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**Added in proof.** The proof of Theorem 2.6 is not correct. In its final part it is asserted that the set $q(A_x^{\infty} \wedge \#)$ is nowhere dense. It is not clear why this should be so since the function $x^{\wedge \#}$ need not be, a priori, constant on equivalence classes. In fact, it can be shown that Theorem 2.6 implies the non-existence of measurable cardinals (cf. the beginning of § 2).

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# The Lie structure of $C^*$ and Poisson algebras


by

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**Abstract.** Associative algebras with a Lie structure are considered. In particular, we describe the form of maximal Lie ideals of $C^*$ algebras, maximal Lie ideals and maximal finite-codimensional Lie subalgebras of Poisson algebras of functions on symplectic manifolds.

**1. Notation and preliminaries.** There are many natural algebraic objects which carry both an associative and a Lie ring structure. For example, every associative ring $A$ can also be regarded as a Lie ring with the Lie bracket $[X, Y] := XY - YX$.

It is easy to see that in this case $\mathrm{ad}_X$ is a derivation of the associative ring $A$ for all $X \in A$, i.e.,

$$[X, YZ] = [X, Y]Z + Y[X, Z]. \tag{1.1}$$

We also have the identity

$$[X, YZ] + [Y, ZX] + [Z, XY] = 0. \tag{1.2}$$

Another example is the associative ring $C^\infty(M)$ of all smooth functions on a symplectic manifold $M$ with a Lie ring structure given by the Poisson bracket. In this case also $\mathrm{ad}_X$ is a derivation of $C^\infty(M)$ for all $X \in C^\infty(M)$.

More generally, by a *Poisson ring* we shall understand an associative commutative ring $A$ equipped with a Lie bracket which makes $A$ a Lie ring and is such that $\mathrm{ad}_X$ is a derivation of the associative ring $A$ for all $X \in A$.

One can check that (1.2) is then also satisfied.

Our aim in this note is to propose a general approach to investigations of such structures (close to the methods used in [1] and [3]), which gives us various results (partially well-known) concerning the relations between the Lie and the associative structures.

The above examples lead to the following definition:

(1.3) Definition. An associative ring (algebra) $A$ equipped with a Lie bracket which makes $A$ a Lie ring (algebra) and satisfies (1.1) and (1.2) will be called an AL-*ring* (*algebra*).

A *topological* AL-*ring* (*algebra*) is defined in the natural way.

(1.4) Definition. An associative ideal $K$ of an AL-ring (algebra) $A$ which is also a Lie ideal of $A$ will be called an AL-*ideal* of $A$. An AL-*homomorphism* of AL-rings (algebras) $A_1$ and $A_2$ is a mapping $\alpha\colon A_1 \to A_2$ which is simultaneously a homomorphism of the associative and the Lie ring (algebra) structures.

1.5. Remark. AL-rings (algebras) with AL-homomorphisms form a category. For an AL-ideal $K$ of an AL-ring (algebra) $A$ the additive group (vector space) $A/K$ has a natural AL-ring (algebra) structure for which the natural projection $\pi\colon A \to A/K$ is an AL-homomorphism.

For subsets $B$ and $C$ of an AL-ring $A$ we shall denote by $[B, C]$, $BC$ and $B+C$ the sets of all finite sums of the elements $[X, Y]$, $XY$ and $X+Y$, respectively, for $X \in B$ and $Y \in C$.

Instead of $[B, \{X\}]$, $B\{X\}$, $\{X\}B$ and $B+\{X\}$ we shall write $[B, X]$, $BX$, $XB$ and $B+X$, respectively.

By *subrings* (*subalgebras*), *left* or *right ideals* and *ideals* of an AL-ring (algebra) $A$ we shall always understand subrings (subalgebras), left or right ideals and two-sided ideals of $A$ with respect to the associative ring (algebra) structure.

Subrings (subalgebras) and ideals of $A$ with respect to the Lie ring (algebra) structure will be called *Lie subrings* (*subalgebras*) and *Lie ideals.*

Let $L$ be a subset of an AL-ring $A$. We shall use the following notation:

$$N(L) := \{X \in A\colon [X, L] \subset L\},$$
$$\mathrm{ad}^{-1}(L) := \{X \in A\colon [X, A] \subset L\},$$
$$P(L) := \{X \in L\colon AX \subset L,\ XA \subset L \text{ and } AXA \subset L\},$$
$$J(L) := P(\mathrm{ad}^{-1}(L)).$$

The following theorem contains a list of rather trivial and practically well-known observations (see for example [1], [7], [8], [12]), but it will be very useful in the sequel.

(1.6) Theorem. *Let $A$ be an AL-ring (algebra) and let $L$ be an additive subgroup (a linear subspace) of $A$. Then:*

(a) *$N(L)$ is a Lie subring (subalgebra) of $A$.*

(b) *$\mathrm{ad}^{-1}(L)$ is an AL-subring (subalgebra) of $A$ and a Lie ideal of $N(L)$.*

(c) *$P(L)$ is the largest ideal of $A$ contained in $L$.*

(d) *If $L$ is a Lie subring (subalgebra) of $A$, then $L \subset N(L)$ and $P(L)$ is a Lie ideal of $N(L)$. Moreover, $J(L)$ is a Lie ideal of $N(L)$.*

(e) *If $L$ is a Lie ideal of $A$, then $L \subset \mathrm{ad}^{-1}(L)$, $N(L) = A$, $\mathrm{ad}^{-1}(L)$ is a Lie ideal of $A$, $J(L)$ is an AL-ideal of $A$ and $[\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)] \subset J(L)$.*

Proof. (a) This follows immediately from the Jacobi identity.

(b) $\mathrm{ad}^{-1}(L)$ is a Lie subring (subalgebra) and a Lie ideal of $N(L)$ by the Jacobi identity. By (1.2) $\mathrm{ad}^{-1}(L)$ is an associative subring (subalgebra).

(c) Trivial.



(d) Let $L$ be a Lie subring (subalgebra). Obviously, $L \subset N(L)$. Since

$$A[P(L), N(L)] \subset [AP(L), N(L)] + [A, N(L)]P(L) \subset L$$

and similarly $[P(L), N(L)]A \subset L$ and $A[P(L), N(L)]A \subset L$, $P(L)$ is a Lie ideal of $N(L)$. By definition, $J(L) = P(\mathrm{ad}^{-1}(L))$, so as above $J(L)$ is a Lie ideal of $N(\mathrm{ad}^{-1}(L))$. Also, $\mathrm{ad}^{-1}(L)$ is by (b) a Lie ideal of $N(L)$, and so $N(L) \subset N(\mathrm{ad}^{-1}(L))$.

(e) Let $L$ be a Lie ideal of $A$. Then obviously $L \subset \mathrm{ad}^{-1}(L)$, $N(L) = A$ and $\mathrm{ad}^{-1}(L)$ is a Lie ideal of $A$ by (b). By (d), $J(L)$ is an AL-ideal of $A$ and $\mathrm{ad}^{-1}(L)$ is a Lie ideal of $A$ and an associative subring (subalgebra) by (b). Then

$$A[\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)] \subset [A\,\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)] + [A, \mathrm{ad}^{-1}(L)]\,\mathrm{ad}^{-1}(L) \subset \mathrm{ad}^{-1}(L).$$

Similarly, $[\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)]A \subset \mathrm{ad}^{-1}(L)$. Hence by (1.2)

$$[A, A[\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)]A] \subset [A, [\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)]AA] + [[\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)], AAA] \subset [A, \mathrm{ad}^{-1}(L)] \subset L$$

and so $A[\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)]A \subset \mathrm{ad}^{-1}(L)$. Thus

$$[\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)] \subset P(\mathrm{ad}^{-1}(L)) = J(L). \ \blacksquare$$

If $A$ is a topological AL-ring and $L$ is closed, then $N(L)$, $\mathrm{ad}^{-1}(L)$, $P(L)$ and $J(L)$ are closed and we can derive the topological version of (1.6).

(1.7) Definition. For an ideal $J$ of an associative ring $A$ and for a natural $n$ we define $J/n := \{X \in A\colon nX \in J\}$. We define the radical of $J$ as

$$r(J) := \{X \in A\colon \text{there is an } m \text{ such that } (XA)(XA)\ldots(XA)\,(m \text{ times}) \subset J\}.$$

(1.8) Remark. It is easy to see that $J/n$ and $r(J)$ are ideals of $A$.

(1.9) Lemma (Herstein). *Let $A$ be an AL-ring (algebra), $J$ an ideal of $A$ and $X \in A$ such that $[X, [X, A]] \subset J$. Then $[X, A]A[X, A] \subset J/2$.*

Proof. Take $Y, Z \in A$. By (1.1)

$$[X, [X, YZ]] = [X, [X, Y]]Z + 2[X, Y][X, Z] + Y[X, [X, Z]].$$

Hence $[X, Y][X, Z] \in J/2$. Putting $Y := VU$, $V, U \in A$, we get by (1.1) $[X, V]U[X, Z] \in J/2$. $\blacksquare$

The following theorem and corollary generalize the classical theorems about associative rings of Zuev [12] and Herstein [6].

(1.10) Theorem (Zuev). *Let $A$ be an AL-ring (algebra) and let $L$ be a Lie ideal of $A$. Then for each $X \in \mathrm{ad}^{-1}(L)$ the square of the ideal $I$ of $A$ generated by $[X, A]$ lies in $J(L)/2$, i.e., $II \subset J(L)/2$. Moreover, $[A, \mathrm{ad}^{-1}(L)] \subset r(J(L)/2)$.*

Proof. $\mathrm{ad}^{-1}(L)$ is a Lie ideal of $A$ and $[\mathrm{ad}^{-1}(L), \mathrm{ad}^{-1}(L)] \subset J(L)$ by (e) of Theorem (1.6), so by Lemma (1.9),

$$[X, A]A[X, A] \subset J(L)/2 \quad \text{for all } X \in \mathrm{ad}^{-1}(L).$$

It is easy to see that for $Z = [X_1, Y_1] + \ldots + [X_n, Y_n]$, where $X_i \in \mathrm{ad}^{-1}(L)$ and $Y_i \in A$, $i = 1, \ldots, n$, we have

$$(ZA)(ZA)\ldots(ZA)(n+1 \text{ times}) \subset J(L)/2.$$

Hence $Z \in r(J(L)/2)$. ∎

(1.11) Corollary (Herstein). *If $A$ is an* AL-*ring (algebra) which is simple as an associative ring (algebra) and is of characteristic $\neq 2$, then for each Lie ideal $L$ of $A$ we have $[A, A] \subset L$ or $L \subset Z(L)$, where $Z(L)$ is the Lie centre of $A$.*

Proof. If $[A, A] \not\subset L$, then $\mathrm{ad}^{-1}(L) \neq A$ and $J(L) = \{0\}$. Since $A$ is of characteristic $\neq 2$, $J(L)/2 = \{0\}$. If there is an $X \in L$ such that $[X, A] \neq \{0\}$, then the ideal $I$ of $A$ generated by $[X, A]$ equals $A$ and by Theorem (1.10), $AA = \{0\}$. Since $A$ is simple, $A$ as an associative ring (algebra) is generated by one element and by (1.1), $[A, A] = \{0\} \subset L$ — a contradiction. ∎

## 2. Lie subalgebras of finite codimension.

(2.1) Proposition. *Let $A$ be an* AL-*algebra. Then:*

(a) *If $L$ is a finite-codimensional Lie subalgebra of $A$, then $\mathrm{ad}^{-1}(L)$ is a finite-codimensional* AL-*subalgebra of $A$.*

(b) *If $L$ is a finite-codimensional associative subalgebra of $A$, then $P(L)$ is a finite-codimensional ideal of $A$.*

(c) *If $L$ is a finite-codimensional Lie subalgebra of $A$, then $J(L)$ is a finite-codimensional ideal of $A$. In particular, if $A$ has no finite codimensional ideals except $A$, then every finite-codimensional Lie subalgebra of $A$ contains $[A, A]$.*

Proof. (a) Since $L \cap \mathrm{ad}^{-1}(L)$ is finite-codimensional in $L$ as the kernel of the adjoint representation of $L$ in the finite-dimensional vector space $A/L$, $\mathrm{ad}^{-1}(L)$ is of finite codimension in $A$.

(b) Put $K = \{X \in L \colon AX \subset L\}$. $K$ is a left ideal of $A$ and it is of finite codimension as the kernel of the natural representation of $L$ in the finite-dimensional vector space $A/L$ by the right multiplication. Moreover, $P(L) \subset K$ and $P(L)$ is finite-codimensional in $K$ as the kernel of the natural representation of $K$ in $A/K$ by the left multiplication.

(c) Since $\mathrm{ad}^{-1}(L)$ is by Theorem (1.6) (b) an associative subalgebra of $A$ and it is by (a) of finite codimension, $J(L) = P(\mathrm{ad}^{-1}(L))$ is of finite codimension in $A$ by (b).

If $J(L) = A$, then clearly $[A, A] \subset L$. ∎

The above proposition allows us to answer P. de la Harpe's question [5] whether the Banach–Lie algebra $\mathrm{gl}(H, C_\infty)$ of all compact linear



operators on a separable Hilbert space $H$ has a nontrivial Lie subalgebra of finite codimension (the answer for the case of closed Lie subalgebras is given in [8]) and to solve the problem of the existence of nontrivial closed Lie subalgebras of finite codimension for other complex classical Banach–Lie algebras of compact operators.

Let us recall what "classical" means above. Let $H$ be a separable infinite-dimensional Hilbert space and let $1 \leqslant p \leqslant +\infty$. By $\mathrm{gl}(H, C_p)$ we denote the Schatten $p$-class of compact operators on $H$ (see [11]). The classes $\mathrm{gl}(H, C_p)$ are ideals of the associative algebra $\mathrm{gl}(H)$ of all bounded operators on $H$ and $\mathrm{gl}(H, C_p) \subset \mathrm{gl}(H, C_q)$ if $p \leqslant q$. In particular, $\mathrm{gl}(H, C_\infty)$ is the ideal of all compact operators, $\mathrm{gl}(H, C_2)$ is the ideal of Hilbert–Schmidt operators and $\mathrm{gl}(H, C_1)$ is the ideal of nuclear operators. Each $\mathrm{gl}(H, C_p)$ is a Banach algebra and a Banach–Lie algebra with respect to the Schatten $p$-norm.

Let $J_R$ be a conjugation and $J_Q$ an anticonjugation of $H$. This means that there are orthonormal bases $(e_n)_{n\in N}$ and $(f_n)_{n\in Z^*}$ of $H$ such that

$$J_R\Big(\sum_{n\in N} x_n e_n\Big) = \sum_{n\in N} \bar{x}_n e_n$$

and

$$J_Q\Big(\sum_{n\in N} x_{-n} f_{-n} + \sum_{n\in N} x_n f_n\Big) = \sum_{n\in N} x_{-n} f_n - \sum_{n\in N} x_n f_{-n}.$$

We denote after [4]:

$$\mathrm{o}(H, J_R, C_p) := \{X \in \mathrm{gl}(H, C_p) \colon J_R X^* J_R = -X\},$$
$$\mathrm{sp}(H, J_Q, C_p) := \{X \in \mathrm{gl}(H, C_p) \colon J_Q X^* J_Q = X\},$$
$$\mathrm{sl}(H, C_1) := \{X \in \mathrm{gl}(H, C_1) \colon \mathrm{tr}(X) = 0\}.$$

(2.2) Definition. The Lie algebras $\mathrm{gl}(H, C_p)$, $\mathrm{sl}(H, C_1)$, $\mathrm{o}(H, J_R, C_p)$ and $\mathrm{sp}(H, J_Q, C_p)$ are called *classical complex Banach–Lie algebras of compact operators.*

(2.3) Theorem (P. de la Harpe). *The classical complex Banach–Lie algebras of compact operators, except $\mathrm{gl}(H, C_1)$, are topologically simple. The only nontrivial closed Lie ideal of $\mathrm{gl}(H, C_1)$ is $\mathrm{sl}(H, C_1)$.*

(2.4) Theorem (P. de la Harpe). *The Lie algebra $\mathrm{gl}(H, C_\infty)$ has no nontrivial finite-codimensional Lie ideals.*

The following theorem generalizes (2.4).

(2.5) Theorem. *The Lie algebra $\mathrm{gl}(H, C_\infty)$ has no nontrivial finite-codimensional Lie subalgebras.*

*The classical complex Banach–Lie algebras of compact operators, except $\mathrm{gl}(H, C_1)$, have no nontrivial closed finite-codimensional Lie subalgebras. The only such Lie subalgebra of $\mathrm{gl}(H, C_1)$ is $\mathrm{sl}(H, C_1)$.*

Proof. Since in associative algebras ideals are also Lie ideals, by (2.4) $\mathrm{gl}(H, C_\infty)$ has no nontrivial ideals of finite codimension and

$[\mathrm{gl}(H, C_\infty), \mathrm{gl}(H, C_\infty)] = \mathrm{gl}(H, C_\infty)$ (see also [10]). By (2.3), if $A$ is a classical complex Banach–Lie algebra of compact operators, then $A$ has no nontrivial closed ideals and $[A, A]$ is dense in $A$, except $A = \mathrm{gl}(H, C_1)$, where $[\mathrm{gl}(H, C_1), \mathrm{gl}(H, C_1)]$ is dense in $\mathrm{sl}(H, C_1)$ and $\mathrm{sl}(H, C_1)$ is 1-codimensional in $\mathrm{gl}(H, C_1)$. So the theorem follows by Proposition (2.1) (c). ∎

(2.6) Remark. Observe that the classical complex Banach–Lie algebras of compact operators, except $\mathrm{gl}(H, C_\infty)$, have many dense finite-codimensional Lie subalgebras, since $[\mathrm{gl}(H, C_{2p}), \mathrm{gl}(H, C_{2p})] \subset \mathrm{gl}(H, C_p)$.

(2.7) Definition. Let $A$ be a (topological) AL-algebra. A (closed) ideal (Lie subalgebra, Lie ideal) $L$ of $A$ will be called *maximal* if $L \neq A$ and if for each (closed) ideal (Lie subalgebra, Lie ideal) $K$ of $A$ such that $L \subsetneq K$ we have $K = A$.

We shall deal in the sequel with AL-algebras which have the following properties:

$(P_1)$ Every maximal finite-codimensional ideal of $A$ is equal to its radical.

$(P_2)$ The Lie normalizer $N(I)$ of each maximal finite-codimensional ideal $I$ of $A$ is a proper finite-codimensional subspace of $A$.

The following theorem is analogous to a result of Atkin [1].

(2.8) Theorem (Atkin). *Let $A$ be an AL-algebra satisfying $(P_1)$ and $(P_2)$. Then $I \mapsto N(I)$ gives us a one-one correspondence between maximal finite-codimensional ideals $I$ of $A$ and maximal finite-codimensional Lie subalgebras of $A$ not containing $[A, A]$. The inverse mapping is of the form $L \mapsto r(P(L))$.*

*Moreover, each finite-codimensional Lie subalgebra of $A$ not containing $[A, A]$ is contained in $N(I)$ for a maximal finite-codimensional ideal $I$ of $A$.*

We shall use in the proof of the above theorem the following lemma.

(2.9) Lemma. *Let $A$ be an AL-algebra, let $J$ be a finite-codimensional ideal of $A$ and let $I$ be an ideal of $A$ containing $J$ and equal to its radical. Then $N(J) \subset N(I)$.*

Proof. Let $X \in N(J)$. Consider the finite-dimensional associative algebra $A' = A/J$. Then $I' = I/J$ is an ideal of $A'$ equal to its radical and since $\mathrm{ad}_X(J) \subset J$, $\mathrm{ad}_X$ can be projected on $A'$ and gives us a derivation $\mathrm{ad}'_X$ of $A'$.

Take $Y \in I'$ and define the descending sequence

$$V_1 = YA' \supset V_2 = YA'YA' \supset V_3 = YA'YA'YA' \supset \ldots$$

of subspaces of $A'$. Since $A'$ is finite-dimensional, there is a natural $n$ such that $V_n = V_{n+1}$. Let $Z = YU_1 YU_2 \ldots YU_n \in V_n$. It is easy to prove by Leibniz's formula that $(\mathrm{ad}'_X)^n(V_{n+1}) \subset I'$, so $(\mathrm{ad}'_X)^n Z \in I'$. On the other hand, $(\mathrm{ad}'_X)^n Z$ is of the form

$$(\mathrm{ad}'_X Y)U_1(\mathrm{ad}'_X Y)U_2 \ldots (\mathrm{ad}'_X Y)U_n + Y',$$

where $Y' \in I'$. Hence $\mathrm{ad}'_X Y \in r(I') = I'$, i.e., $[X, I] \subset I$. ∎



Proof of Theorem (2.8). Let $I$ be a maximal finite-codimensional ideal of $A$. Observe that $I \cap N(I)$ is an ideal of $A$, since

$$[A(N(I) \cap I), I] \subset [A, I]I + A[N(I), I] \subset I$$

and similarly $(N(I) \cap I)A \subset N(I)$. This ideal is of finite codimension by $(P_2)$ and it is easy to see that it contains $II$.

Thus if $L$ is a Lie subalgebra of $A$, $L \neq A$, such that $N(I) \subset L$, we have

$$II \subset I \cap N(I) \subset P(L), \quad \text{so that} \quad I \subset r(P(L)).$$

$P(L)$ is a finite-codimensional ideal of $A$, $P(L) \subset L \neq A$, so $r(P(L)) \neq A$ ($P(L)$ is contained in some maximal finite-codimensional ideal of $A$ which is equal to its radical). Hence $I = r(P(L))$ by the maximality of $I$. In particular, $I = r(P(N(I)))$.

By (d) of Theorem (1.6), $P(L)$ is a Lie ideal of $N(L)$. Thus

$$N(I) \subset L \subset N(L) \subset N(P(L))$$

and, by Lemma (2.9), $N(P(L)) \subset N(I)$, so that $N(I) = L = N(L) = N(N(I))$. This shows that $N(I)$ is a finite-codimensional maximal Lie subalgebra of $A$. Since $N(N(I)) = N(I) \neq A$, $N(I)$ does not contain $[A, A]$.

If $N(I) = N(I')$ for a maximal finite-codimensional ideal $I'$ of $A$, then as above

$$I = r(P(N(I))) = r(P(N(I'))) = I'.$$

Conversely, let $L$ be a finite-codimensional Lie subalgebra of $A$ which does not contain $[A, A]$. Then the ideal $J(L)$ does not equal $A$ and it is of finite codimension by Proposition (2.1). By Theorem (1.6) (d), $J(L)$ is a Lie ideal of $N(L)$, whence $L \subset N(J(L))$. $J(L)$ is contained in a maximal finite-codimensional ideal $I$ of $A$ which by $(P_1)$ equals its radical. Thus by Lemma (2.9), $L \subset N(J(L)) \subset N(I)$ and the theorem follows. ∎

(2.10) Corollary. *Let $M$ be a $C^\infty$ ($\mathbf{R}$-analytic, Stein) manifold with a $C^\infty$ ($\mathbf{R}$-analytic, holomorphic) symplectic structure. Let $A$ be the Poisson algebra of all $C^\infty$ functions on $M$ with compact support or the Poisson algebra of all $C^\infty$ ($\mathbf{R}$-analytic, holomorphic) functions on $M$. Then*

$$M \ni x \mapsto L_x = \{f \in A\colon df(x) = 0\}$$

*gives us a one-one correspondence between points of $M$ and maximal finite-codimensional Lie subalgebras of $A$ not containing $[A, A]$.*

Proof. The maximal finite-codimensional ideals of $A$ are of the form $I_x = \{f \in A\colon f(x) = 0\}$ for $x \in M$ (see [3]), so $A$ has the property $(P_1)$.

It is easy to see that

$$N(I_x) = \{f \in A\colon df(x) = 0\}.$$

Thus the property $(P_2)$ is also satisfied and the corollary follows by Theorem (2.8). ∎

### 3. Lie ideals.

(3.1) Definition. An ideal $L$ of a Lie ring (algebra) $A$ we shall call *perfect* if $L \neq A$ and $\mathrm{ad}^{-1}(L) = L$.

For a (topological) AL-algebra $A$ denote by $R_a(A)$ the set of all (closed) ideals of $A$ different from $A$ and equal to their radicals and by $R_l(A)$ the set of all (closed) perfect Lie ideals of $A$.

(3.2) Theorem. *Let $A$ be a (topological) associative algebra over a field of characteristic $\neq 2$ and such that the (closed) ideal generated by $[A, A]$ equals $A$. Then we have a one-one mapping $\alpha$ from $R_a(A)$ into $R_l(A)$ given by*

$$R_a(A) \ni I \mapsto \mathrm{ad}^{-1}(I) \in R_l(A).$$

*The inverse mapping is of the form $L \mapsto P(L)$. In particular, if every (closed) ideal of $A$ equals its radical, then $\alpha$ is also "onto".*

Proof. Let $I \in R_a(A)$. Since $[A, \mathrm{ad}^{-1}(I)] \subset I \subset \mathrm{ad}^{-1}(I)$ (the second inclusion follows since $A$ is an associative algebra), $\mathrm{ad}^{-1}(I)$ is a Lie ideal of $A$ and $\mathrm{ad}^{-1}(I) \neq A$ by the assumptions. If $X \in \mathrm{ad}^{-1}(\mathrm{ad}^{-1}(I))$, then $[X, [X, A]] \subset I$ and by Lemma (1.9), $[X, A] \subset r(I) = I$, i.e., $\mathrm{ad}^{-1}(I)$ is perfect. Since $I \subset \mathrm{ad}^{-1}(I)$, $I \subset P(\mathrm{ad}^{-1}(I))$. On the other hand,

$$P(\mathrm{ad}^{-1}(I))[A, A] \subset [A, P(\mathrm{ad}^{-1}(I))A] + [A, P(\mathrm{ad}^{-1}(I))]A \subset I.$$

The (closed) ideal generated by $[A, A]$ equals $A$, so $P(\mathrm{ad}^{-1}(I))A \subset I$. Thus $P(\mathrm{ad}^{-1}(I)) \subset r(I) = I$ and thus $P(\mathrm{ad}^{-1}(I)) = I$.

Suppose that every (closed) ideal of $A$ equals its radical and let $L \in R_l(A)$. Then $J(L) = P(\mathrm{ad}^{-1}(L)) = P(L)$ and by Theorem (1.10), $[A, L] \subset r(P(L)) = P(L)$, i.e., $L \subset \mathrm{ad}^{-1}(P(L))$. On the other hand, from $P(L) \subset L$ it follows that $\mathrm{ad}^{-1}(P(L)) \subset \mathrm{ad}^{-1}(L) = L$. ∎

(3.3) Remark. Note that for each $C^*$-algebra $A$ every closed ideal $I$ of $A$ is self-adjoint (see [2], § 1) and thus $r(I) = I$, because if $X \in A/I$ and $X(A/I) \ldots X(A/I)(n \text{ times}) = \{0\}$, then $(XX^*)^n = 0$.

(3.4) Corollary. *If $A$ is a $C^*$-algebra such that the ideal generated by $[A, A]$ is dense in $A$, then for $A$ as a topological AL-algebra the mapping*

$$R_a(A) \ni I \mapsto \mathrm{ad}^{-1}(I) \in R_l(A)$$

*is a one-one correspondence.*

For a (topological) AL-algebra $A$ denote by $M_a(A)$ the set of all maximal ideals of $A$ and by $M_l(A)$ the set of all maximal perfect Lie ideals of $A$.

(3.5) Theorem. *Let $A$ be a (topological) associative algebra over a field of characteristic $\neq 2$ such that every (closed) ideal of $A$ different from $A$ is contained in some ideal from $R_a(A)$ and that the (closed) vector space generated by $[A, A]$ equals $A$.*



*Then the mapping $I \mapsto \mathrm{ad}^{-1}(I)$ is a bijection from $M_a(A)$ onto $M_l(A)$. The inverse mapping is of the form $L \mapsto P(L)$.*

Proof. It is easy to see that $M_a(A) \subset R_a(A)$, i.e., each maximal ideal of $A$ equals its radical. Thus by Theorem (3.2)

$$M_a(A) \ni I \mapsto \mathrm{ad}^{-1}(I) \in R_l(A)$$

is a one-one mapping.

We shall prove that $\mathrm{ad}^{-1}(I)$ is a maximal Lie ideal of $A$ for $I \in M_a(A)$.

Suppose that for a (closed) Lie ideal $L$ of $A$, $L \neq A$, we have $\mathrm{ad}^{-1}(I) \subset L$. Then $\mathrm{ad}^{-1}(L) \neq A$ by the assumptions and thus $J(L) \neq A$. Since $J(L) \neq A$, $J(L)$ is contained in some ideal from $R_a(A)$, so $r(J(L)) \neq A$ ($\mathrm{cl}(r(J(L))) \neq A$) and by Theorem (1.10), $[A, L] \subset r(J(L))$.

Observe that $I \cap \mathrm{ad}^{-1}(I)$ is an ideal of $A$ because of the inclusions

$$[A(I \cap \mathrm{ad}^{-1}(I)), A] \subset [A, A]I + A[\mathrm{ad}^{-1}(I), A] \subset I$$

and

$$[(I \cap \mathrm{ad}^{-1}(I))A, A] \subset I[A, A] + [\mathrm{ad}^{-1}(I), A]A \subset I.$$

We have $\mathrm{ad}^{-1}(I) \subset L \subset \mathrm{ad}^{-1}(L)$ and thus $I \cap \mathrm{ad}^{-1}(I) \subset J(L)$ by the definition of $J(L)$. Since $II \subset I \cap \mathrm{ad}^{-1}(I)$, $I \subset r(J(L))$. By the maximality of $I$, $I = r(J(L))$ ($I = \mathrm{cl}(r(J(L)))$) and thus $[A, L] \subset I$, i.e., $L \subset \mathrm{ad}^{-1}(I)$, which proves the maximality of $\mathrm{ad}^{-1}(I)$.

It suffices to prove now that the mapping in question is "onto". Take $L \in M_l(A)$. By Theorem (1.10), $[A, L] \subset r(J(L))$. Since $J(L) \neq A$, there is an $I \in R_a(A)$ containing $J(L)$ and hence containing $r(J(L))$. Thus $L \subset \mathrm{ad}^{-1}(I)$ and, by the maximality of $L$, $L = \mathrm{ad}^{-1}(I)$. Since $J(L)$ is the largest ideal of $A$ contained in $\mathrm{ad}^{-1}(L)$ and $I \subset \mathrm{ad}^{-1}(I) = L$, $I = J(L) = r(J(L))$.

If $I$ is not maximal, it is contained in some ideal $I' \in R_a(A)$, $I \neq I'$, and as above $I' = J(L) = I$ — a contradiction. ∎

(3.6) Corollary. *If $A$ is a $C^*$-algebra such that $[A, A]$ is dense in $A$, then $I \mapsto \mathrm{ad}^{-1}(I)$ gives us a one-one correspondence between maximal ideals $I$ of $A$ and maximal perfect Lie ideals of $A$.*

For a more general class of AL-algebras than those mentioned in the above theorems we can prove a weaker result.

Denote

$$\mathrm{ad}^{-\infty}(L) := \bigcap_{i=1}^{\infty} \mathrm{ad}^{-i}(L), \quad \text{where} \quad \mathrm{ad}^{-(i+1)}(L) = \mathrm{ad}^{-1}(\mathrm{ad}^{-i}(L)).$$

It is easy to see that if $L$ is a linear subspace of an AL-algebra $A$, then $\mathrm{ad}^{-\infty}(L)$ is a Lie ideal of $A$.

(3.7) Theorem. *Let $A$ be an AL-algebra over a field of characteristic $\neq 2$. Suppose that $r(J) \neq A$ for each ideal $J$ of $A$ different from $A$. Let $L$ be a Lie* *ideal of $A$ which does not contain $[A, A]$. Then there is an ideal $J$ of $A$, $J \neq A$, such that $L \subset \mathrm{ad}^{-\infty}(J)$.*

*If additionally the ideal of $A$ generated by $[A, A]$ equals $A$ and each ideal of $A$ different from $A$ is contained in some maximal ideal of $A$, then each maximal Lie ideal of $A$ not containing $[A, A]$ is of the form $\mathrm{ad}^{-\infty}(I)$ for a maximal ideal $I$ of $A$. In other words,*

$$M_a(A) \ni I \mapsto \mathrm{ad}^{-\infty}(I) \in R_l(A)$$

*contains $M_l(A)$ in its image.*

Proof. Let $L$ be a Lie ideal of $A$ which does not contain $[A, A]$. Then by Theorem (1.10), $[A, L] \subset r(J(L))$, $J(L) \subset \mathrm{ad}^{-1}(L) \neq A$, and hence $r(J(L)) \neq A$. Thus $L \subset \mathrm{ad}^{-1}(r(J(L)))$ and since $L$ is a Lie ideal of $A$, proceeding by induction we get $L \subset \mathrm{ad}^{-\infty}(r(J(L)))$. Suppose $L$ is maximal. Let $I$ be a maximal ideal of $A$ containing $r(J(L))$. Then $L \subset \mathrm{ad}^{-\infty}(I)$. Since $[A, A] \not\subset I$, $\mathrm{ad}^{-\infty}(I) \neq A$ and $L = \mathrm{ad}^{-\infty}(I)$ by the maximality of $L$. ∎

(3.8) COROLLARY. *Let $M$ be a $C^\infty$ (**R**-analytic, Stein) connected manifold with a $C^\infty$ (**R**-analytic, holomorphic) symplectic structure. Let $A$ be the AL-algebra of all $C^\infty$ (**R**-analytic, holomorphic) functions on $M$ with the Poisson bracket. Let $L$ be a Lie ideal of $A$ which does not contain $[A, A]$. Then for each $f \in L$ there is a sequence $p_1, p_2, \ldots$ of points of $M$ such that $df(p_i) = \ldots = d^i f(p_i) = 0$, $i = 1, 2, \ldots$.*

Proof. The proof is similar to the proof of Proposition 6.6 in [3]. Since $A$ has a unit, every proper ideal of $A$ is contained in some maximal and thus prime ideal of $A$. Hence $r(J) \neq A$ for each ideal $J$ of $A$, $J \neq A$, and by Theorem (3.7), $L \subset \mathrm{ad}^{-\infty}(J)$ for an ideal $J$ of $A$, $J \neq A$. There are $f_1, f_2, \ldots, f_n \in A$ such that $df_1, \ldots, df_n$ span the cotangent bundle of $M$. Then the hamiltonian vector fields $X_1 = \mathrm{s\,grad}(f_1), \ldots, X_n = \mathrm{s\,grad}(f_n)$ span the tangent bundle of $M$.

Suppose that there are an $f \in L$ and a natural $m$ such that $d^{k_p} f(p) = 0$ for each $p \in M$ and some natural $k_p \leqslant m$. Then, as in [3], one can prove that the ideal of $A$ generated by the finite set

$$\Omega = \{(X_{i_k} \circ \ldots \circ X_{i_1}) f \colon i_1, \ldots, i_k = 1, \ldots, n,\ k = 1, \ldots, m\}$$

equals $A$.

But $X_i(h) = [f_i, h]$ for all $h \in A$ and $i = 1, \ldots, n$, and thus $\Omega \subset J$, since $L \subset \mathrm{ad}^{-\infty}(J)$ — a contradiction. ∎

(3.9) COROLLARY. *Let $A$ be the Poisson algebra of all $C^\infty$ functions with compact support on a $C^\infty$ symplectic manifold $M$ or the Poisson algebra of all **R**-analytic functions on an **R**-analytic compact symplectic manifold $M$. Then each Lie ideal of $A$ which does not contain $[A, A]$ is contained in the Lie ideal $L_p = \{f \in A\colon 0 = df(p) = d^2 f(p) = \ldots\}$ for a $p \in M$.*

*In particular, in the **R**-analytic and connected case the only such nontrivial Lie ideal consists of constant functions.*



*In the $C^\infty$ case the Lie ideals $L_p$ are maximal.*

Proof. Each ideal of $A$ different from $A$ is contained in the ideal $I_p = \{f \in A\colon f(p) = 0\}$ for some $p \in M$. It is easy to see that

$$\mathrm{ad}^{-n}(I_p) = \{f \in A\colon df(p) = \ldots = d^n f(p) = 0\},$$

so by Theorem (3.7), $L \subset \mathrm{ad}^{-\infty}(I_p) = L_p$.

To prove that in the $C^\infty$ case $L_p$ is maximal, it suffices to show that $[A, A] + L_p = A$. Let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be coordinates in a neighbourhood of $p$ in which the symplectic form can be written as $\sum_{i=1}^{n} dx_i \wedge dy_i$. Then in a neighbourhood of $p$ we have

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right).$$

Choose $h \in A$, $f \in A$ such that $\frac{\partial f}{\partial x_1} = h$ in a neighbourhood of $p$, and $g \in A$ such that $g = y_1$ in a neighbourhood of $p$. Then $[f, g] = h$ in a neighbourhood of $p$, which proves that $[A, A] + L_p = A$. ∎

Note that the $C^\infty$ version of the above corollary is due to Omori [9].

(3.10) PROPOSITION. *Let $A$ be one of the Poisson algebras mentioned in Corollary (3.8). Then $A$ has no finite-dimensional or Lie-commutative Lie ideals except the Lie ideal of constant functions.*

Proof. It is easy to see that the Lie-centre of $A$ consists of constant functions. Assume that $f \in A$ is not constant. Then there are $p \in M$ and $g \in A$ such that $[f, g](p) \neq 0$. If $f$ is in a Lie ideal $L$, then $[g^n, f] = ng^{n-1}[g, f] \in L$ and we can choose $g$ such that $L$ cannot be finite-dimensional.

If $L$ is commutative, then $[f, g[f, g]] = [f, g]^2 + g[f, [f, g]] = 0$ and since we can choose $g$ such that $g(p) = 0$,

$$[f, g[f, g]](p) = ([f, g](p))^2 \neq 0$$

— a contradiction. ∎

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# Uniformly non-$l_n^{(1)}$ Orlicz spaces with Luxemburg norm

by

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**Abstract.** K. Sundaresan [15] has given a criterion for an Orlicz space $L^{\Phi}(\mu)$ over an atomless measure $\mu$ and generated by an Orlicz function $\Phi$ satisfying the corresponding condition $\Delta_2$ to be uniformly non-$l_n^{(1)}$. This paper gives some simpler criteria for this property of Orlicz spaces over an atomless as well as a purely atomic measure $\mu$ and generated by arbitrary Orlicz functions (the necessity of the corresponding condition $\Delta_2$ is proved here).

**0. Introduction.** $N$ is the set of positive integers, $R$ is the set of real numbers, $(T, \mathscr{T}, \mu)$ is a space of positive measure. A function $\Phi\colon R \to [0, +\infty]$ is said to be an *Orlicz function* if it is not identically zero and is even, convex, and vanishing and continuous at zero. The *Orlicz space* $L^{\Phi}(\mu)$ is then defined as the set of all equivalence classes of $\mathscr{T}$-measurable functions $x\colon T \to R$ such that $\int_T \Phi(kx(t))\,d\mu < +\infty$ for some $k > 0$ depending on $x$. Under the so-called Luxemburg norm $\|\ \|_{\Phi}$ defined by

$$\|x\|_{\Phi} = \inf\{r > 0\colon \int_T \Phi(r^{-1}x(t))\,d\mu \leqslant 1\}$$

the Orlicz space $L^{\Phi}(\mu)$ is a Banach space (see [12, 13]).

Let us write $I(x) = I_{\Phi}(x) = \int_T \Phi(x(t))\,d\mu$ for any $x \in L^{\Phi}(\mu)$. The functional $I$ is a convex modular on $L^{\Phi}(\mu)$ (see [14]).

We define the subspace $E^{\Phi}(\mu)$ of the Orlicz space $L^{\Phi}(\mu)$ by

$$E^{\Phi}(\mu) = \{x \in L^{\Phi}(\mu)\colon I(kx) < +\infty \text{ for any } k > 0\}.$$

Recall that an Orlicz function $\Phi$ *satisfies condition* $\Delta_2$ *for all* $u \in R$ (*at infinity*) [*at zero*] if the inequality $\Phi(2u) \leqslant K\Phi(u)$ holds for all $u \in R$ (for $u$ satisfying $|u| \geqslant v_0$) [for $u$ satisfying $|u| \leqslant v_0$], where $K$ and $v_0$ are some positive constants and $\Phi(v_0) > 0$ (see [12, 13]).

0.1. Lemma (see [4, 5] and [10]). *Let $\Phi$ be an Orlicz function and $x \in L^{\Phi}(\mu)$. The condition $I(x) = 1$ iff $\|x\|_{\Phi} = 1$ holds iff $\Phi$ satisfies condition $\Delta_2$ for all $u \in R$ (at infinity) [at zero] in the case of a measure space atomless and infinite (atomless and finite) [purely atomic with measure of atoms equal to one], respectively*(¹).

(¹) In the purely atomic case we assume that $\Phi(c) = 1$ for some $c > 0$.