Unique continuation for Schrödinger equations in dimensions three and four
by
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Abstract. The unique continuation property for solutions of the Schrödinger equations (or inequalities) in dimensions three and four is proved. The $L_p$-conditions assumed for the strong uniqueness are shown to be optimal. The main result follows from the Carleman type inequalities, which are obtained from $L_p$-estimates of integral operators.

1. Introduction. The study of the unique continuation property of solutions to differential equations with nonanalytic coefficients began with a paper by Carleman [4] (in 2 dimensions), in which he introduced the important method of weighted inequalities. Then the unique continuation theorem was proved for the second order elliptic equations under the assumption that the coefficients of the lower order terms are of class $L^p$. The case where the leading part is the Laplacian was considered by Müller [12] and Heinz [8] while Aronszajn [2] and Cordes [5] studied equations whose leading part had variable coefficients. The sufficiency of the Lipschitz continuity was shown by Hörmander [9] and in a stronger form by Aronszajn, Krzywicki and Szarski [3]. An example given by Pilis [13] shows that it is almost a necessary condition.

There are partial results on the unique continuation for the equations with unbounded lower order coefficients. Sufficient conditions are given e.g. by Schechter and Simon [14], Amrein, Berthier and Georgescu [1] and Hörmander [10]. Our result on the strong uniqueness concerns rather special cases but the $L_p$-conditions on the coefficients are sharp.

In this paper we consider differential inequalities of the form

\[(1.1) \quad |\Delta u(x)| \leq a(x)|u(x)| \]

where the function $a(x)$ is assumed to be of class $L^\infty$, $u$ is a function on a connected subset $\Omega$ of $\mathbb{R}^n$ ($n \geq 3$), and $\Delta$ is the Laplacian. Without loss of generality one may replace inequality (1.1) by the Schrödinger equation $\Delta u + Vu = 0$ with $|F'(x)| \leq a(x)$.

We prove the strong uniqueness for solutions of (1.1) under the assumption $\sigma \geq n/2$ for $n = 3$ or 4. This improves a result of [1], where it is
assumed that \( \sigma > n/2 \). We also give some examples that show the theorem does not hold for \( \sigma < n/2 \) (\( n \geq 3 \)).

We use the Carleman type inequalities of the form
\[
(\int |x|^{-\alpha} u(x)^p dx)^{1/p} \leq C (\int |x|^{-1} \Delta x u(x)^q dx)^{1/q}
\]
where \( u \) is a function of class \( C^2_0 (\mathbb{R}^n - \{0\}) \) and \( k = 1, 2, \ldots \). The uniqueness theorems are derived from (1.2) in a routine manner (Lemma 1.1).

**Definition 1.1.** A function \( u \) is said to have zero of infinite order in \( L_q \)-sense at \( x_0 \in \Omega \) if for any positive integer \( k \) the functions \( |x-x_0|^{-k} u(x) \) are of class \( L_{q,loc} \). If \( q < \infty \) can be arbitrary we say that \( u \) has zero of infinite order.

One says that (1.1) has the strong unique continuation property if every function \( u \) satisfying (1.1) and having zero of infinite order must vanish identically. Inequality (1.1) has the unique continuation property if every function \( u \) satisfying (1.1) and equal to zero on some open set is identically zero on \( \Omega \).

By saying that a function \( u \) satisfies (1.1) we mean here that \( u \in L_{q,loc} \) for any \( q < \infty \). \( \Delta u \) (distributionally) is a locally integrable function and inequality (1.1) is fulfilled almost everywhere in \( \Omega \).

The following lemma is proved in Section 6.

**Lemma 1.1.** Let \( a \in L_{q,loc} \), \( 1 \leq q < \infty \). Inequality (1.1) has the strong unique continuation property provided there exist real numbers \( p, q, C, R \) such that \( 1 \leq p \leq q < \infty \), \( 1/a = 1/p - 1/q \), \( R > 0 \) and inequality (1.2) holds for all \( \phi \in C^\infty_0 (B_R - \{0\}) \) where \( B_R = \{x: |x| < R\} \).

We prove (1.2) with \( 1/p - 1/q = 2/n \) for \( n = 3, 4 \). The proof is based on \( L_p \)-estimates of some majorizing integral operators given in Section 3. (The case \( n = 4 \) was presented in [15]).

Remark 1.1. By a homothetic change of variable in (1.2) we conclude that if inequality (1.2) holds for all \( \phi \in C^\infty_0 (B_{R_0} - \{0\}) \) with a constant \( C = C_0 \), then (1.2) holds for all \( \phi \in C^\infty_0 (B_R - \{0\}) \) with the constant \( C = C_0 (R/R_0)^{1/(n-1)} \). Hence it follows that inequality (1.2) cannot be proved if \( 1/p - 1/q > n/2 \). Moreover, if \( 1/p - 1/q < n/2 \) then the constant \( C \) may be taken arbitrarily small.

Remark 1.2. The condition that \( u \) has zero of infinite order in the \( L_q \)-sense at \( x_0 \in \Omega \) is equivalent to the following: for any positive integer \( k \)
\[
(\int |x-x_0|^k u(x)^p dx)^{1/p} \leq C (\int |x-x_0| u(x)^q dx)^{1/q}
\]

2. Estimates of Taylor remainders. Now we prove some estimates of Taylor remainders which give us estimates of the kernels of the integral operators involved in our considerations.

We introduce the following operators in the complex domain:
\[
J_1 f(z) = \text{Re} \int_0^{2\pi} \frac{1}{(1-r^2)^{1/2}} f(\text{Re} z + it \text{Im} z) dt,
\]
\[
J_2 f(z) = \text{Re} \int_0^{2\pi} \frac{1}{(1-r^2)^{1/2}} f(\text{Re} z + it \text{Im} z) dt,
\]
which allow us to reduce the problem to an investigation of the complex geometric series.

**Lemma 2.1.** (i) \( J_m \text{Re}(z^m) \) \( m = 1, 2 \) are homogeneous polynomials of degree \( l \) in the variable \( z \) \( l = 0, 1, \ldots \).

(ii) If \( f(z) \) is an analytic function with real values for real arguments then
\[
J_1 f'(z) = \text{Im} f(z) / \text{Im} z \quad (\text{Im} z \neq 0).
\]

(iii) The following equalities hold:
\[
J_1 ((1-z)^{-1}) = \text{sign} (1-\text{Re} z) |1-z|^{-1},
\]
\[
J_2 ((1-z)^{-2}) = |1-z|^{-2}.
\]

(iv) The following inequality is true:
\[
J_1 |(1-z)^{-1}| \leq \left( \log \left( \frac{1+|\text{Im} z|}{1-\text{Re} z} \right) + 1 \right)|1-z|^{-1}.
\]

**Proof.** It is easy to check (i) and (ii), then (2.5) follows from (ii). We get (2.4) substituting \( z = \text{Im} z (1-\text{Re} z) \) in the formula
\[
2\pi^{-1} \int_0^{2\pi} \frac{|1-r^2|^{1/2}}{(1+r^2 t^2)^{1/2}} dt = (1+\alpha^2)^{-1/2}
\]

which can be obtained by means of the well-known methods of integration.

(iv): The following inequality is obtained by means of integration by parts and by applying the estimate \( \text{arc} \sin t \leq \pi/2 \):
\[
2\pi^{-1} \int_0^{2\pi} \frac{|1-r^2|^{1/2}}{(1+r^2 t^2)^{1/2}} dt \leq (1+\alpha^2)^{-1/2} + \int_0^{2\pi} \frac{1}{(1+\alpha^2 t^2)^{3/2}} dt.
\]

For the last term we have
\[
\int_0^{2\pi} \frac{1}{|t(1+\alpha^2 t^2)^{1/2}} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|t|^{3/2}} dt 
\]
\[
\leq (1+\alpha^2)^{1/2} + 2\log(1+|\alpha|)(1+\alpha^2)^{-1/2}.
\]

Putting \( \alpha = \text{Im} z (1-\text{Re} z) \) in these inequalities we get (iv).

One can observe that the operator \( J_1 \) can be expressed as a composition of the operator of fractional integration of order \( \frac{1}{2} \) along the imaginary axis and some elementary operators.
LEMMA 2.2. Let $\Phi_{n,k}$ be the k-th Taylor polynomial at the origin of the function $1/[1-z]^n$ ($m = 1, 2, k = 0, 1, \ldots$). Then

$$1 - z^{-1} - \Phi_{1,k}(z) \leq 2 \left(1 + \log \left(1 + \frac{\text{Im} z}{1 - \text{Re} z}\right)\right) \frac{|z|^{k+1}}{|1 - z|^2},$$

(2.7)

$$1 - z^{-2} - \Phi_{2,k}(z) \leq |z|^{k+2} |1 - z|^{-3} |\text{Im} z|^{-1}. $$

(2.8)

Proof. By Lemma 2.1 (i)-(iii)

$$\Phi_{1,k}(z) = J_1(1 + \ldots + z^k), \quad \Phi_{2,k}(z) = J_2 \left(\frac{d}{dz}(1 + \ldots + z^{k+1})\right).$$

If $\text{Re} z < 1$ we have

$$1 - z^{-1} - \Phi_{1,k}(z) = J_1 (z^{k+1} (1 - z)^{-1})$$

and for $\text{Re} z > 1$

$$1 - z^{-1} - \Phi_{1,k}(z) = J_1 (z^{k+1} (1 - z)^{-1}) + 2 |1 - z|^{-1}.$$  

Then (2.7) is a consequence of (2.6) and the obvious properties of the operator $J_1$.

Estimate (2.8) follows immediately from the identity

$$1 - z^{-2} - \Phi_{2,k}(z) = (\text{Im} z)^{-1} \text{Im} (z^{k+2} (1 - z)^{-1}),$$

which results from (2.5) and (2.3).

LEMMA 2.3. Let $h_{n,k}$ be the k-th Taylor polynomial at the origin of the function $x \mapsto |x-y|^{-n}$ in $\mathbb{R}^{n+2}$ ($y \neq 0$). Then

(i) $h_{n,k}(x, y)$ is harmonic in the variable $y$,

(ii) $h_{n,k}(x, y) = |y|^{-n} \Phi_{n,k}(y)$, where $\text{Re} z = xy|y|^{-2}$, $\text{Im} z = (x^2 y^2 - (xy)^2)^{1/2} |y|^{-2}$, and $\Phi_{n,k}$ are the polynomials introduced in Lemma 2.2.

(iii) $|x - y|^{-n} h_{n,k}(x, y)$

$$\leq 2 \left(1 + \frac{|x^2 y^2 - (xy)^2|^{1/2}}{|y^2|^{1/2}}\right) + 2 \left(\frac{|x|^{k+1}}{|y|^{k+1}} \frac{|x|}{|y|} |x - y|^{-1},

|y|^{k+1} |x - y|^{-1} - h_{n,k}(x, y)\right).$$

Proof. (i) is a consequence of the harmonicity of the derivatives of $|x-y|^{-n}$.

For any $\lambda > 0$

$$h_{n,k}(\lambda x, \lambda y) = \lambda^{-n} h_{n,k}(x, y),$$

and for any orthogonal matrix $A$

$$h_{n,k}(Ax, Ay) = h_{n,k}(x, y).$$

(2.9)

(2.10)

There exists an orthogonal matrix $A$ such that $Ay = \{y(1, 0, 0)$ and $Ax = y(\text{Re} z, \text{Im} z, 0) (0$ is the origin in $\mathbb{R}^3$). Now by (2.9) and (2.10) and the obvious equality

$$\Phi_{n,k}(z) = h_{n,k}(\text{Re} z, \text{Im} z, 0, (1, 0, 0))$$

we get (ii).

Substituting the identities $|z - 1| = |x - y|/|y|$, $|z| = |x|/|y|$ and $\Phi_{n,k}(z) = |y|^{n} h_{n,k}(x, y)$ into (2.7) and (2.8) we obtain (iii).

3. $L_p$-estimates of the majorizing operators. We are interested in $L_p$-estimates of the integral operators $M_1$, $M_2$ in $\mathbb{R}$ given formally by

$$M_p f(x) = \int \left(\frac{x^2 y^2 - (xy)^2}{|y^2 - xy|}\right)^{1/2} |x - y|^{-n} f(y) dy,$$

where $k(t) = 2 \log(1 + t) + 4$,

$$M_2 f(x) = \int |x - y|^{-n+1} \left[|x|^{-1} |x^2 y^2 - (xy)^2|^{1/2} - |y|^{-1} f(y) dy,$$

where $1 < a, a_1 < \infty$ and $1 < a_2 \leq \infty$, $n \geq 3$.

Remark that $|x|^{-1} |x^2 y^2 - (xy)^2|^{1/2}$ is the distance from the point $y$ to the straight line spanned by $x$ and the origin.

First we prove two technical lemmas. If $F$ is a function on the interval $[0, 2]$ we denote by $\|F(S^{-1})\|_p$ the $L_p(S^{-1})$-norm of the function $x \mapsto F(|x| - x_0)$ (this norm does not depend on $x_0$). We shall use the following formula ($n = 2, 3, \ldots$)

$$\int_{S^{-1}} F(|x| - x_0) dx_0 = c_{n-1} \int_{0}^{2} F(t) t^{2-n} (1 + \frac{1}{2} t)^{n-2} - t^{n-1} dt,$$

(3.3)

where $dx_0$ is the surface element of $S^{-1}$ and $c_2$ is the surface measure of the unit sphere $S^{n-1}$, $c_0 \in S^{n-1}$.

This formula can easily be derived e.g. from the integral formula for a plane wave function over the unit sphere ([11], Ch.1).

LEMMA 3.1. Let $k(t) \geq 0$ be an increasing function on $R_+$, $n \geq 3$ and $\sigma > 1$. Put

$$k_{n,k}(t) = k(2 (t^{1/2} + v) (t^{1/2} + 2 r^{1/2}))^{-n/2}$$

where $t, t, v > 0$ and $v = 2 (q - n)/r$. There exists a constant $C = C(\sigma, n)$ such that for any $a, \beta > 0$, $a + \beta = n - 1$

$$||k_{n,k}(S^{-1})||_p \leq C \omega \xi |r - q|^{-1} r^{-\sigma} q^{-\beta},$$

(3.4)

where

$$
\xi = \left(\frac{q}{\lambda}\right)^{(\beta (r^2 + 1)^{-1})} dt.
\right)$$
Proof. By (3.3)
\[ \|k_\alpha\|_{L^2(S^n)} \leq c_{n-1} \int_0^1 \kappa_\alpha^2 t^{n-2} dt. \]
Put
\[ k_\alpha(t) = k^n(2t|t^2 - |v|^2|^{-1}) \left( t^2 + |v|^2 \right)^{-1}. \]
Denote by \( I_1, I_2 \) the integrals of the function \( k_\alpha(t) \) over the intervals \( 0 \leq t \leq \frac{1}{2} |v|^2 \) and \( \frac{1}{2} |v|^2 \leq t \leq 2 \), respectively.
If \( t \leq \frac{1}{2} |v|^2 \) then \( t^2 |t^2 - |v|^2|^{-1} \leq 2 |v|^2 |v|^2^{-1} \). By the change of the variable \( t = s|v| \) and the last estimate we obtain
\[ I_1 \leq |v|^{-2} \int_0^{|v|} k^n(2s^2) (s^2 + |v|^2)^{-1} ds. \]
If \( \frac{1}{2} |v|^2 \leq t \leq 2 \) then \( (t^2 + |v|^2)^{-1} \leq 2 |v|^2 |v|^2^{-1} \). Since \( t^2 |t^2 - |v|^2|^{-1} \leq |v|^{-2} \), then we have
\[ I_2 \leq 2 |v|^{-2} \int_2^{|v|} k^n(2|v|^{-1}) dt \leq C_\alpha |v|^{-2}. \]
Adding \( I_1 \) and \( I_2 \) we get
\[ \int_0^{|v|} k_\alpha(t) dt \leq C_\alpha |v|^{-2}. \]
If \( |v| \leq 2 \) then \( |v| \leq 2 \sqrt{2} |r - |v||^{-1/2} \). Therefore
\[ t^{n-2} (t^2 - |v|^2 + |v|^2)^{-1} \leq 8 (r^2 - |v|^2 + |v|^2)^{-1}. \]
From this and (3.5) we obtain
\[ \int_0^{|v|} k_\alpha(t) t^{n-2} dt \leq C_\alpha r^{n/2 - n/2} |r - |v||^{-1}. \]
and (3.4) follows from \( \int_2^{|v|} k^n_s(t) t^{n-2} dt \leq C_\alpha |r - |v||^{-n/2} \).
If \( |v| > \frac{1}{2} \) or \( |v| > 2 \) then \( r^2 |v|^2 \leq (2|r - |v||)^{-1} \) and it is enough to prove that
\[ \int_0^{|v|} k_\alpha(t) t^{n-2} dt \leq C_\alpha |r - |v||^{-n/2}, \]
which immediately results from the inequality
\[ k_\alpha(t) \leq |r - |v||^{-n/2} k^n(2t|t^2 - |v|^2|^{-1}). \]
Lemma 3.2. Let
\[ k_\alpha(t) = (t^2 - |v|^2 + |v|^2)^{-n/2} \]
along the interval \([0, 2]\). The integral from 0 to 1 may be estimated directly from (3.7), and that from 1 to 2 can be reduced to the first one by a suitable estimation.

The \( L^p \)-estimates of the invariant integral operators on the unit sphere will be derived from the following variant of the Young inequality ([6], Th. 9.5.1).

Let \( K \) be a symmetric function on a set \( X \times X \) and let \( \mathcal{F} \) be the integral operator corresponding to the kernel \( K \) and some measure \( \mu \) on \( X \). If \( 1 < p, q, \sigma < \infty \) and \( 1/p - 1/q = 1 - 1/\sigma \) then
\[ \|K\|_p \|f\|_q \leq \text{ess sup} \{ \int K(\xi, \eta) d\mu(\eta) \}^{1/s}. \]
We shall also make use of the Hardy–Littlewood inequality ([7]):
\[ \int_0^w \frac{\varphi(t) \psi'(t) dt}{t} \leq C \|\varphi\|_p \|\psi\|_q, \]
where \( \lambda = 2 - 1/p - 1/s, 1/p + 1/s > 1, 1 < p, q < \infty \) and \( -1 + 1/s < \gamma < 1 - 1/p \). The constant \( C \) depends only on \( p, s, q, \) and \( \gamma \).

Theorem 3.1. Let \( M_1, M_2 \) be integral operators in \( L^p \) \((n \geq 3)\) given by (3.1) and (3.2).
If \( 1 < p < \sigma (\sigma - 1), 1/q = 1/p - (\sigma - 1)/\sigma \) then there exists a constant \( C = C(p, q, n) \) such that
\[ \|M_1 f\|_{L^2(S^n)} \leq C \|f\|_{L^p(S^n)} \]
and
\[ \|M_2 f\|_{L^1(S^n)} \leq C \|f\|_{L^p(S^n)} \]
If $\sigma_1$ and $\sigma_2$ satisfy the assumptions of Lemma 3.2 and

$$\frac{1}{\sigma_1} \leq \frac{1}{p} < 1 - \frac{1}{\sigma_2} = \frac{1}{q} > 1 - \frac{1}{\sigma_2},$$

then there exists a constant $C = C(p, q, n, \sigma_1, \sigma_2)$ such that

$$||M_2 f||_{L_p^p(\mathbb{R}^n)} \leq C ||f||_{L_q^q(\mathbb{R}^n)}.$$  

(3.10)

Proof. Denote by $K_1(x, y)$ the kernel of the operator $M_1$. Then

$$K_1(\xi, \eta) = k \left( \frac{\xi - \eta}{\|\xi - \eta\|^2 + v} \right) = e^{2\pi i \langle \xi - \eta, y \rangle} \int_{\mathbb{R}^n} e^{2\pi i \langle \xi, \theta \rangle} d\nu(x).$$

From the Young inequality (3.8)

$$\int |K_1(x, y) f(x) g(x) dx| dy \leq ||K_1||_{L^1(\mathbb{R}^n)} ||f||_{L^p(\mathbb{R}^n)} ||g||_{L^q(\mathbb{R}^n)}$$

where $1/p + 1/q = 1$. By using (3.8) and (3.11) we have for some $\gamma$

$$\int |K_1(x, y) f(x) g(x) dx| dy \leq C \int |f(x)| \int |g(y)| \psi(x) \psi(y) dx dy$$

where $\psi$ is a function satisfying $\psi(x) \leq C \int \psi(x) dx$. Combining this with (3.12) and using the Hardy–Littlewood inequality we obtain

$$\int |K_1(x, y) f(x) g(x) dx| dy \leq C \int |f(x)| ||K_1||_{L^1(\mathbb{R}^n)} ||g||_{L^q(\mathbb{R}^n)}.$$

Thus the proof of (3.10) goes similarly but we cannot allow $\gamma$ to be 0 in inequality (3.12).

4. Weighted inequalities and unique continuation.

**Theorem 4.1.** If $1/p = 1/q - 2/n$, $n = 3, 4$, and

$$1 < p < \frac{3}{2},$$

then there exists a constant $C = C(p, q)$ such that the weighted inequalities (1.2) are satisfied for all functions $f \in L_p^p(\mathbb{R}^n)$ and $k = 0, 1, \ldots$

Proof. Put $v_k(x) = |x|^{-k} v(x)$, $f_k(y) = |y|^{-k} d\nu (y)$ and

$$g_k(x, y) = (x - y)^{-k - n} h_{(x, y)} (|x|/|y|).$$

where $n = m + 2$. By the harmonicity of $h_{(x, y)} (|x|/|y|)$ in the variable $y$

$$v_k(x) = -\frac{1}{m c_n} \int_{|x| = |y|} (x - y)^{-k - n} h_{(x, y)} (|x|/|y|) d\nu(y) dy$$

or

$$v_k(x) = -\frac{1}{m c_n} \int_{|x| = |y|} g_k(x, y) f_k(y) dy.$$

From this and from Lemma 2.3 (iii) one gets

$$|v_k(x)| \leq \frac{1}{m c_n} \int |f_k(y)| dy (m = 1, 2).$$

Now, our weighted inequalities (1.2) are consequences of Theorem 3.1 ($a = n = 3$ or $\sigma_1 = \sigma_2 = n = 4$).

**Remark 4.1.** The weighted inequalities (1.2) can be proved even for $k = 1, 2, \ldots$ Instead of $h_{(x, y)} (|x|/|y|)$ one should use $h_{(x, y)} (|x|/|y|)$ (a duality argument). Taking $g_k(x, y)$ with the factor $|x|/|y|$ we may prove inequality (1.2) for noninteger $k$.

As a corollary to Theorem 4.1 and Lemma 1.1 we get the strong uniqueness theorem.

**Theorem 4.2.** Let $\Omega$ be a connected open subset of $\mathbb{R}^n$, $n = 3, 4$. If $a \in L_1(\mathbb{R}^n)$ then any function $u$ satisfying (1.1) and having zero of infinite order in $\Omega$ vanishes identically in $\Omega$.

5. Example. We give an example of a differential equation which shows that the assumption on the function $a$ in Theorem 4.2 cannot be weakened.

**Theorem 5.1.** There exists a function $a$ such that $a \in L_1(\mathbb{R}^n)$ for any $p < q/2$ and a function $u \in C^\infty(\mathbb{R}^n)$ such that $u(x) > 0$ for $x \neq 0$, $u(x)$ has zero of infinite order at the origin, and

$$a u + au = 0.$$  

Proof. We put $u(x) = \varphi(|x|)$ where $\varphi(r) = \int_0^r q^{-1/2} e^{-q^{1/2} t} \varphi(t) dt$. Then

$$a(x) = -\Delta u(x) = 2|\varphi'(x)|^{1/2}.$$  

The integrability assumption on the function $a(x)$ is satisfied because for any $\epsilon > 0$

$$r^{1+\epsilon} |\varphi(r)| = \int_0^r e^{q t^{1/2}} dt \to 0 \quad \text{as } r \to 0.$$  

6. Appendix. Proof of Lemma 1.1. Let $\zeta(x)$ be a $C^\infty$-function such that $\zeta(x) = 0$ for $|x| < 1$ and $\zeta(x) = 1$ for $|x| > 2$. 

$$r^{1+\epsilon} \varphi(r) \to 0 \quad \text{as } r \to 0.$$
We prove (1.2) under the assumption that \( v \in H^{2, \frac{p}{2}} \cap L_q \) and \( v, \partial v \) have zero of infinite order at the origin in \( L_q \)-sense and \( L_q \)-sense, respectively. If \( v(x) = 0 \) for small \( |x| \) then this follows by a density argument. Put \( \zeta_v(x) = \zeta(\epsilon^{-1} x) \epsilon(x) \) for \( \epsilon > 0 \). Setting \( e = v \), (1.2) and taking \( s \to 0 \) we get (1.2) in the general case. The convergence follows from (1.3).

We prove that if \( a(x) \) satisfies (1.1) and has zero of infinite order at \( x_0 \in \Omega \) then \( u = 0 \) in any ball \( B_r = \{ x : |x - x_0| < r \} \) where \( r \) is such that \( B_r \subset B_\delta \),

\[
\left( \int_{B_r} |a(x)|^p \, dx \right)^{1/p} \leq \frac{1}{2C},
\]

and \( C \) is the constant from inequality (1.2). One can assume that \( x_0 = 0 \). Let \( \zeta \) be a \( C^\infty \)-function such that \( \zeta (x) = 1 \) for \( x \in B_1 \), \( \supp \zeta = B_1 \). Put \( u_k(x) = \zeta(x) u(x) |x|^{-k}, f_k(x) = |x|^{-k} \partial u_0(x) \) \( (k = 0, 1, \ldots) \). Denote by \( \gamma \) the characteristic function of \( B_1 \). By Theorem 4.1 and (1.1)

\[
\|u_k\|_p \leq C \|f_k\|_p \leq C \left( \|u_0\|_p + \left( \int (1 - \gamma) f_k \right)^p \right).
\]

From the Hölder inequality \( \|u_k\|_p \leq \|u_0\|_p \|u_k\|_F \) and from (6.1)

\[
\|u_k\|_p \leq 2C \left( \int (1 - \gamma) f_k \right)^p,
\]

Therefore

\[
\left( \int_{|x| < q} |u(x)|^p \, dx \right)^{1/p} \leq \text{const} \cdot (q/r)^p
\]

for any \( q < r \). Taking \( k \to \infty \) we obtain \( u = 0 \) on \( B_r \). By a standard connectedness argument \( u = 0 \) in the whole \( \Omega \).

Added in proof. Professor Lars Hörmander has recently informed the author that David Jerison and Carlos E. Kenig established the unique continuation property for Schrödinger equations in any dimension with optimal exponents.

References


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