r)=\Big(\int_{S^{n-1}}|g(r\xi)|^{q'}\,d\omega_\xi\Big)^{1/q'}.$$

From the Young inequality (3.8)

$$\iint k_{r,\varrho}(|\xi-\eta|)f(\varrho\eta)g(r\xi)\,d\omega_\xi\,d\omega_\eta\leqslant\|k_{r,\varrho}|S^{n-1}\|_\sigma\,\varphi(\varrho)\psi(r) \tag{3.11}$$

where $1/\sigma+1/p+1/q'=2$. By (3.4) we have for some γ

$$\begin{aligned} r^{n-1}\varrho^{n-1}\|k_{r,\varrho}|S^{n-1}\|_\sigma\,\varphi(\varrho)\psi(r) &\\ \leqslant C(r/\varrho)^\gamma|r-\varrho|^{-1/\sigma}\varrho^{(n-1)/p}\varphi(\varrho)\,r^{(n-1)/q'}\psi(r).& \end{aligned}\tag{3.12}$$

Putting $\alpha=(n-1)\sigma/q$, $\beta=(n-1)\sigma/p'$ we get $\gamma=0$. Combining this with (3.11) and using the Hardy–Littlewood inequality we obtain

$$\iint K_1(x,y)f(x)g(y)\,dx\,dy\leqslant C\|f\|_p\|g\|_{q'},$$

so that (3.9) is proved.

The proof of (3.10) goes similarly but we cannot allow γ to be 0 in inequality (3.12).

4. Weighted inequalities and unique continuation.

THEOREM 4.1. *If $1/q=1/p-2/n$, $n=3,4$,*

$$1<p<\tfrac{3}{2}\quad \textit{for } n=3,$$

and

$$\tfrac{4}{3}<p<2\quad \textit{for } n=4,$$

then there exists a constant $C=C(p,q)$ such that the weighted inequalities (1.2) are satisfied for all functions $v\in C_0^\infty(\mathbf{R}^n-\{0\})$ and $k=0,1,\ldots$

Proof. Put $v_k(x)=|x|^{-k}v(x)$, $f_k(y)=|y|^{-k}\Delta v(y)$ and

$$g_k(x,y)=\left(|x-y|^{-m}-h_{m,k-1}(x,y)\right)(|y|/|x|)^k$$

where $n=m+2$. By the harmonicity of $h_{m,k}(x,y)$ in the variable y

$$v(x)=-\frac{1}{mc_n}\int\left(|x-y|^{-m}-h_{m,k-1}(x,y)\right)\Delta v(y)\,dy$$

or

$$v_k(x)=-\frac{1}{mc_n}\int g_k(x,y)f_k(y)\,dy.$$

From this and from Lemma 2.3 (iii) one gets

$$|v_k(x)|\leqslant\frac{1}{mc_n}\int M_m(|f_k(y)|)\,dy\qquad(m=1,2).$$

Now, our weighted inequalities (1.2) are consequences of Theorem 3.1 ($\sigma=n=3$ or $\sigma_1=\sigma_2=n=4$).

Remark 4.1. The weighted inequalities (1.2) can be proved even for $k=-1,-2,\ldots$ Instead of $h_{m,k}(x,y)$ one should use $h_{m,k}(y,x)$ (a duality argument). Taking $g_k(x,y)$ with the factor $(|y|/|x|)^\gamma$ we may prove inequality (1.2) for noninteger k.

As a corollary to Theorem 4.1 and Lemma 1.1 we get the strong uniqueness theorem.

THEOREM 4.2. *Let Ω be a connected open subset of $\mathbf{R}^n$, $n=3,4$. If $a\in L_{n/2,\mathrm{loc}}(\Omega)$ then any function u satisfying (1.1) and having zero of infinite order in Ω vanishes identically in Ω.*

5. Example. We give an example of a differential equation which shows that the assumption on the function a in Theorem 4.2 cannot be weakened.

THEOREM 5.1. *There exists a function a such that $a\in L_{p,\mathrm{loc}}(\mathbf{R}^n)$ for any $p<n/2$ and a function $u\in C^\infty(\mathbf{R}^n)$ such that $u(x)>0$ for $x\neq0$, $u(x)$ has zero of infinite order at the origin, and*

$$\Delta u+au=0.$$

Proof. We put $u(x)=\varphi(|x|)$ where $\varphi(r)=\int_0^r\varrho^{1-n+\log(1/\varrho)}\,d\varrho$. Then

$$a(x)=-\Delta u(x)/u(x)=2|x|^{-1}\log|x|\frac{\varphi'(|x|)}{\varphi(|x|)}.$$

The integrability assumption on the function $a(x)$ is satisfied because for any $\varepsilon>0$

$$r^{1+\varepsilon}\varphi'(r)/\varphi(r)\to0\quad\text{as } r\to0.$$

6. Appendix. Proof of Lemma 1.1. Let $\zeta(x)$ be a C^∞-function such that $\zeta(x)=0$ for $|x|\leqslant1$ and $\zeta(x)=1$ for $|x|\geqslant2$.

We prove (1.2) under the assumption that $v \in H_0^{2,p} \cap L_q$ and v, Δv have zero of infinite order at the origin in L_q-sense and L_p-sense, respectively. If $v(x) = 0$ for small $|x|$ then this follows by a density argument. Put $v_\varepsilon(x) = \zeta(\varepsilon^{-1} x) v(x)$ for $\varepsilon > 0$. Setting $v = v_\varepsilon$ in (1.2) and taking $\varepsilon \to 0$ we get (1.2) in the general case. The convergence follows from (1.3).

We prove that if $a \in L_{\sigma,\mathrm{loc}}$, u satisfies (1.1) and has zero of infinite order at $x_0 \in \Omega$ then $u = 0$ in any ball $B_r = \{x: |x - x_0| < r\}$ where r is such that $\bar{B}_r \subset B_R$,

$$(\int_{B_r} |a(x)|^\sigma \, dx)^{1/\sigma} \leqslant \frac{1}{2C} \tag{6.1}$$

and C is the constant from inequality (1.2). One can assume that $x_0 = 0$. Let ζ_1 be a C_0^∞-function such that $\zeta_1(x) = 1$ for $x \in B_r$, supp $\zeta_1 \subset B_R$. Put $v_k(x) = \zeta_1(x) u(x) |x|^{-k}$, $f_k(x) = |x|^{-k} \Delta v_0(x)$ $(k = 0, 1, \ldots)$. Denote by χ the characteristic function of B_r. By Theorem 4.1 and (1.1)

$$\|\chi v_k\|_q \leqslant C \|f_k\|_p \leqslant C(\|\chi a v_k\|_p + \|(1-\chi) f_k\|_p).$$

From the Hölder inequality $\|a v_k \chi\|_p \leqslant \|\chi a\|_\sigma \|\chi v_k\|_q$ and from (6.1)

$$\|\chi v_k\|_q \leqslant 2C \|(1-\chi) f_k\|_p.$$

Therefore

$$(\int_{|x|<\varrho} |u(x)^q \, dx)^{1/q} \leqslant \mathrm{const} \cdot (\varrho/r)^k$$

for any $\varrho < r$. Taking $k \to \infty$ we obtain $u = 0$ on B_r. By a standard connectedness argument $u = 0$ in the whole Ω.

Added in proof. Professor Lars Hörmander has recently informed the author that David Jerison and Carlos E. Kenig established the unique continuation property for Schrödinger equations in any dimension with optimal exponents.

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