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Uniformly non-$l_n^{(1)}$ Orlicz spaces with Luxemburg norm

by

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Abstract. K. Sundaresan [15] has given a criterion for an Orlicz space $L^{\Phi}(\mu)$ over an atomless measure μ and generated by an Orlicz function Φ satisfying the corresponding condition Δ_2 to be uniformly non-$l_n^{(1)}$. This paper gives some simpler criteria for this property of Orlicz spaces over an atomless as well as a purely atomic measure μ and generated by arbitrary Orlicz functions (the necessity of the corresponding condition Δ_2 is proved here).

0. Introduction. $\boldsymbol{N}$ is the set of positive integers, $\boldsymbol{R}$ is the set of real numbers, $(T, \mathscr{T}, \mu)$ is a space of positive measure. A function $\Phi\colon \boldsymbol{R}\to[0, +\infty]$ is said to be an *Orlicz function* if it is not identically zero and is even, convex, and vanishing and continuous at zero. The *Orlicz space* $L^{\Phi}(\mu)$ is then defined as the set of all equivalence classes of $\mathscr{T}$-measurable functions $x\colon T\to\boldsymbol{R}$ such that $\int_T \Phi(kx(t))\,d\mu<+\infty$ for some $k>0$ depending on x. Under the so-called Luxemburg norm $\|\ \|_{\Phi}$ defined by

$$\|x\|_{\Phi}=\inf\{r>0\colon \int_T \Phi(r^{-1}x(t))\,d\mu\leqslant 1\}$$

the Orlicz space $L^{\Phi}(\mu)$ is a Banach space (see [12, 13]).

Let us write $I(x)=I_{\Phi}(x)=\int_T \Phi(x(t))\,d\mu$ for any $x\in L^{\Phi}(\mu)$. The functional I is a convex modular on $L^{\Phi}(\mu)$ (see [14]).

We define the subspace $E^{\Phi}(\mu)$ of the Orlicz space $L^{\Phi}(\mu)$ by

$$E^{\Phi}(\mu)=\{x\in L^{\Phi}(\mu)\colon I(kx)<+\infty \text{ for any } k>0\}.$$

Recall that an Orlicz function Φ *satisfies condition* Δ_2 *for all* $u\in\boldsymbol{R}$ (*at infinity*) [*at zero*] if the inequality $\Phi(2u)\leqslant K\Phi(u)$ holds for all $u\in\boldsymbol{R}$ (for u satisfying $|u|\geqslant v_0$) [for u satisfying $|u|\leqslant v_0$], where K and v_0 are some positive constants and $\Phi(v_0)>0$ (see [12, 13]).

0.1. LEMMA (see [4, 5] and [10]). *Let* Φ *be an Orlicz function and* $x\in L^{\Phi}(\mu)$. *The condition* $I(x)=1$ *iff* $\|x\|_{\Phi}=1$ *holds iff* Φ *satisfies condition* Δ_2 *for all* $u\in\boldsymbol{R}$ (*at infinity*) [*at zero*] *in the case of a measure space atomless and infinite* (*atomless and finite*) [*purely atomic with measure of atoms equal to one*], *respectively* (1).

(1) In the purely atomic case we assume that $\Phi(c)=1$ for some $c>0$.

An arbitrary Banach space $(X, \|\ \|)$ is said to be *uniformly non-*$l_n^{(1)}$ $(n\in N,\ n\geq 2)$ if there exists an $\varepsilon>0$ such that for any members $x_1, \ldots, x_n$ in X with $\|x_i\|\leq 1$ we have for some choice of signs

$$\|x_1\pm\ldots\pm x_n\|\leq n(1-\varepsilon).$$

Uniformly non-$l_2^{(1)}$ Banach spaces are called *uniformly non-square* (see [8] and [16]).

A Banach space $(X, \|\ \|)$ is called *uniformly convex* if for each $\varepsilon\in(0, 2)$ there exists a $\delta(\varepsilon)\in(0, 1)$ such that $\|x+y\|\leq 2(1-\delta(\varepsilon))$ whenever $x, y\in X$, $\max(\|x\|, \|y\|)\leq 1$ and $\|x-y\|\geq\varepsilon$.

A Banach space $(X, \|\ \|)$ is said to be *B-convex* (see [2]) if it is uniformly non-$l_n^{(1)}$ for some integer $n\geq 2$.

A Banach space $(X, \|\ \|)$ is said to be *strictly convex* (*rotund*) if for any elements $x, y\in X$ such that $x\neq y$ and $\|x\|=\|y\|=1$ we have $\|x+y\|<2$.

Each uniformly convex Banach space $(X, \|\ \|)$ is uniformly non-square, because for any members x, y in X such that $\max(\|x\|, \|y\|)\leq 1$,

$$\left\|\frac{x-y}{2}\right\|\leq\frac{1}{4}\quad\text{or}\quad\left\|\frac{x+y}{2}\right\|\leq 1-\delta\left(\frac{1}{2}\right).$$

Taking $\varepsilon=\min(\frac{3}{4}, \delta(\frac{1}{2}))$, we have $\min(\|(x+y/2)\|, \|(x-y)/2\|)\leq 1-\varepsilon$.

The inverse statement is not true. In fact, there exists a uniformly non-square Banach space which is not rotund. For let $X=\mathbf{R}^2$ and $\|x\|=\|(x_1, x_2)\|=\max(2|x_1|, |x_1|+|x_2|)$. We have

$$B(1)=\{x\in\mathbf{R}^2\colon \|x\|\leq 1\}$$
$$=\{x\in\mathbf{R}^2\colon |x_1|\leq\tfrac{1}{2}\wedge|x_2|\leq 1\}\cap\{x\in\mathbf{R}^2\colon |x_1|+|x_2|\leq 1\}.$$

Since the unit ball $B(1)$ is compact, and $\min(\|x+y\|, \|x-y\|)<2$ for any $x, y\in B(1)$,

$$\sup_{x,y\in B(1)}\min\left(\left\|\frac{x+y}{2}\right\|, \left\|\frac{x-y}{2}\right\|\right)<1,$$

i.e., the norm $\|\ \|$ is uniformly non-square. It is obvious that this norm is not rotund.

Any uniformly non-$l_n^{(1)}$ Banach space $(X, \|\ \|)$ is uniformly non-$l_{n+1}^{(1)}$. Indeed, let $\varepsilon>0$ be the number from the definition of the term "uniformly non-$l_n^{(1)}$" and let $x_1, \ldots, x_{n+1}$ be members of the unit ball in X. Then, for some choice of signs, we have $\|x_1\pm\ldots\pm x_n\|\leq n(1-\varepsilon)$. So

$$\|x_1\pm\ldots\pm x_n+x_{n+1}\|\leq n(1-\varepsilon)+1=(n+1)\left[1-\frac{n\varepsilon}{n+1}\right].$$

1. Uniformly non-$l_n^{(1)}$ Orlicz spaces. First, we shall prove an auxiliary lemma.

1.1. LEMMA. *Any Orlicz function Φ satisfying condition Δ_2 for all $u\in\mathbf{R}$ (at infinity) [at zero] satisfies the condition*

$$\lim_{k\to+\infty}\{\Phi((1+1/k)u)/\Phi(u)\}=1$$

uniformly with respect to $u\in\mathbf{R}$ (with respect to large values of $|u|$) [with respect to small values of $|u|$].

Proof. We shall prove only the case of condition Δ_2 at infinity. Let $K, u_0>0$ be such that $\Phi(2u)\leq K\Phi(u)$ for all $|u|\geq u_0$. Let φ denote the right-hand derivative of Φ. Since

$$\frac{u}{2}\varphi\left(\frac{u}{2}\right)\leq\Phi(u)\leq u\varphi(u)$$

for all $u\in\mathbf{R}$, φ also satisfies condition Δ_2 for $|u|\geq u_0$ with some positive constant K_1. We have for all $k\in N$ and $|u|\geq u_0$

$$\Phi((1+1/k)u)/\Phi(u)=1+\int_{|u|}^{(1+1/k)|u|}\varphi(t)\,dt/\Phi(u)\leq 1+\frac{|u|\,\varphi((1+k^{-1})|u|)}{k\Phi(u)}$$
$$\leq 1+\frac{K_1|u|\,\varphi(|u|)}{k\Phi(u)}\leq 1+\frac{K_1\,\Phi(2u)}{k\Phi(u)}\leq 1+\frac{K_1 K}{k},$$

and the proof is finished.

In the following we will use the inequality

$$(+)\qquad \Phi\left(\frac{u}{n}\right)\leq\frac{\sigma\Phi(u)}{n},\quad\text{where}\quad \sigma\in(0, 1),\ n\in\mathbf{N},\ n\geq 2,\ u\in\mathbf{R}.$$

1.2. THEOREM. *Let Φ be an Orlicz function. The Orlicz space $L^{\Phi}(\mu)$ is uniformly non-$l_n^{(1)}$ iff:*

(i) *Φ satisfies condition Δ_2 for all $u\in\mathbf{R}$ and condition $(+)$ for all $u\in\mathbf{R}$ if μ is an atomless and infinite measure,*

(ii) *Φ is finite, satisfies condition Δ_2 at infinity and inequality $(+)$ for $u\geq\Phi^{-1}(n/\mu(T))=\sup\{v\geq 0\colon \Phi(v)\leq n/\mu(T)\}$ if μ is an atomless and finite measure,*

(iii) *Φ satisfies condition Δ_2 at zero and inequality $(+)$ in some interval $[0, u_0]$, where $u_0>0$, if μ is a purely atomic measure with infinite and countable number of atoms of measure 1, and $\Phi(c)=1$ for some $c>0$.*

Proof. Sufficiency. (i) First, note that it follows from inequality $(+)$ that

$$\Phi\left(\frac{u_1\pm\ldots\pm u_n}{n}\right)\leq\frac{\sigma}{n}\sum_{i=1}^{n}\Phi(u_i)$$

for any $u_1, \ldots, u_n \in \boldsymbol{R}$ and some choice of signs. Indeed, choose the signs in such a manner that $|u_1 \pm \ldots \pm u_n| \leqslant \max_{1 \leqslant i \leqslant n} |u_i|$. Then we have by (+)

$$\Phi\left(\frac{u_1 \pm \ldots \pm u_n}{n}\right) \leqslant \Phi\left(\frac{\max_i |u_i|}{n}\right) \leqslant \frac{\sigma}{n} \Phi(\max_i |u_i|) = \frac{\sigma}{n} \max_i \Phi(u_i) \leqslant \frac{\sigma}{n} \sum_{i=1}^{n} \Phi(u_i).$$

Hence, we get

$$\sum \Phi\left(\frac{u_1 \pm \ldots \pm u_n}{n}\right) \leqslant \frac{2^{n-1}\alpha}{n} \sum_{i=1}^{n} \Phi(u_i), \tag{1.1}$$

where $\alpha = 1-(1-\sigma)/2^{n-1}$. Here and in the following the symbol $\sum$ denotes the summation operator over all possible choices of signs.

Now, let $x_i \in L^{\Phi}(\mu)$ and $\|x_i\|_{\Phi} \leqslant 1$ for $i = 1, \ldots, n$. From (1.1), we get

$$\sum \Phi\left(\frac{x_1(t) \pm \ldots \pm x_n(t)}{n}\right) \leqslant \frac{2^{n-1}\alpha}{n} \sum_{i=1}^{n} \Phi(x_i(t)) \tag{1.2}$$

for any $t \in T$. Let $\varepsilon > 0$ be such that

$$\Phi(u/(1-\varepsilon)) \leqslant \Phi(u)/\alpha \tag{1.3}$$

for all $u \in \boldsymbol{R}$. By Lemma 1.1 such a number ε exists. Applying conditions (1.2) and (1.3), we get

$$\sum \left(\frac{x_1(t) \pm \ldots \pm x_n(t)}{n(1-\varepsilon)}\right) \leqslant \frac{2^{n-1}}{n} \sum_{i=1}^{n} \Phi(x_i(t))$$

for all $t \in T$. Integrating both sides of this inequality over T, we get

$$\sum I\left(\frac{x_1 \pm \ldots \pm x_n}{n(1-\varepsilon)}\right) \leqslant 2^{n-1}.$$

Thus $I((x_1 \pm \ldots \pm x_n)/n(1-\varepsilon)) \leqslant 1$, i.e., $\|x_1 \pm \ldots \pm x_n\|_{\Phi} \leqslant n(1-\varepsilon)$ for some choice of signs.

(ii) Since Φ is finite, it is continuous. Hence, there exists a constant $\theta \in (0, 1)$ such that

$$\Phi(u) \geqslant n\theta/\mu(T) \quad \text{implies} \quad \Phi\left(\frac{u}{n}\right) \leqslant \frac{\sqrt{\sigma}}{n} \Phi(u). \tag{1.4}$$

Now, we shall prove that there exists a constant $\eta \in (0, 1)$ such that, for any $u_1, \ldots, u_n \in \boldsymbol{R}$, we have

$$\sum_{i=1}^{n} \Phi(u_i) \geqslant \frac{\sqrt{\theta}\, n}{\mu(T)} \quad \text{implies} \quad \sum \Phi\left(\frac{u_1 \pm \ldots \pm u_n}{n}\right) \leqslant \frac{2^{n-1}\eta}{n} \sum_{i=1}^{n} \Phi(u_i). \tag{1.5}$$

Let us consider two cases.

1° $\max_i \Phi(u_i) \geqslant n\theta/\mu(T)$. Choose $n-1$ signs in such a manner that $|u_1 \pm \ldots \pm u_n| \leqslant \max_i |u_i|$. Then, by (1.4), we have for these signs

$$\Phi\left(\frac{u_1 \pm \ldots \pm u_n}{n}\right) \leqslant \Phi(\max_i |u_i|/n) \leqslant \frac{\sqrt{\sigma}}{n} \Phi(\max_i |u_i|) \leqslant \frac{\sqrt{\sigma}}{n} \sum_{i=1}^{n} \Phi(u_i).$$

Hence, by the convexity of Φ for all other choices of signs, we get

$$\sum \Phi\left(\frac{u_1 \pm \ldots \pm u_n}{n}\right) \leqslant \frac{2^{n-1}\alpha}{n} \sum_{i=1}^{n} \Phi(u_i),$$

where $\alpha = 1-(1-\sqrt{\sigma})/2^{n-1}$.

2° $\max_i \Phi(u_i) < n\theta/\mu(T)$. Let

$$a = \min\left(\frac{n\sqrt{\theta}-n\theta}{(n-1)\mu(T)}, \frac{n}{2\mu(T)}\right).$$

Then at least two of the numbers $\Phi(u_i)$, $i = 1, \ldots, n$, have values no smaller than a. Otherwise, we have

$$\sum_{i=1}^{n} \Phi(u_i) < \frac{n\theta}{\mu(T)} + (n-1)a \leqslant \frac{n\theta}{\mu(T)} + (n-1)\frac{n\sqrt{\theta}-n\theta}{(n-1)\mu(T)} = \frac{n\sqrt{\theta}}{\mu(T)},$$

a contradiction. Let v_0 and v_1 be positive numbers such that $\Phi(v_0) = a$ and $\Phi(v_1) = n\theta/\mu(T)$ (it may be assumed that $\theta \geqslant \frac{1}{2}$). Let

$$Q = [v_0, v_1] \times [v_0, v_1] \times \underbrace{[0, v_1] \times \ldots \times [0, v_1]}_{n-2\,\text{times}}.$$

The set Q is compact in $\boldsymbol{R}^n$ and, moreover, the function

$$f(\boldsymbol{u}) = \max_i \Phi(u_i) / \sum_{i=1}^{n} \Phi(u_i), \qquad \boldsymbol{u} = (u_1, \ldots, u_n),$$

is continuous and has values smaller than 1 on the set $Q_1 = \{\boldsymbol{u} \in \boldsymbol{R}^n\colon |\boldsymbol{u}| \in Q\}$ where $|\boldsymbol{u}| = (|u_1|, \ldots, |u_n|)$. Thus, there exists a $\xi \in Q$ such that

$$\sup\{f(\boldsymbol{u})\colon \boldsymbol{u} \in Q_1\} = \sup\{f(\boldsymbol{u})\colon \boldsymbol{u} \in Q\} = f(\xi) < 1.$$

We have $|u_1 \pm \ldots \pm u_n| \leqslant \max_i |u_i|$ for some choice of signs. Hence

$$\Phi\left(\frac{u_1 \pm \ldots \pm u_n}{n}\right) \leqslant \Phi(\max_i |u_i|/n) \leqslant \frac{1}{n} \max_i \Phi(u_i) \leqslant \frac{f(\xi)}{n} \sum_{i=1}^{n} \Phi(u_i).$$

Hence, we get

$$\sum \Phi\left(\frac{u_1 \pm \ldots \pm u_n}{n}\right) \leqslant \frac{2^{n-1}\beta}{n} \sum_{i=1}^{n} \Phi(u_i),$$

where $\beta = 1-(1-f(\xi))/2^{n-1} \in (0, 1)$. Putting $\eta = \max(\alpha, \beta)$, we obtain condition (1.5).

We may restrict ourselves in the definition of the uniform non-$l_n^{(1)}$ property of a Banach space X to elements $x_1, \ldots, x_n$ of the unit sphere of X (for the case $n = 2$, see [16]). So, let $\|x_1\|_\Phi = \ldots = \|x_n\|_\Phi = 1$. We have, by Lemma 1.1, $I(x_1) = \ldots = I(x_n) = 1$. Define

$$E = \{t \in T: \sum_{i=1}^{n} \Phi(x_i(t)) \geq n\sqrt{\theta}/\mu(T)\}.$$

Since $\sum_{i=1}^{n} I(x_i \chi_{T\setminus E}) \leq n\sqrt{\theta}$, $\sum_{i=1}^{n} I(x_i \chi_E) \geq n - n\sqrt{\theta}$. Hence and from (1.5) we get

$$\begin{aligned}2^{n-1} - \sum I\left(\frac{x_1 \pm \ldots \pm x_n}{n}\right) &= \frac{2^{n-1}}{n}\sum_{i=1}^{n} I(x_i) - \sum I\left(\frac{x_1 \pm \ldots \pm x_n}{n}\right)\\ &\geq \frac{2^{n-1}}{n}\sum_{i=1}^{n} I(x_i\chi_E) - \sum I\left(\frac{x_1 \pm \ldots \pm x_n}{n}\chi_E\right)\\ &\geq \frac{2^{n-1}(1-\eta)}{n}\sum_{i=1}^{n} I(x_i\chi_E) \geq 2^{n-1}(1-\eta)(1-\sqrt{\theta}).\end{aligned}$$

This means that

$$\sum I\left(\frac{x_1 \pm \ldots \pm x_n}{n}\right) \leq 2^{n-1}(1-(1-\eta)(1-\sqrt{\theta})) = 2^{n-1}(1-\delta),$$

where $\delta = (1-\eta)(1-\sqrt{\theta})$. Applying Lemma 1.1 to the interval $[v_0, +\infty)$ for sufficiently small v_0, we conclude that there exists a constant $\varepsilon > 0$, independent of x_i, such that

$$\sum I\left(\frac{x_1 \pm \ldots \pm x_n}{n(1-\varepsilon)}\right) \leq 2^{n-1},$$

i.e., $\|x_1 \pm \ldots \pm x_n\|_\Phi \leq n(1-\varepsilon)$ for some choice of signs.

(iii) Assume without loss of generality that $\Phi(u_0) < 1$. It is clear that

$$\Phi\left(\frac{u}{n}\right) < \frac{1}{n}\Phi(u) \tag{1.6}$$

for every $u > 0$ such that $\Phi(u) > 0$. Otherwise, we have $\Phi(u/n) = \Phi(u)/n$ for some $u > 0$, and so (see e.g. [5], Lemma 1.8) $\Phi(\min(u, u_0)/n) = \Phi(\min(u, u_0))/n$, a contradiction with inequality (+) for $u \in [0, u_0]$.

Let us consider on the interval $[u_0, c]$ the function

$$f(u) = n\Phi(u/n)/\Phi(u).$$

By condition (1.6) we can see that $f(u) < 1$ on this interval. Since Φ is continuous, there exists a number $\xi \in [u_0, c]$ such that

$$\sup\{f(u): u \in [u_0, c]\} = f(\xi) < 1.$$

Hence and from (+) for the interval $[0, u_0]$ we get

$$\Phi\left(\frac{u}{n}\right) \leq \frac{\alpha\Phi(u)}{n} \tag{1.7}$$

for any $u \in [0, c]$, where $\alpha = \max(\sigma, f(\xi)) \in (0, 1)$. By Lemma 1.1 we get

$$\forall \eta > 1\ \exists \xi > 1\ \forall u \in [0, c]: \quad \Phi(\xi u) \leq \eta\Phi(u). \tag{1.8}$$

Let x_i, $i = 1, \ldots, n$, be elements of the unit sphere of $L^\Phi(\mu)$. In the same manner as in the proof of sufficiency in case (i), we get

$$\sum I\left(\frac{x_1 \pm \ldots \pm x_n}{n}\right) \leq \frac{2^{n-1}\gamma}{n}\sum_{i=1}^{n} I(x_i) \leq 2^{n-1}\gamma,$$

where $\gamma = 1-(1-\alpha)/2^{n-1} \in (0, 1)$. Applying condition (1.8), we get

$$\sum I\left(\frac{x_1 \pm \ldots \pm x_n}{n(1-\varepsilon)}\right) \leq 2^{n-1},$$

so for some choice of signs

$$\|x_1 \pm \ldots \pm x_n\|_\Phi \leq n(1-\varepsilon)$$

with absolute $\varepsilon > 0$. The proof of sufficiency is finished.

Necessity. If Φ does not satisfy the corresponding condition Δ_2 or if Φ is infinite in the case of μ atomless, then $L^\Phi(\mu)$ contains an isometric copy of l^∞ (see [5, 6] and [11]), and so it is not uniformly non-$l_n^{(1)}$, because l^∞ is not such (see [3]).

Now, we shall prove the necessity of the inequality (+) in the corresponding intervals. We shall consider two cases.

(i) or (ii). If Φ does not satisfy (+) for all $u \in \mathbf{R}$ (or for $u \in [\Phi^{-1}(n/\mu(T)), +\infty)$ if $\mu(T) < +\infty$), then for every sequence (σ_k), $\sigma_k \uparrow 1$ as $k \uparrow +\infty$, there exist sequences (u_k) of positive numbers and $A_1, \ldots, A_n$ of $\mathscr{T}$-measurable and pairwise disjoint subsets of T such that

$$\mu(A_i) = \frac{1}{\Phi(u_k)} \quad \text{and} \quad \Phi\left(\frac{u_k}{n}\right) > \frac{\sigma_k}{n}\Phi(u_k)$$

for $k \in N$ and $i = 1, \ldots, n$. Defining $x_i = u_k\chi_{A_i}$ for $i = 1, \ldots, n$, we have

$$I\left(\frac{x_1 \pm \ldots \pm x_n}{n\sigma_k}\right) \geq \frac{1}{\sigma_k}\sum_{i=1}^{n} I\left(\frac{x_i}{n}\right) = \frac{n}{\sigma_k} I\left(\frac{x_1}{n}\right) = \frac{n}{\sigma_k}\int_{A_1}\Phi\left(\frac{u_k}{n}\right)d\mu > 1,$$

and so $\|x_1 \pm \ldots \pm x_n\|_\Phi \geqslant n\sigma_k$ for any choice of signs. This means that $L^\Phi(\mu)$ is not uniformly non-$l_n^{(1)}$.

(iii). Assume that Φ satisfies inequality (+) in no neighbourhood of zero. Let us put

$$u_k = \Phi^{-1}\left(\frac{1}{k}\right), \quad \sigma_k = 1-\frac{1}{k}; \quad k = 1, 2, \ldots$$

Then there exists a sequence (v_k), $0 < v_k \leqslant u_k$, such that

$$\Phi\left(\frac{v_k}{n}\right) > \frac{\sigma_k}{n}\Phi(v_k) \tag{1.9}$$

for $k = 1, 2, \ldots$ Let m_k denote the integer part of $1/\Phi(v_k)$ for $k = 1, 2, \ldots$ Then

$$m_k \Phi(v_k) \leqslant 1 \quad \text{and} \quad (m_k+1)\Phi(v_k) > 1 \tag{1.10}$$

for any $k \in N$. Next, we define

$$x_i = \sum_{j=1}^{m_k} v_k e_{j+(i-1)m_k}, \quad i = 1, \ldots, n,$$

for an arbitrary fixed $k \in N$, where $e_j = (0, \ldots, 0, 1, 0, \ldots)$ is the jth basic sequence. Obviously, $x_i \in L^\Phi(\mu)$ for $i = 1, \ldots, n$, and moreover

$$\sigma_k \leqslant I(x_i) \leqslant 1, \quad i = 1, \ldots, n. \tag{1.11}$$

Applying conditions (1.9) and (1.11), we get for any choice of signs

$$I\left(\frac{x_1 \pm \ldots \pm x_n}{n}\right) = \sum_{i=1}^{nm_k} \Phi\left(\frac{v_k}{n}\right) > \frac{\sigma_k}{n}\sum_{i=1}^{nm_k} \Phi(v_k) > \sigma_k^2. \tag{1.12}$$

Since $I(x) \leqslant \|x\|_\Phi$ for any $x \in L^\Phi(\mu)$ with $I(x) \leqslant 1$, we get by (1.12) $\|x_1 \pm \ldots \pm x_n\|_\Phi \geqslant n\sigma_k^2$ for any choice of signs. So the space $L^\Phi(\mu)$ is not uniformly non-$l_n^{(1)}$. The proof of the theorem is complete.

2. B-convexity of Orlicz spaces. The author and A. Kamińska in [7] have proved that the Musielak–Orlicz space $L^\Phi(\mu)$ is B-convex iff it is reflexive. Thus, every reflexive Orlicz space is uniformly non-$l_n^{(1)}$ for some integer $n \geqslant 2$. However, it does not follow from these results for what n this holds. We shall give below a solution of this problem in the case of an atomless and infinite measure μ. This method may also be used in all other cases of measure μ.

2.1. THEOREM. *Let μ be an atomless and infinite measure and let Φ be an Orlicz function. Then the space $L^\Phi(\mu)$ is B-convex iff it is reflexive, i.e., the function Φ and its complementary function Ψ satisfy condition Δ_2 for all $u \in R$.*

Moreover, if $K = \sup_{u>0}[\Psi(2u)/\Psi(u)] < +\infty$ and Φ satisfies condition Δ_2 for all $u \in R$, then the space $L^\Phi(\mu)$ is at least uniformly non-$l_n^{(1)}$, where $n = E(K)+1$ ($E(K)$ denotes here the integer part of K).

Finally, if $K = \sup_{u>0}[\Psi(2u)/\Psi(u)] \leqslant 4$ and Φ satisfies condition Δ_2 for all $u \in R$, then the space $L^\Phi(\mu)$ is uniformly non-square; thus it is uniformly non-$l_n^{(1)}$ for any integer $n \geqslant 2$ (see the note added in proof).

Proof. Let $L^\Phi(\mu)$ be reflexive. Then the Orlicz functions Φ and Ψ satisfy condition Δ_2 for all $u \in R$ (see e.g. [13]). Denote $K = \sup_{u>0}[\Psi(2u)/\Psi(u)]$. We have for all $u \in R$

$$\Phi\left(\frac{2u}{K}\right) = \sup_{v>0}\left[\frac{2uv}{K} - \Psi(v)\right] \leqslant \frac{1}{K}\sup_{v>0}[2uv - \Psi(2v)] = \frac{\Phi(u)}{K}. \tag{2.1}$$

Hence, for any $u \in R$,

$$\Phi\left(\frac{u}{K}\right) = \Phi\left(\frac{1}{2}\cdot\frac{2u}{K}\right) \leqslant \frac{1}{2}\Phi\left(\frac{2u}{K}\right) \leqslant \frac{\Phi(u)}{2K}. \tag{2.2}$$

Let $n = E(K)+1$. We have by (2.2)

$$\Phi\left(\frac{u}{n}\right) = \Phi\left(\frac{K}{n}\cdot\frac{u}{K}\right) \leqslant \frac{K}{n}\Phi\left(\frac{u}{K}\right) \leqslant \frac{\Phi(u)}{2n} \tag{2.3}$$

for all $u \in R$, i.e., Φ satisfies the assumption of Theorem 1.2 (i) with $\sigma = \frac{1}{2}$. So, the space $L^\Phi(\mu)$ is uniformly non-$l_n^{(1)}$ with $n = E(K)+1$.

If, additionally, $K \leqslant 4$, we have by (2.1)

$$\Phi\left(\frac{u}{2}\right) = \Phi\left(\frac{K}{4}\cdot\frac{2u}{K}\right) \leqslant \frac{K}{4}\Phi\left(\frac{2u}{K}\right) \leqslant \frac{\Phi(u)}{4} = \frac{\sigma}{2}\Phi(u)$$

for all $u \in R$, where $\sigma = \frac{1}{2}$. Thus, it follows from Theorem 1.2 (i) that $L^\Phi(\mu)$ is uniformly non-square.

Conversely, let $L^\Phi(\mu)$ be B-convex, i.e., uniformly non-$l_n^{(1)}$ for some integer $n \geqslant 2$. Then, by Theorem 1.2 (i), Φ satisfies condition Δ_2 for all $u \in R$ and condition (+) for all $u \in R$. Hence for all $u \in R$

$$\begin{aligned}\Psi\left(\frac{u}{\sigma}\right) &= \sup_{v>0}\left[\frac{uv}{\sigma} - \Phi(v)\right] \leqslant \sup_{v>0}\left[\frac{uv}{\sigma} - \frac{n}{\sigma}\Phi\left(\frac{v}{n}\right)\right] \\ &= \frac{n}{\sigma}\sup_{v>0}\left[\frac{uv}{n} - \Phi\left(\frac{v}{n}\right)\right] = \frac{n}{\sigma}\Psi(u).\end{aligned} \tag{2.4}$$

Let $l \in N$ be such that $2 \leqslant \sigma^{-l}$. Then, by (2.4),

$$\Psi(2u) \leqslant \Psi(\sigma^{-l}u) \leqslant \left(\frac{n}{\sigma}\right)^l \Psi(u),$$

for all $u \in R$, i.e., Ψ satisfies condition Δ_2 for all $u \in R$, and so the space $L^\Phi(\mu)$ is reflexive (see [13]).

3. Corollaries and examples.

3.1. COROLLARY. *Let Φ_1 and Φ_2 be Orlicz functions and let $\Psi = \Phi_1 \cdot \Phi_2$. Then the Orlicz space $L^{\Psi}(\mu)$ is uniformly non-square iff Φ_1 and Φ_2 satisfy the corresponding condition Δ_2 (see Theorem 1.2).*

Proof. Note that Ψ satisfies the corresponding condition Δ_2 iff both functions Φ_1 and Φ_2 satisfy it. Moreover, for any $u \in \boldsymbol{R}$

$$\Psi\left(\frac{u}{2}\right) = \Phi_1\left(\frac{u}{2}\right)\Phi_2\left(\frac{u}{2}\right) \leqslant \left(\frac{1}{2}\Phi_1(u)\right)\left(\frac{1}{2}\Phi_2(u)\right) = \frac{\Psi(u)}{4},$$

i.e., the function Ψ satisfies inequality (+) for all $u \in \boldsymbol{R}$ with $\sigma = \frac{1}{2}$.

3.2. LEMMA. *If φ is a nonnegative and nondecreasing function defined on an interval $[a, b] \subset \boldsymbol{R}$, then for each number $c \in [a, b]$ the following inequality holds:*

$$\frac{1}{b-c}\int_c^b \varphi(t)\,dt \geqslant \frac{1}{b-a}\int_a^b \varphi(t)\,dt.$$

Proof. Since the function $f(u) = u/(u+v)$ is increasing for $v > 0$,

$$\frac{\int_c^b \varphi(t)\,dt}{\int_a^b \varphi(t)\,dt} = \frac{\int_c^b \varphi(t)\,dt}{\int_a^c \varphi(t)\,dt + \int_c^b \varphi(t)\,dt} \geqslant \frac{(b-c)\varphi(c)}{\int_a^c \varphi(t)\,dt + (b-c)\varphi(c)}$$

$$\geqslant \frac{(b-c)\varphi(c)}{(c-a)\varphi(c) + (b-c)\varphi(c)} = \frac{b-c}{b-a},$$

and the proof is finished.

3.3. LEMMA. *Every Orlicz function Φ vanishing only at zero, which has the right-hand derivative φ satisfying the condition*

$$\varphi(ku) \geqslant k\varphi(u) \tag{3.1}$$

for all $u \in \boldsymbol{R}$ with an absolute constant $k \geqslant 2$, satisfies (+) for all $u \in \boldsymbol{R}$, with $n = k$ if k is an integer and with $n = E(k)+1$ otherwise.

Proof. We have, for every $u > 0$,

$$\Phi(u)/\Phi(u/k) = \Big(\Phi(u/k) + \int_{u/k}^{u} \varphi(t)\,dt\Big)/\Phi(u/k) = 1 + \Big(\int_{u/k}^{u} \varphi(t)\,dt\Big)/\Phi(u/k). \tag{3.2}$$

Putting $t = ks$ in the last integral, we get

$$\int_{u/k}^{u} \varphi(t)\,dt = k\int_{u/k^2}^{u/k} \varphi(ks)\,ds \geqslant k^2 \int_{u/k^2}^{u/k} \varphi(s)\,ds. \tag{3.3}$$

Applying Lemma 3.2 with $a = 0$, $b = u/k$ and $c = u/k^2$, we get

$$\int_{u/k^2}^{u/k} \varphi(s)\,ds \geqslant \frac{k-1}{k}\int_0^{u/k} \varphi(s)\,ds.$$

Hence, by (3.3), we obtain

$$\int_{u/k}^{u} \varphi(t)\,dt \geqslant k(k-1)\int_0^{u/k} \varphi(t)\,dt = k(k-1)\Phi(u/k)$$

for any $u > 0$. Applying this inequality in condition (3.2), we get

$$\Phi\left(\frac{u}{k}\right) \leqslant \frac{k}{k(k-1)+1}\cdot\frac{1}{k}\Phi(u)$$

for all $u \in \boldsymbol{R}$, i.e., Φ satisfies inequality (+) for all $u \in \boldsymbol{R}$ with $\sigma = k/\big(k(k-1)+1\big)$.

Let k be noninteger. Denoting $n = E(k)+1$, where $E(k)$ is the integer part of k, we get

$$\Phi\left(\frac{u}{n}\right) = \Phi\left(\frac{k}{n}\cdot\frac{u}{k}\right) \leqslant \frac{k}{n}\Phi\left(\frac{u}{k}\right) \leqslant \frac{\sigma}{n}\Phi(u)$$

for all u, where $\sigma = k/\big(k(k-1)+1\big)$. Thus, the proof is finished.

3.4. COROLLARY. *Every Orlicz function Φ satisfying the corresponding condition Δ_2 and possessing a convex right-hand derivative φ generates a uniformly convex (hence uniformly non-square) Orlicz space.*

Indeed, since φ vanishes at zero and is convex,

$$\varphi\big((1+\varepsilon)u\big) \geqslant (1+\varepsilon)\varphi(u) \tag{3.4}$$

for any ε, $u > 0$. Thus, the function Φ is uniformly convex on $\boldsymbol{R}$ (see [1]) and, by the corresponding condition Δ_2, the space $L^{\Phi}(\mu)$ is uniformly convex (see [9]). Uniform non-squareness of $L^{\Phi}(\mu)$ can also be obtained immediately by (3.4) and Theorem 1.2.

3.5. EXAMPLE. *For each integer $n \geqslant 2$ there exists a uniformly non-$l_n^{(1)}$ Orlicz space $L^{\Phi}(\mu)$ which is not rotund.*

Indeed, define the right-hand derivative φ of Φ by

$$\varphi(t) = \begin{cases} n^k & \text{for } t \in [n^k, n^{k+1}), \quad k = 0, 1, 2, \ldots, \\ n^{-k} & \text{for } t \in [n^{-k}, n^{-k+1}), \quad k = 1, 2, \ldots, \\ 0 & \text{for } t = 0. \end{cases}$$

We have $\varphi(nt)/\varphi(t) = n$ for any $t > 0$. Thus, Φ satisfies condition Δ_2 for all $u \in \boldsymbol{R}$ and inequality (+) for all $u \in \boldsymbol{R}$ (see Lemma 3.3). Hence, $L^{\Phi}(\mu)$ is uniformly non-$l_n^{(1)}$ for an arbitrary positive measure μ. However, this space is not strictly convex, because the function Φ is not strictly convex (see [4], [10], [17] and [18]).

3.6. EXAMPLE. *There exists a strictly convex Orlicz space $L^{\Phi}(\mu)$ which is not B-convex.*

Indeed, let $\Phi(u)=|u|\log(1+|u|)$ for $u\in\mathbf{R}$. This function satisfies condition Δ_2 for all $u\in\mathbf{R}$ and is strictly convex on the interval $[0,+\infty)$. So, $L^{\Phi}(\mu)$ is a strictly convex (rotund) space for any positive measure μ (see [4], [10] and [17, 18]). However, by L'Hospital's formula we have $\lim_{u\to+\infty}\big(n\Phi(u/n)/\Phi(u)\big)=1$ for any $n\in\mathbf{N}$, i.e., Φ does not satisfy condition (+) for large values of u. So, $L^{\Phi}(\mu)$ is not B-convex for any atomless measure μ. Let us note that $L^{\Phi}(\mu)$ is uniformly non-square for a purely atomic measure as in Theorem 1.2, because

$$\lim_{u\to 0}\big(2\Phi(u/2)/\Phi(u)\big)=\tfrac{1}{2}.$$

3.7. COROLLARY. *Let Φ be an Orlicz function and μ a positive measure. The subspace $E^{\Phi}(\mu)$ is uniformly non-$l_n^{(1)}$ iff $L^{\Phi}(\mu)$ possesses this property.*

Proof. It suffices to prove that $L^{\Phi}(\mu)$ is uniformly non-$l_n^{(1)}$ if $E^{\Phi}(\mu)$ possesses this property. Assume that $E^{\Phi}(\mu)$ is uniformly non-$l_n^{(1)}$. Then $E^{\Phi}(\mu)$ is reflexive, because any B-convex Banach space with unconditional Schauder basis is reflexive (see [3]). So, Φ and its complementary function Ψ satisfy the corresponding condition Δ_2. Moreover, it follows from the proof of Theorem 1.2 that Φ satisfies inequality (+) in the corresponding interval according to the measure μ. Thus, $L^{\Phi}(\mu)$ is uniformly non-$l_n^{(1)}$ by Theorem 1.2.

3.8. COROLLARY. *Call a modular $I=I_{\Phi}$ uniformly non-$l_n^{(1)}$ if there exists a constant $\varepsilon>0$ such that for each $x_i\in L^{\Phi}(\mu)$ with $I(x_i)=1$ for $i=1,\ldots,n$ we have $I\big((x_1\pm\ldots\pm x_n)/n\big)\leqslant 1-\varepsilon$ for some choice of signs. Theorem 1.2 is true with the modular I in place of the norm $\|\ \|_{\Phi}$ without the corresponding condition Δ_2.*

This follows immediately from the proof of Theorem 1.2.

3.9. COROLLARY. *Let μ be a finite and atomless measure. For any integer $n\geqslant 2$ there exists an Orlicz function Φ such that the space $L^{\Phi}(\mu)$ is B-convex and is not uniformly non-$l_n^{(1)}$.*

It suffices to assume that $\Phi(u)=0$ for $0\leqslant u\leqslant n/\mu(T)$ and $\Phi(u)=u^2-\big(n/\mu(T)\big)^2$ for $u\geqslant n/\mu(T)$ (see Theorem 1.2 (ii)).

4. Added in proof. The following theorem (stronger than Th. 2.1) holds:

4.1. THEOREM. *B-convexity and uniform non-squareness of Orlicz spaces over an atomless infinite measure and over a purely atomic measure as in Theorem 1.2 (iii) coincide.*

Proof. We shall prove only the sequence case. The proof in the case of an atomless infinite measure is analogous (but simpler), so it is omitted here. Let l^{Φ} be a B-convex Orlicz space. Then l^{Φ} is reflexive (see [7]), and so Φ and its complementary function Ψ satisfy

condition Δ_2 at zero (see [13]). We may assume that Φ and Ψ have finite values and satisfy the condition

$$\lim_{u\to\infty}(\Phi(u)/u)=\lim_{u\to\infty}(\Psi(u)/u)=\infty. \tag{4.1}$$

Otherwise we define a new Orlicz function Φ_1 by $\Phi_1(u)=\Phi(u)$ for $\Phi(u)\leqslant 1$ and $\Phi_1(u)=bu^2+c$ for $\Phi(u)\geqslant 1$, where b and c are positive numbers such that Φ_1 is a convex and continuous function. Obviously, the Orlicz function Φ_1 and its complementary function Ψ_1 satisfy condition (4.1), and the spaces $(L^{\Phi},\|\ \|_{\Phi})$ and $(L^{\Phi_1},\|\ \|_{\Phi_1})$ are isometric.

Take arbitrary number $u_0>0$. By condition (4.1) there exists a number $v_0>0$ such that

$$\Psi(v)\geqslant u_0 v \quad \text{for any } v\geqslant v_0. \tag{4.2}$$

Moreover, by Lemma 1.1, we have

$$\exists a>1\ \forall 0\leqslant v\leqslant v_0:\quad \Phi(au)\leqslant 2a\Psi(u). \tag{4.3}$$

Applying conditions (4.2) and (4.3), we get for $|u|\leqslant u_0$

$$\Phi(u/2)=\sup_{v\geqslant 0}[uv/2-\Psi(v)]=\sup_{0\leqslant v\leqslant v_0}[uv-\Psi(v)]\leqslant\sup_{0\leqslant v\leqslant v_0}[uv/2-\Psi(av)/(2a)]$$
$$=(2a)^{-1}\sup_{0\leqslant v\leqslant v_0}[uav-\Psi(av)]=(2a)^{-1}\Phi(u).$$

Now, it suffices to apply Theorem 1.2 (iii).

4.2. COROLLARY. *In the case of measure as in Theorem 4.1 uniform non-squareness of Orlicz spaces coincides with reflexivity.*

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Invariant states for positive operator semigroups

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Abstract. A characterization is given of the semigroups of normal positive contractive operators on a von Neumann algebra that admit a faithful family of normal states invariant under the action of the semigroup. It is shown that the family (when it exists) may be chosen to consist of one single state if the algebra is σ-finite.

Introduction. In the study of *-automorphism groups on C^*-algebras, invariant states play an important role. Especially the existence of a faithful invariant state or set of states has been a useful tool in the investigation of such groups. In more recent years the existence of a faithful invariant state has been assumed also in connection with positive operator semigroups (see e.g. [6], [7], [15]). Therefore the question arises as to which positive semigroups of operators actually admit a single invariant faithful state, or at least a faithful family of invariant states. In 1972 Størmer [16] characterized the automorphism groups with a faithful family of invariant normal states on a von Neumann algebra. In this paper we will do the same for positive operator semigroups.

In fact we shall give three conditions which are equivalent to the existence of a faithful family of normal invariant states. Since we prove that such a family (if it exists) can be chosen orthogonal, it follows that we may assume that the family consists of one state only if the algebra is σ-finite. If the algebra is not σ-finite, no faithful normal state can exist. Our results generalize and improve results in [11], [12] and [16].

For the readers not especially interested in semigroups, we remark that our results apply to normal positive contractive operators as well. If namely π is such an operator, $\{\pi^n \mid n \in \boldsymbol{N}\}$ is a semigroup of the form considered in this paper.

We first recall some definitions and fix the notation.

(1) Let $\mathscr{M}$ be a von Neumann algebra acting on the Hilbert space $\mathscr{H}$ and $\mathscr{S}$ a semigroup of positive normal contractive operators on $\mathscr{M}$. By $\mathscr{L}(\mathscr{M})$ we denote the space of bounded operators on $\mathscr{M}$ and by $\mathscr{L}_*(\mathscr{M})$ the subspace of $\mathscr{L}(\mathscr{M})$ consisting of normal operators. A bar $^-$ will denote the closure in the point-weak topology of $\mathscr{L}(\mathscr{M})$. By [10] the unit ball $\mathscr{L}(\mathscr{M})_1$ is compact in this topology.