The Lie structure of $C^*$ and Poisson algebras

by

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Abstract. Associative algebras with a Lie structure are considered. In particular, we describe the form of maximal Lie ideals of $C^*$ algebras, maximal Lie ideals and maximal finite-codimensional Lie subalgebras of Poisson algebras of functions on symplectic manifolds.

1. Notation and preliminaries. There are many natural algebraic objects which carry both an associative and a Lie ring structure. For example, every associative ring $A$ can also be regarded as a Lie ring with the Lie bracket $[X, Y] := XY - YX$.

It is easy to see that in this case $ad_X$ is a derivation of the associative ring $A$ for all $X \in A$, i.e.,

(1.1) \[ [X, YZ] = [X, Y]Z + Y[X, Z]. \]

We also have the identity

(1.2) \[ [X, Y] + [Y, ZX] + [Z, XY] = 0. \]

Another example is the associative ring $C^\infty(M)$ of all smooth functions on a symplectic manifold $M$ with a Lie ring structure given by the Poisson bracket. In this case also $ad_X$ is a derivation of $C^\infty(M)$ for all $X \in C^\infty(M)$.

More generally, by a Poisson ring we shall understand an associative commutative ring $A$ equipped with a Lie bracket which makes $A$ a Lie ring and is such that $ad_X$ is a derivation of the associative ring $A$ for all $X \in A$.

One can check that (1.2) is then also satisfied.

Our aim in this note is to propose a general approach to investigations of such structures (close to the methods used in [1] and [3]), which gives us various results (partially well-known) concerning the relations between the Lie and the associative structures.

The above examples lead to the following definition:

(1.3) Definition. An associative ring (algebra) $A$ equipped with a Lie bracket which makes $A$ a Lie ring (algebra) and satisfies (1.1) and (1.2) will be called an AL-ring (algebra).

A topological AL-ring (algebra) is defined in the natural way.

(1.4) Definition. An associative ideal $K$ of an AL-ring (algebra) $A$ which is also a Lie ideal of $A$ will be called an AL-ideal of $A$. An AL-homomorphism

\[ 
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\]
of AL-rings (algebras) $A_1$ and $A_2$ is a mapping $\alpha: A_1 \to A_2$ which is simultaneously a homomorphism of the associative and the Lie ring (algebra) structures.

1.5. Remark. AL-rings (algebras) with AL-homomorphisms form a category. For an AL-ideal $K$ of an AL-ring (algebra) $A$ the additive group (vector space) $A/K$ has a natural AL-ring (algebra) structure for which the natural projection $\pi: A \to A/K$ is an AL-homomorphism.

For subsets $B$ and $C$ of an AL-ring $A$ we shall denote by $[B, C], BC$ and $B + C$ the sets of all finite sums of the elements $[X, Y], XY$ and $X + Y$, respectively, for $X \in B$ and $Y \in C$.

Instead of $[B, \{X\}], B \{X\}, \{X\} B$ and $B + \{X\}$ we shall write $[B, X], BX, XB$ and $B + X$, respectively.

By subrings (subalgebras), left or right ideals and ideals of an AL-ring (algebra) $A$ we shall always understand subrings (subalgebras), left or right ideals and two-sided ideals of $A$ with respect to the associative ring (algebra) structure.

Subrings (subalgebras) and ideals of $A$ with respect to the Lie ring (algebra) structure will be called Lie subrings (subalgebras) and Lie ideals.

Let $L$ be a subset of an AL-ring $A$. We shall use the following notation:

$$
N(L):= \{X \in A: [X, L] \subseteq L\},
$$
$$
ad^{-1}(L):= \{X \in A: [X, A] \subseteq L\},
$$
$$
P(L):= \{X \in L: AX \subseteq L, XA \subseteq L \text{ and } AXA \subseteq L\},
$$
$$
J(L):= P(ad^{-1}(L)).
$$

The following theorem contains a list of rather trivial and practically well-known observations (see for example [1], [7], [8], [12]), but it will be very useful in the sequel.

1.6. Theorem. Let $A$ be an AL-ring (algebra) and let $L$ be an additive subgroup (a linear subspace) of $A$. Then:

(a) $N(L)$ is a Lie subring (subalgebra) of $A$.
(b) $ad^{-1}(L)$ is an AL-subring (subalgebra) of $A$ and a Lie ideal of $N(L)$.
(c) $P(L)$ is the largest ideal of $A$ contained in $L$.
(d) If $L$ is a Lie subring (subalgebra) of $A$, then $L \subseteq N(L)$ and $P(L)$ is a Lie ideal of $N(L)$. Moreover, $J(L)$ is a Lie ideal of $N(L)$.
(e) If $L$ is a Lie ideal of $A$, then $L \subseteq ad^{-1}(L)$, $N(L) = L$, $ad^{-1}(L)$ is a Lie ideal of $A$, $J(L)$ is an AL-ideal of $A$ and $[ad^{-1}(L), ad^{-1}(L)] = J(L)$.

Proof. (a) This follows immediately from the Jacobi identity.
(b) $ad^{-1}(L)$ is a Lie subring (subalgebra) and a Lie ideal of $N(L)$ by the Jacobi identity. By (1.2) $ad^{-1}(L)$ is an associative subring (subalgebra).
(c) Trivial.

(d) Let $L$ be a Lie subring (subalgebra). Obviously, $L \subseteq N(L)$. Since

$$
A[P(L), N(L)] = [AP(L), N(L)] + [A, N(L)]P(L) \subseteq L
$$

and similarly

$$
P(L), N(L)[A \subseteq L \text{ and } A[P(L), N(L)]A \subseteq L, P(L) \text{ is a Lie ideal of } N(L).\text{ By definition, } J(L) = P(ad^{-1}(L)), \text{ so as above } J(L) \text{ is a Lie ideal of } N(ad^{-1}(L)).\text{ Also, } ad^{-1}(L) \text{ is by (b) a Lie ideal of } N(L), \text{ and so } N(L) = N(ad^{-1}(L)).
$$

(e) Let $L$ be a Lie ideal of $A$. Then obviously $L \subseteq ad^{-1}(L)$, $N(L) = L$ and $ad^{-1}(L)$ is a Lie ideal of $A$ by (b). By (d), $J(L)$ is an AL-ideal of $A$ and $ad^{-1}(L)$ is a Lie ideal of $A$ and an associative subring (subalgebra) by (b). Then

$$
A[ad^{-1}(L), ad^{-1}(L)] \subseteq [Aad^{-1}(L), ad^{-1}(L)] + [A, ad^{-1}(L)]ad^{-1}(L) \subseteq ad^{-1}(L).
$$

Similarly, $ad^{-1}(L), ad^{-1}(L)]A \subseteq ad^{-1}(L)$. Hence by (1.2)

$$
[A, A[ad^{-1}(L), ad^{-1}(L)]A] = [A, ad^{-1}(L), ad^{-1}(L)] AA +
$$

$$
+[(ad^{-1}(L), ad^{-1}(L)), AAA] = [A, ad^{-1}(L)] AA \subseteq L
$$

and so $A[ad^{-1}(L), ad^{-1}(L)]A \subseteq ad^{-1}(L)$. Thus

$$
[ad^{-1}(L), ad^{-1}(L)] = P(ad^{-1}(L)) = J(L).
$$

If $A$ is a topological AL-ring and $L$ is closed, then $N(L), ad^{-1}(L), P(L)$ and $J(L)$ are closed and we can derive the topological version of (1.6).

(1.7) Definition. For an ideal $J$ of an associative ring $A$ and for a natural $n$ we define $J^n := \{X \in A: nX \in J\}$. We define the radical of $J$ as

$$
r(J) = \{X \in A: \text{ there is an } n \text{ such that } (X)(A)(X)...(X)(A)(m \text{ times }) \subseteq J\}.
$$

(1.8) Remark. It is easy to see that $J/n$ and $r(J)$ are ideals of $A$.

(1.9) Lemma (Herstein). Let $A$ be an AL-ring (algebra), $J$ an ideal of $A$ and $X, Y \in A$ such that $[X, [X, A]] \subseteq J$. Then $[X, A][X, A] \subseteq J/2$.

Proof. Take $Y, Z \in A$. By (1.1)

$$
$$

Hence $[X, Y][X, Z] \subseteq J/2$. Putting $Y = VU$, $V, U \in A$, we get by (1.1)

$$
[X, V][X, Z] \subseteq J/2.
$$

The following theorem and corollary generalize the classical theorems about associative rings of Zuev [12] and Herstein [6].

1.10. Theorem (Zuev). Let $A$ be an AL-ring (algebra) and let $L$ be a Lie ideal of $A$. Then for each $X \in ad^{-1}(L)$ the square of the ideal $I$ of $A$ generated by $[X, A]$ lies in $J(L)/2$, i.e., $II \subseteq J(L)/2$. Moreover, $[A, ad^{-1}(L)] = r(J(L)/2)$.
Proof. $\text{ad}^{-1}(L)$ is a Lie ideal of $A$ and $[\text{ad}^{-1}(L), \text{ad}^{-1}(L)] \subseteq J(L)$ by (e) of Theorem (1.6), so by Lemma (1.9),

$$[X, A] A [X, A] \subseteq J(L)/2$$

for all $X \in \text{ad}^{-1}(L)$.

It is easy to see that for $Z = [X_1, Y_1] + \ldots + [X_n, Y_n]$, where $X_i \in \text{ad}^{-1}(L)$ and $Y_i \in A$, $i = 1, \ldots, n$, we have

$$(ZA)(ZA) \ldots (ZA)(n + 1 \text{ times}) \subseteq J(L)/2.$$ 

Hence $Z \in r(J(L)/2)$. 

(1.11) Corollary (Herstein). If $A$ is an AL-ring (algebra) which is simple as an associative ring (algebra) and is of characteristic $\neq 2$, then for each Lie ideal $L$ of $A$ we have $[A, A] \subseteq L$ or $L \subseteq Z(L)$, where $Z(L)$ is the Lie centre of $A$.

Proof. If $[A, A] \subseteq L$, then $\text{ad}^{-1}(L) \neq A$ and $J(L) = 0$. Since $A$ is of characteristic $\neq 2$, $J(L)/2 = 0$. If there is an $X \in L$ such that $[X, A] \neq 0$, then the ideal $I$ of $A$ generated by $[X, A]$ equals $A$ and by Theorem (1.10), $AA = 0$. Since $A$ is simple, $A$ as an associative ring (algebra) is generated by one element and by (1.1), $[A, A] = 0 = L$ — a contradiction. 

2. Lie subalgebras of finite codimension.

(2.1) Proposition. Let $B$ be an AL-algebra. Then:

(a) If $L$ is a finite-dimensional Lie subalgebra of $A$, then $\text{ad}^{-1}(L)$ is a finite-dimensional AL-subalgebra of $A$.

(b) If $L$ is a finite-dimensional associative subalgebra of $A$, then $P(L)$ is a finite-dimensional ideal of $A$.

(c) If $L$ is a finite-dimensional Lie subalgebra of $A$, then $J(L)$ is a finite-dimensional ideal of $A$. In particular, if $A$ has no finite codimensional ideals except $A$, then every finite codimensional Lie subalgebra of $A$ contains $[A, A]$.

Proof. (a) Since $L \cap \text{ad}^{-1}(L)$ is finite-codimensional in $L$ and the kernel of the adjoint representation of $L$ in the finite-dimensional vector space $A(L) \cap \text{ad}^{-1}(L)$ is of finite codimension in $A$.

(b) Put $K = \{X \in L : AX \subseteq L\}$. $K$ is a left ideal of $A$ and it is of finite codimension as the kernel of the natural representation of $L$ in the finite-dimensional vector space $A(L)$ by the right multiplication. Moreover, $P(L) \subseteq K$ and $P(L)$ is finite-codimensional in $K$ as the kernel of the natural representation of $K$ in $A/K$ by the left multiplication.

(c) Since $\text{ad}^{-1}(L)$ is by Theorem (1.6) (b) an associative subalgebra of $A$ and it is by (a) of finite codimension, $J(L) = P(\text{ad}^{-1}(L))$ is of finite codimension in $A$ by (b).

If $J(L) = A$, then clearly $[A, A] \subseteq L$. 

The above proposition allows us to answer P. de la Harpe’s question [5] whether the Banach–Lie algebra $gl(H, C_w)$ of all compact linear operators on a separable Hilbert space $H$ has a nontrivial Lie subalgebra of finite codimension (the answer for the case of closed Lie subalgebras is given in [8]) and to solve the problem of the existence of nontrivial closed Lie subalgebras of finite codimension for other complex classical Banach–Lie algebras of compact operators.

Let us recall what “classical” means above. Let $H$ be a separable infinite-dimensional Hilbert space and let $1 \leq p \leq \infty$. By $gl(H, C_w)$ we denote the Schatten $p$-class of compact operators on $H$ (see [11]). The classes $gl(H, C_w)$ are ideals of the associative algebra $gl(H)$ of all bounded operators on $H$ and $gl(H, C_w) \subset gl(H, C_w)$ if $p \leq q$. In particular, $gl(H, C_w)$ is the ideal of all compact operators, $gl(H, C_w)$ is the ideal of Hilbert–Schmidt operators and $gl(H, C_w)$ is the ideal of nuclear operators. Each $gl(H, C_w)$ is a Banach algebra and a Banach–Lie algebra with respect to the Schatten $p$-norm.

Let $J_p$ be a conjugation and $J_0$ an anticonjugation of $H$. This means that there are orthonormal bases $(e_{n})_{n=0}^{\infty}$ of $H$ such that

$$J_p(\sum_{n=0}^{\infty} e_{n} e_{n}) = \sum_{n=0}^{\infty} e_{-n} e_{n}$$

and

$$J_0(\sum_{n=0}^{\infty} e_{n} f_{n}) = \sum_{n=0}^{\infty} e_{n} f_{-n} - \sum_{n=0}^{\infty} e_{-n} f_{n}.$$ 

We denote after [4]:

$$o(H, J_p, C_w) := \{X \in gl(H, C_w) : J_p X^* X = -X\},$$

$$\text{sp}(H, J_0, C_w) := \{X \in gl(H, C_w) : J_0 X^* J_0 = X\},$$

$$\text{sl}(H, C_w) := \{X \in gl(H, C_w) : \text{tr}(X) = 0\}.$$ 

(2.2) Definition. The Lie algebras $gl(H, C_w)$, $\text{sl}(H, C_w)$, $o(H, J_p, C_w)$ and $\text{sp}(H, J_0, C_w)$ are called complex classical Banach–Lie algebras of compact operators.

(2.3) Theorem (P. de la Harpe). The classical complex Banach–Lie algebras of compact operators, except $gl(H, C_w)$, are topologically simple. The only nontrivial closed Lie ideal of $gl(H, C_w)$ is $\text{sl}(H, C_w)$.

(2.4) Theorem (P. de la Harpe). The Lie algebra $gl(H, C_w)$ has no nontrivial finite-dimensional Lie ideals.

The following theorem generalizes (2.4).

(2.5) Theorem. The Lie algebra $gl(H, C_w)$ has no nontrivial finite-dimensional Lie subalgebras.

The classical complex Banach–Lie algebras of compact operators, except $gl(H, C_w)$, have no nontrivial closed finite-dimensional Lie subalgebras. The only such Lie subalgebra of $gl(H, C_w)$ is $\text{sl}(H, C_w)$.

Proof. Since in associative algebras ideals are also Lie ideals, by (2.4) $gl(H, C_w)$ has no nontrivial ideals of finite codimension
[gl(H, C_w), gl(H, C_w)] = gl(H, C_w) (see also [10]). By (2.3), if A is a classical complex Banach-Lie algebra of compact operators, then A has no nontrivial closed ideals and [A, A] is dense in A, except A = gl(H, C_w), where [gl(H, C_w), gl(H, C_w)] is dense in sl(H, C_w) and sl(H, C_w) is 1-codimensional in gl(H, C_w). So the theorem follows by Proposition (2.1) (c).

(2.6) Remark. Observe that the classical complex Banach-Lie algebras of compact operators, except gl(H, C_w), have many dense finite-codimensional Lie subalgebras, since [gl(H, C_w), gl(H, C_w)] ⊂ gl(H, C_w).

(2.7) Definition. Let A be a (topological) AL-algebra. A (closed) ideal (Lie subalgebra, Lie ideal) L of A will be called maximal if L ≠ A and if for each (closed) ideal (Lie subalgebra, Lie ideal) K of A such that L ⊆ K we have K = A.

We shall deal in the sequel with AL-algebras which have the following properties:

(P_1) Every maximal finite-codimensional ideal of A is equal to its radical.

(P_2) The Lie normalizer N(I) of each maximal finite-codimensional ideal I of A is a proper finite-codimensional subspace of A.

The following theorem is analogous to a result of Atkin [1].

(2.8) Theorem (Atkin). Let A be an AL-algebra satisfying (P_1) and (P_2). Then I ∩ N(I) gives us a one-one correspondence between maximal finite-codimensional ideals I of A and maximal finite-codimensional Lie subalgebras of A not containing [A, A]. The inverse mapping is of the form L → r(P(L)).

Moreover, each finite-codimensional Lie subalgebra of A not containing [A, A] is contained in N(I) for a maximal finite-codimensional ideal I of A.

We shall use in the proof of the above theorem the following lemma.

(2.9) Lemma. Let A be an AL-algebra, let J be a finite-codimensional ideal of A and let I be an ideal of A containing J and equal to its radical. Then N(I) ⊆ N(J).

Proof. Let x ∈ N(J). Consider the finite-dimensional associative algebra A' := A/J. Then I' = I/J is an ideal of A' equal to its radical and since ad_x(J) ⊆ I, ad_x can be projected on A' and gives us a derivation ad_x of A'.

Take Y ∈ I and define the descending sequence

V_0 = Y' ≥ V_1 = Y'Y ≥ V_2 = Y'Y'Y ≥ V_3 = Y'Y'Y'Y ≥ ...

of subspaces of A'. Since A' is finite-dimensional, there is a natural n such that V_n = V_{n+1}. Let Z = Y U_1 Y U_2 ... Y U_n V_n. It is easy to prove by Leibniz's formula that (ad_x)^n(Y U_n+1) = Y U_n+1, so (ad_x)^n(Y U_n) = Y U_n. On the other hand, (ad_x)^n(Y U_1 Y U_2 ... Y U_n) = Y U_1 Y U_2 ... Y U_n + Y',

where Y' ∈ I'. Hence ad_x Y ∈ r(I') = I', i.e., [X, I'] ⊆ I.

Proof of Theorem (2.8). Let I be a maximal finite-codimensional ideal of A. Observe that I ∩ N(I) is an ideal of A, since [A, N(I)] ∩ I = [A, I] + A [N(I)] ⊆ I and similarly (N(I) ∩ I)A ⊆ N(I). This ideal is of finite codimension by (P_2) and it is easy to see that it contains I.

Thus if L is a Lie subalgebra of A, L ≠ A, such that N(I) ⊆ L, we have

I ≤ I ∩ N(I) ⊆ N(I) ≤ L,

so that I ⊆ r(P(L)).

P(L) is a finite-codimensional ideal of A, P(L) ≠ L, so r(P(L)) ≠ A (P(L) is contained in some maximal finite-codimensional ideal of A which is equal to its radical). Hence I = r(P(L)) by the maximality of I. In particular, I = r(P(N(I))).

By (d) of Theorem (1.6), P(I) is a Lie ideal of N(L). Thus

N(I) ⊆ L ⊆ N(L) ⊆ N(P(L))

and by Lemma (2.9), N(P(L)) = N(I), so that N(I) = L = N(L) = N(N(I)). This shows that N(I) is a finite-codimensional maximal Lie subalgebra of A.

Since N(N(I)) = N(I), N(I) does not contain [A, A].

If N(I) = N(I') for a maximal finite-codimensional ideal I' of A, then as above

I' = r(P(N(I'))) = r(P(N(I))) = I'.

Conversely, let L be a finite-codimensional subalgebra of A which does not contain [A, A]. Then the ideal J(L) does not equal A and it is of finite codimension by Proposition (2.1). By Theorem (1.6) (d), J(L) is a Lie ideal of N(L), whence L ⊆ N(J(L)). J(L) is contained in a maximal finite-codimensional ideal I of A which by (P_1) equals its radical. Thus by Lemma (2.9), L = N(J(L)) = N(I) and the theorem follows.

(2.10) Corollary. Let M be a C^∞ (R-algebraic, Stein) manifold with a C^∞ (R-algebraic, holomorphic) symplectic structure. Let A be the Poisson algebra of all C^∞ functions on M with compact support or the Poisson algebra of all C^∞ (R-algebraic, holomorphic) functions on M. Then

M x L_x = {f ∈ A: df(x) = 0}

gives us a one-one correspondence between points of M and maximal finite-codimensional Lie subalgebras of A not containing [A, A].

Proof. The maximal finite-codimensional ideals of A are of the form

I_x = {f ∈ A: df(x) = 0}

for x ∈ M (see [3]), so A has the property (P_1). It is easy to see that

N(I_x) = {f ∈ A: df(x) = 0}.

Thus the property (P_2) is also satisfied and the corollary follows by Theorem (2.8).
3. Lie ideals.

(3.1) Definition. An ideal \( L \) of a Lie ring (algebra) \( A \) we shall call perfect if \( L \neq 0 \) and \( \text{ad}^{-1}(L) = L \).

For a (topological) \( \mathbb{A} \)-algebra \( A \) denote by \( R_{\mathbb{A}}(A) \) the set of all (closed) ideals of \( A \) different from \( 0 \) and equal to their radicals and by \( R_0(A) \) the set of all (closed) perfect Lie ideals of \( A \).

(3.2) Theorem. Let \( A \) be a (topological) associative algebra over a field of characteristic \( \neq 2 \) and such that the (closed) ideal generated by \([A, A]\) equals \( A \). Then we have a one-one mapping \( \varphi \) from \( R_0(A) \) into \( R_0(A) \) given by

\[
R_0(A) \ni I \mapsto \text{ad}^{-1}(I) \in R_0(A).
\]

The inverse mapping is of the form \( L \mapsto P(L) \). In particular, if every (closed) ideal of \( A \) equals its radical, then \( A \) is also "onto".

Proof. Let \( I \in R_0(A) \). Since \([A, \text{ad}^{-1}(I)] \subset I \subset \text{ad}^{-1}(I) \) (the second inclusion follows since \( A \) is an associative algebra) and \( \text{ad}^{-1}(I) \) is a Lie ideal of \( A \) and \( \text{ad}^{-1}(I) \neq A \) by the assumptions. If \( X \in \text{ad}^{-1}(\text{ad}^{-1}(I)) \), then \([X, [X, A]] \subset I \) and by Lemma (1.9), \([X, A] \subset r(I) = I \), i.e., \( \text{ad}^{-1}(I) \) is perfect. Since \([I, [A, A]] = I = r(\text{ad}^{-1}(I)) \), \( P(\text{ad}^{-1}(I) \cap [A, P(\text{ad}^{-1}(I))] \subset I \).

The (closed) ideal generated by \([A, A] \) equals \( A \), so \( P(\text{ad}^{-1}(I) \cap [A, P(\text{ad}^{-1}(I))] \subset I \).

Thus \( P(\text{ad}^{-1}(I) \cap [A, P(\text{ad}^{-1}(I))] \subset I \). Suppose that every (closed) ideal of \( A \) equals its radical and let \( I \in R_0(A) \). Then \( J(L) = P(\text{ad}^{-1}(I)) = P(L) \) and by Theorem (1.10), \([A, L] \subset r(P(L)) = P(L) \), i.e., \( L \subset \text{ad}^{-1}(P(L)) \). On the other hand, from \( P(L) \in L \) it follows that \( \text{ad}^{-1}(P(L)) \subset \text{ad}^{-1}(L) \).

(3.3) Remark. Note that for each \( \mathbb{A} \)-algebra \( A \) every closed ideal \( I \) of \( A \) is self-adjoint (see [2], § 1) and thus \( r(I) = I \). Let \( X \in A \) and \( X = X_1 \ldots X_n \) (n times), then \( (XX^*)^n = 0 \).

(3.4) Corollary. If \( A \) is a \( \mathbb{A} \)-algebra such that the ideal generated by \([A, A]\) is dense in \( A \), then for \( A \) as a topological \( \mathbb{A} \)-algebra the mapping

\[
R_0(A) \ni I \mapsto \text{ad}^{-1}(I) \in R_0(A)
\]

is a one-one correspondence.

For a (topological) \( \mathbb{A} \)-algebra \( A \) denote by \( M_0(A) \) the set of all maximal ideals of \( A \) and by \( M_1(A) \) the set of all maximal perfect Lie ideals of \( A \).

(3.5) Theorem. Let \( A \) be a (topological) associative algebra over a field of characteristic \( \neq 2 \) such that every (closed) ideal of \( A \) different from \( A \) is contained in some ideal from \( R_0(A) \) and that the (closed) vector space generated by \([A, A]\) equals \( A \). Then the mapping \( I \mapsto \text{ad}^{-1}(I) \) is a bijection from \( M_0(A) \) onto \( M_1(A) \). The inverse mapping is of the form \( L \mapsto P(L) \).

Proof. It is easy to see that \( M_0(A) \subset R_0(A) \), i.e., each maximal ideal of \( A \) equals its radical. Thus by Theorem (3.2)

\[
M_0(A) \ni I \mapsto \text{ad}^{-1}(I) \in R_0(A)
\]

is a one-one mapping.

We shall prove that \( \text{ad}^{-1}(I) \) is a maximal ideal of \( A \) for \( I \in M_0(A) \). Suppose that for a (closed) Lie ideal \( L \) of \( A \), \( L \neq A \), we have \( \text{ad}^{-1}(I) \subset L \). Then \( \text{ad}^{-1}(L) \neq A \) by the assumptions and thus \( J(L) \neq L \). Since \( J(L) \neq A \), \( J(L) \) is contained in some ideal from \( R_0(A) \), so \( r(J(L)) \neq A \). By Theorem (1.10), \([A, L] \subset r(J(L)) \). Hence there is some \( I \in R_0(A) \) such that \([A, A] I + A \text{ad}^{-1}(I), A \subset I \) and

\[
[I \cap \text{ad}^{-1}(I), A] \subset [A, A] I + A \text{ad}^{-1}(I), A \subset I.
\]

We have \( \text{ad}^{-1}(I) \subset L \subset \text{ad}^{-1}(L) \) and thus \( I \cap \text{ad}^{-1}(I) \subset J(L) \) by the definition of \( J(L) \). Since \( I \subset J(L) \), \( I \subset J(L) \) by the maximality of \( I \), \( I = r(J(L)) \) (i.e., \( J(L) \neq A \)). Hence \([A, L] \subset I \), i.e., \( L \subset \text{ad}^{-1}(L) \), which proves the maximality of \( \text{ad}^{-1}(L) \).

(3.6) Corollary. If \( A \) is a \( \mathbb{A} \)-algebra such that \([A, A]\) is dense in \( A \), then \( I \mapsto \text{ad}^{-1}(I) \) gives us a one-one correspondence between maximal ideals of \( A \) and maximal perfect Lie ideals of \( A \).

For a more general class of \( \mathbb{A} \)-algebras than those mentioned in the above theorems we can prove a weaker result.

Denote

\[
\text{ad}^{-1}(L) = \bigcap_{i=1}^{\infty} \text{ad}^{-1}(L), \quad \text{where} \quad \text{ad}^{-1}(n+1)(L) = \text{ad}^{-1}(n)(L).
\]

It is easy to see that if \( L \) is a linear subspace of an \( \mathbb{A} \)-algebra \( A \), then \( \text{ad}^{-1}(L) \) is a Lie ideal of \( A \).

(3.7) Theorem. Let \( A \) be an \( \mathbb{A} \)-algebra over a field of characteristic \( \neq 2 \). Suppose that \( r(J) \neq A \) for each ideal \( J \) of \( A \) different from \( A \). Let \( L \) be a Lie
ideal of $A$ which does not contain $[A, A]$. Then there is an ideal $J$ of $A, J \neq A$, such that $L \subset \text{ad}^{-1}(J)$.

If additionally the ideal of $A$ generated by $[A, A]$ equals $A$ and each ideal of $A$ different from $A$ is contained in some maximal ideal of $A$, then each maximal Lie ideal of $A$ not containing $[A, A]$ is of the form $\text{ad}^{-1}(I)$ for a maximal ideal $I$ of $A$. In other words,

$$M_1(A) = \{ I \mapsto \text{ad}^{-1}(I) \in R_1(A) \}$$

contains $M_1(A)$ in its image.

Proof. Let $L$ be a Lie ideal of $A$ which does not contain $[A, A]$. Then by Theorem (1.10), $[A, L] = r(J(L)), J(L) \subset \text{ad}^{-1}(L) \neq A$, and hence $r(J(L)) \neq A$. Thus $L \subset \text{ad}^{-1}(r(J(L)))$ and since $L$ is a Lie ideal of $A$, proceeding by induction we get $L \subset \text{ad}^{-n}(r(J(L)))$. Suppose $L$ is maximal. Let $I$ be a maximal ideal of $A$ containing $r(J(L))$. Then $L \subset \text{ad}^{-n}(I)$. Since $[A, A] \notin I$, $\text{ad}^{-n}(I) \neq A$ and $L = \text{ad}^{-n}(I)$ by the maximality of $L$.

(3.8) Corollary. Let $M$ be a $C^n$ ($R$-analytic, Stein) connected manifold with a $C^\infty$ ($R$-analytic, holomorphic) symplectic structure. Let $A$ be the $\mathcal{A}$-algebra of all $C^\infty$ ($R$-analytic, holomorphic) functions on $M$ with the Poisson bracket. Let $L$ be a Lie ideal of $A$ which does not contain $[A, A]$. Then for each $f \in L$, there is a sequence $p_1, p_2, \ldots$ of points of $M$ such that $df(p_1) = \ldots = df(p_i) = 0$.

Proof. The proof is similar to the proof of Proposition 6.6 in [3]. Since $A$ has an unit, every proper ideal of $A$ is contained in some maximal and thus prime ideal of $A$. Hence $r(J) \neq A$ for each ideal $J$ of $A, J \neq A$, and by Theorem (3.7), $L \subset \text{ad}^{-n}(J)$ for an ideal $J$ of $A, J \neq A$. There are $f_1, f_2, \ldots, f_n \in A$ such that $df_1, \ldots, df_n$ span the cotangent bundle of $M$. Then the hamiltonian vector fields $X_{f_1} = s \text{grad}(f_1), \ldots, X_{f_n} = s \text{grad}(f_n)$ span the tangent bundle of $M$.

Suppose that there are $f \in L$ and a natural number $m$ such that $df^m(p) = 0$ for each $p \in M$ and some natural number $k_m \leq m$. Then, as in [3], one can prove that the ideal of $A$ generated by the finite set

$$\Omega = \{ (X_{f_1} \circ \cdots \circ X_{f_n}) f : i_1, \ldots, i_k = 1, \ldots, n, k = 1, \ldots, m \}$$

equals $A$.

But $X_{f_i}(h) = [f_i, h]$ for all $h \in A$ and $i = 1, \ldots, n$, and thus $\Omega \subset J$, since $L \subset \text{ad}^{-m}(J)$ -- a contradiction.

(3.9) Corollary. Let $A$ be the Poisson algebra of all $C^\infty$ functions with compact support on a $C^\infty$ symplectic manifold $M$ or the Poisson algebra of all $R$-analytic functions on an $R$-analytic compact symplectic manifold $M$. Then each Lie ideal of $A$ which does not contain $[A, A]$ is contained in the Lie ideal $L_\rho = \{ f \in A : 0 = df(p) = df(p) = \cdots \}$.

In particular, in the $R$-analytic and connected case the only such nontrivial Lie ideal consists of constant functions.

In the $C^n$ case the Lie ideals $L_\rho$ are maximal.

Proof. Each ideal of $A$ different from $A$ is contained in the ideal $L_\rho = \{ f \in A : f(p) = 0 \}$ for some $p \in M$. It is easy to see that

$$L_\rho = \{ f \in A : df(p) = \cdots = d^n f(p) = 0 \},$$

so by Theorem (3.7), $L_\rho = \text{ad}^{-n}(J)$.

To prove that in the $C^n$ case $L_\rho$ is maximal, it suffices to show that $[A, A] + L_\rho = A$. Let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be coordinates in a neighborhood of $p$ in which the symplectic form can be written as

$$\sum_{i=1}^n dx_i \wedge dy_i.$$ Then in a neighborhood of $p$ we have

$$[f, g] = \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right).$$

Choose $h \in A, f \in A$ such that $\frac{\partial f}{\partial x_i} = h$ in a neighborhood of $p$, and $g \in A$ such that $g = f_1$ in a neighborhood of $p$. Then $[f, g] = h$ in a neighborhood of $p$, which proves that $[A, A] + L_\rho = A$.

Note that the $C^n$ version of the above corollary is due to Omori [9].

(3.10) Proposition. Let $A$ be one of the Poisson algebras mentioned in Corollary (3.8). Then $A$ has no finite-dimensional or $L$-commutative Lie ideals except the Lie ideal of constant functions.

Proof. It is easy to see that the Lie-centre of $A$ consists of constant functions. Assume that $f \in A$ is not constant. Then there are $p \in M$ and $g \in A$ such that $[f, g](p) = 0$. If $f$ is in a Lie ideal $L$, then $[g, f] = \eta_{L_0}([g, f]) \in L$ and we can choose $g$ such that $L$ cannot be finite-dimensional.

If $L$ is commutative, then $[f, g(f, g)](p) = [f, g]^2 + g[f, f, g] = 0$ and since we can choose $g$ such that $g(p) = 0$,

$$[f, g(f, g)](p) = ([f, g](p))^2 = 0$$

is a contradiction.

References

Uniformly non-$\ell^1_n$ Orlicz spaces with Luxemburg norm

by

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Abstract. K. Sundaresan [15] has given a criterion for an Orlicz space $L^\Phi(\mu)$ over an atomless measure $\mu$ and generated by an Orlicz function $\Phi$ satisfying the corresponding condition $\Delta_2$ to be uniformly non-$\ell^1_n$. This paper gives some simpler criteria for this property of Orlicz spaces over an atomless as well as a purely atomic measure $\mu$ and generated by arbitrary Orlicz functions (the necessity of the corresponding condition $\Delta_2$ is proved here).

0. Introduction. $N$ is the set of positive integers, $R$ is the set of real numbers, $(T, \mathcal{F}, \mu)$ is a space of positive measure. A function $\Phi: R \to [0, +\infty]$ is said to be an Orlicz function if it is not identically zero and is even, convex, and vanishing and continuous at zero. The Orlicz space $L^\Phi(\mu)$ is then defined as the set of all equivalence classes of $\mathcal{F}$-measurable functions $x: T \to R$ such that $\int_T \Phi(kx(t))d\mu < +\infty$ for some $k > 0$ depending on $x$. Under the so-called Luxemburg norm $\|x\|_\Phi$ defined by

$$\|x\|_\Phi = \inf \{ r > 0 : \int_T \Phi(r^{-1}x(t))d\mu \leq 1 \}$$

the Orlicz space $L^\Phi(\mu)$ is a Banach space (see [12, 13]).

Let us write $I(x) = I_\Phi(x) = \int_T \Phi(x(t))d\mu$ for any $x \in L^\Phi(\mu)$. The functional $I$ is a convex modular on $L^\Phi(\mu)$ (see [14]).

We define the subspace $E^\Phi(\mu)$ of the Orlicz space $L^\Phi(\mu)$ by

$$E^\Phi(\mu) = \{ x \in L^\Phi(\mu) : I(kx) < +\infty \text{ for any } k > 0 \}.$$

Recall that an Orlicz function $\Phi$ satisfies condition $\Delta_2$ for all $u \in R$ (at infinity) [at zero] if the inequality $\Phi(2u) \leq K\Phi(u)$ holds for all $u \in R$ (for $u$ satisfying $|u| \geq v_0$) [for $u$ satisfying $|u| \leq v_0$], where $K$ and $v_0$ are some positive constants and $\Phi(v_0) > 0$ (see [12, 13]).

0.1. Lemma (see [4, 5] and [10]). Let $\Phi$ be an Orlicz function and $x \in L^\Phi(\mu)$. The condition $I(x) = 1$ if and only if $x \in L^\Phi(\mu)$ satisfies condition $\Delta_2$ for all $u \in R$ (at infinity) [at zero] in the case of a measure space atomless and infinite (atomless and finite) (purely atomic with measure of atoms equal to one), respectively$(^1)$.

$(^1)$ In the purely atomic case we assume that $\Phi(c) = 1$ for some $c > 0.$