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Commuting C_0 groups and the Fuglede–Putnam theorem

by

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Abstract. The following generalization of the Fuglede–Putnam theorem is known (see [5]): If A, B are commuting Hermitian operators on a Banach space X and if $(A+iB)^2 x = 0$ for some $x \in X$, then $Ax = Bx = 0$. We generalize this result further, proving that if A_k $(k = 1, \ldots, n, n \geqslant 2)$ are commuting Hermitian operators on X and if $P(t_1, \ldots, t_n)$ is a complex polynomial with at most one real zero at the origin, then $P(A_1, \ldots, A_n)x = 0$ for some $x \in X$ implies $A_k x = 0$ $(k = 1, \ldots, n)$. This result holds also when iA_k are (unbounded) generators of certain one-parameter groups of operators on X. Our considerations are based on a generalization of the classical Liouville theorem for harmonic functions.

Preliminaries. Let H be a complex Hilbert space and $B(H)$ the Banach space of bounded linear operators on H. Let $a, b, c, d \in B(H)$ be self-adjoint operators such that $[a, b] = 0$, $[c, d] = 0$. The Fuglede–Putnam theorem says that if $x \in B(H)$ and $x(a+ib) = (c+id)x$, then $x(a-ib) = (c-id)x$ (see [11], § 1.6; [12], Theorem 12.16). One way to generalize this theorem is to relax the conditions $[a, b] = 0$, $[c, d] = 0$ (see [2], [10] and the references there). Another – to relax the condition $x(a+ib) = (c+id)x$ (see [1], [9] and the references there). We give here a generalization relaxing this condition and passing to a larger class of operators.

The above theorem can be reformulated as follows: Let A, B be the bounded linear operators on $B(H)$ defined by $Ax = xa - cx$, $Bx = xb - dx$, $x \in B(H)$. Then $[A, B] = 0$ and A, B are Hermitian operators in the sense of Vidav (see [3]), because the one-parameter groups e^{itA}, e^{itB} $(t \in \boldsymbol{R})$ are groups of isometries on $B(H)$ (as $e^{itA}x = e^{-itc}xe^{ita}$, $e^{itB}x = e^{-itd}xe^{itb}$, $x \in B(H)$, $t \in \boldsymbol{R}$ – see for instance [9], p. 186). The Fuglede–Putnam theorem states that if $x \in B(H)$ and $(A+iB)x = 0$, then $Ax = Bx = 0$. In this form it can be generalized to arbitrary Banach spaces, as has been done by a number of authors ([7], [8]):

(1) *Let A, B be commuting Hermitian operators on a complex Banach space X. If $x \in X$ and $(A+iB)x = 0$, then $Ax = Bx = 0$.*

Another theorem about commutation properties of Hilbert space operators is the following: If c, d are normal operators on a Hilbert space H and $T_{c,d}x = xc - dx$, $x \in B(H)$ is the generalized commutator operator on $B(H)$, then $T_{c,d}^2 x = 0$ for some $x \in B(H)$ implies $T_{c,d}x = 0$ (see [7], Corollary 6, and [1]). This result can be generalized for operators on a Banach space X

in the same way as the Fuglede–Putnam theorem. Note that a bounded linear operator N on X is called normal if $N = A+iB$ with A, B commuting Hermitian operators on X. It is easy to see ([7], Corollary 6) that the operator $T_{c,d}$ defined above is a normal operator on $B(H)$. The following theorem, which is obviously a generalization of (1), was proved in [5]:

(2) *If $N = A+iB$ (A, B commuting Hermitian) is a normal operator on a Banach space X and if $N^2 x = 0$ for some $x \in X$, then $Ax = Bx = 0$.*

The aim of this short paper is to give a generalization of (2) which holds for a larger class of unbounded operators. Our considerations are based on the theory of one-parameter groups of operators on Banach spaces and on a generalization of the classical Liouville theorem for harmonic functions.

Let e^{tA}, e^{sB} ($t, s \in \mathbf{R}$) be two commuting and bounded C_0 groups of operators on a Banach space X with generators A, B. In [4] it was shown that if $x \in D(A) \cap D(B)$ and $(A+iB)x = 0$, then $Ax = Bx = 0$. The idea of proof is as follows: We consider the complex function of two real variables $F(t, s) = f(e^{tA} e^{sB} x)$ ($t, s \in \mathbf{R}$) for some arbitrary $f \in X'$ (the dual of X). The above condition implies $(\partial/\partial t + i\partial/\partial s) F(t, s) = 0$ on $\mathbf{R}^2$. Hence the Cauchy–Riemann equations hold for F and F is a holomorphic function of $z = t+is$ on the whole plane. As F is bounded, it is a constant and therefore $F'_t(0, 0) = f(Ax) = F'_s(0, 0) = f(Bx) = 0$.

In the same way, if $x \in D(A^2) \cap D(B^2)$ is such that $Ax, Bx \in D(A) \cap D(B)$, and $(A^2+B^2)x = 0$, we obtain $Ax = Bx = 0$. For the function F defined above we have $\Delta F = 0$ and the Liouville theorem for harmonic functions implies that F is a constant. Combining this result with the above one, we can easily obtain a generalization of (2). However, we are now going to show that these considerations can be made in a more general setting to obtain a stronger result.

Notation and results. Let $P(t_1, \ldots, t_n)$ be a polynomial of degree m ($m \geqslant 1$) on $\mathbf{R}^n$ ($n \geqslant 2$) with complex coefficients and with at most one real zero at the origin, i.e. $P(t_1, \ldots, t_n) \neq 0$ when $t_k \in \mathbf{R}$ ($k = 1, \ldots, n$) and $\sum_{k=1}^{n} t_k^2 > 0$. We consider the differential operator $P(D_1, \ldots, D_n)$ obtained by replacing t_k with $D_k = i(\partial/\partial t_k)$. The following result from the theory of distributions is a generalization of the classical Liouville theorem:

LEMMA. *Let the distribution $F(t_1, \ldots, t_n)$ be a solution of the differential equation $P(D_1, \ldots, D_n) F = 0$ on $\mathbf{R}^n$ and let F be of at most polynomial growth at infinity. Then F is a polynomial.*

(See [13], Ch. III, § 16, problem 5, p. 168.) The idea of proof is as follows: by taking the Fourier transform of the above differential equation, it is easy to see that the Fourier transform $\hat{F}$ of F has its support in at most one point — the origin. According to a well-known theorem in the distribution theory, $\hat{F}$ is a linear combination of the Dirac δ-function and its derivatives (see for instance [13], Ch. II, § 10.6). Hence F is a polynomial.

Let now X be a complex Banach space and let $e^{it_k A_k}$ ($t_k \in \mathbf{R}$, $k = 1, \ldots, n$, $n \geqslant 2$) be n mutually commuting C_0 groups on X with generators iA_k (see [6], Ch. VIII). We can as well take X to be a sequentially complete locally convex topological linear space and $e^{it_k A_k}$ to be commuting equicontinuous C_0 groups on X in the sense of [14], Ch. IX. By X' we denote the dual of X.

For a fixed element $x \in \bigcap_{k=1}^{n} D(A_k^m)$ (recall that m is the degree of the polynomial P introduced above) and for the groups $e^{it_k A_k}$ ($k = 1, \ldots, n$) we assume that the following conditions hold:

(3) The operator $P(A_1, \ldots, A_n)$ is defined for x. For every $f \in X'$, the complex function of n real variables

$$F(t_1, \ldots, t_n) = f(e^{-it_1 A_1} \ldots e^{-it_n A_n} x)$$

has on $\mathbf{R}^n$ partial derivatives of mth order, so that:

$$P(D_1, \ldots, D_n) F(t_1, \ldots, t_n) = f(e^{-it_1 A_1} \ldots e^{-it_n A_n} P(A_1, \ldots, A_n) x) \quad \text{on } \mathbf{R}^n$$

(i.e., the element x must be "sufficiently smooth").

(4) For every $f \in X'$, the function F defined in (3) satisfies:

$$F(t_1, \ldots, t_n)\Big(\sum_{k=1}^{n} t_k^2\Big)^{-1/2} \to 0 \quad \text{as} \quad \sum_{k=1}^{n} t_k^2 \to \infty.$$

Under these assumptions we have the following

THEOREM. *Let $P(A_1, \ldots, A_n)x = 0$. Then $A_k x = 0$ ($k = 1, \ldots, n$).*

Proof. For every $f \in X'$ we consider on $\mathbf{R}^n$ the continuous complex function $F(t_1, \ldots, t_n)$ defined in (3). We have $P(D_1, \ldots, D_n) F = 0$ on $\mathbf{R}^n$ and applying the lemma we conclude that F is a polynomial. According to condition (4), F is a constant. Therefore $D_k F(0, \ldots, 0) = f(A_k x) = 0$ ($k = 1, \ldots, n$). As f is arbitrary, the proof is complete.

Remark. When A_k are (bounded) Hermitian operators in the sense of Vidav, the groups e^{-itA_k} ($t \in \mathbf{R}$) consist of isometries, so that (4) is automatically satisfied; and (3) follows from the power series expansion. So we obtain in particular the result (2) mentioned at the beginning.

COROLLARY. *Let a_k, $k = 1, \ldots, n$, $n \geqslant 2$, be commuting Hermitian elements in a unital Banach algebra U (see [3]) and let P be a polynomial as above. If $P(a_1, \ldots, a_n)x = 0$ (or $xP(a_1, \ldots, a_n) = 0$) for some $x \in U$, then $a_k x = 0$ (resp. $xa_k = 0$), $k = 1, \ldots, n$.*

Here we use the fact that the left and right multiplication operators

$A_k = L_{a_k}$, $B_k = R_{a_k}$, $k = 1, \ldots, n$ are Hermitian operators on U (as for every $a \in U$, $e^{itL_a}x = e^{ita}x$ and $e^{itR_a}x = xe^{ita}$, $t \in \mathbf{R}$) and $P(A_1, \ldots, A_n)x = P(a_1, \ldots, a_n)x$, $P(B_1, \ldots, B_n)x = xP(a_1, \ldots, a_n)$.

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Bemerkung zu einem Satz von Akcoglu und Krengel

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Abstract. For measurable $f\colon \mathbf{R} \to \mathbf{R}$ let $\|f\|_{\mathrm{t.v.}}$ be the total variation and

$$\|f\|_{\mathrm{ess.t.v.}} := \overline{\lim_{t \to 0+0}}\, t^{-1} \int |f(x+t) - f(x)|\, dx.$$

If $\|f\|_{\mathrm{ess.t.v.}} < \infty$,

$$g(x) := \lim_{t \to 0+0} t^{-1} \int_x^{x+t} f(u)\, du$$

is well-defined, continuous from the right, $f = g$ a.e., $\|g\|_{\mathrm{t.v.}} = \|f\|_{\mathrm{ess.t.v.}}$, and for every $\bar{f}$ satisfying $f = \bar{f}$ λ-a.e. and $\|\bar{f}\|_{\mathrm{t.v.}} = \|f\|_{\mathrm{ess.t.v.}}$, $\bar{f}(x)$ lies between $g(x)$ and $g(x-0)$ for all $x \in \mathbf{R}$. This result sharpens a theorem of Akcoglu and Krengel.

Für eine meßbare Funktion $f\colon \mathbf{R} \to \mathbf{R}$ bezeichne $\|f\|_{\mathrm{t.v.}}$ die Totalvariation und

$$\|f\|_{\mathrm{ess.t.v.}} := \overline{\lim_{t \to 0+0}}\, t^{-1} \int |f(x+t) - f(x)|\, dx \tag{1}$$

die essentielle Totalvariation von f. (Der Grenzwert in (1) existiert, was wir aber nicht verwenden werden.) λ sei das Lebesguemaß auf $\mathbf{R}$. Akcoglu und Krengel beweisen in [1] den folgenden interessanten

SATZ. *Für jede meßbare Funktion $f\colon \mathbf{R} \to \mathbf{R}$ und jedes $\bar{f}\colon \mathbf{R} \to \mathbf{R}$ mit $f = \bar{f}$ λ-f.ü. gilt $\|f\|_{\mathrm{ess.t.v.}} \leqslant \|\bar{f}\|_{\mathrm{t.v.}}$. Es gibt ein $\bar{f}$ mit $f = \bar{f}$ λ-f.ü. und $\|f\|_{\mathrm{ess.t.v.}} = \|\bar{f}\|_{\mathrm{t.v.}}$.*

Ziel dieser Note ist der Beweis der folgenden Verschärfung des obigen Satzes:

Sei $\|f\|_{\mathrm{ess.t.v.}} < \infty$. Dann existiert für jedes $x \in \mathbf{R}$

$$g(x) := \lim_{t \to 0+0} t^{-1} \int_x^{x+t} f(u)\, du. \tag{2}$$

g ist rechtsstetig, $f = g$ λ-f.ü., $\|g\|_{\mathrm{t.v.}} = \|f\|_{\mathrm{ess.t.v.}}$, und für jedes $\bar{f}\colon \mathbf{R} \to \mathbf{R}$ mit $f = \bar{f}$ λ-f.ü. und $\|\bar{f}\|_{\mathrm{t.v.}} = \|f\|_{\mathrm{ess.t.v.}}$ gibt es ein $\alpha\colon \mathbf{R} \to [0, 1]$ mit

$$\bar{f}(x) = \alpha(x) g(x) + (1 - \alpha(x)) g(x-0). \tag{3}$$

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