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On boundedly order-complete locally solid Riesz spaces

by

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Abstract. This is a continuation of [L]. In particular, the terminology used therein will be kept here. The paper is devoted to a study of boundedly order-complete spaces, i.e., spaces having the Bounded Order-Completeness property (BOC)(1).

§ 1 is preliminary and relates BOC to other conditions of similar type; Corollary 1.4 is perhaps of some independent interest. The main result of § 2 (Th. 2.6) asserts that *Hausdorff locally solid universally complete Riesz spaces have BOC*. In § 3 spaces in which BOC already implies local order-closedness are investigated. In particular, the following result (Th. 3.1) is proved:

Let (L, τ) *be a locally bounded-solid Riesz space having BOC. If L admits another order-continuous metrizable locally solid topology, then* (L, τ) *is locally solid-order-complete*(2).

Some applications are given in § 4, where the BOC property is characterized in terms of different "lateral" nets.

This paper should be treated as a continuation of [L]. In particular, the same terminology will be used and, to some extent, developed further. Notational conventions and the terminology not explained in [L] or here are as in [A&B]. Recall only that the adjectives "bounded" and "boundedly" always refer to *topological* boundedness, other notions of boundedness being qualified.

As the general Archimedean Riesz spaces are concerned, the same point of view as in [L], § 4 is adopted, i.e., whenever convenient, a space L is automatically identified with an order dense subspace of its universal completion L^u, and L^u is taken to be $C^\infty(\Omega_L)$.

§ 1. Preliminaries. In this section a discussion relating the Bounded Order-Completeness property (BOC) to the Dedekind Completeness (DC) and the parallel "pseudo-properties" is given.

A locally solid tRs (L, τ) is said to have

Bounded Order-Boundedness property (*BOB*) or to be

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(1) This is the "Levi property" of Aliprantis and Burkinshaw [A&B] and the "weak Fatou property" of Luxemburg and Zaanen [6].

(2) i.e., a Nakano space in [F], [A&B].

B_1 *boundedly order-bounded*(3) if:
$0 \leqslant x_\alpha \uparrow$, (x_α) τ-bounded $\Rightarrow (x_\alpha)$ is order-bounded,
i.e., $\exists x \in L$ so that $x_\alpha \leqslant x$ for all α.

It is fairly clear that (L, τ) having BOB must necessarily be Hausdorff and therefore Archimedean. Assuming additionally that L is DC, BOB becomes:

B_2 *Bounded Order-Completeness property*:
$0 \leqslant x_\alpha \uparrow$, (x_α) τ-bounded $\Rightarrow x_\alpha \uparrow x$ (i.e., $\sup x_\alpha = x$ exists in L).

In fact, it is easy to see that the following holds.

1.1. Proposition. *DC+BOB = BOC, i.e., (L, τ) is boundedly order-complete iff it is DC and boundedly order-bounded.*

"Pseudo-analogues" of the above properties may also be considered (cf. [L], § 0), namely:

P_1 *Pseudo-Order-Boundedness property (POB)*:
$0 \leqslant x_\alpha \uparrow$, (x_α) τ-Cauchy $\Rightarrow \exists x \in L$, $x_\alpha \leqslant x$ for each α,

whereas under DC one has

P_2 *Pseudo-Order-Completeness property (POC)*:
$0 \leqslant x_\alpha \uparrow$, (x_α) τ-Cauchy $\Rightarrow x_\alpha \uparrow x$.

Similarly to BOB, POB implies Hausdorffness. On the other hand, POC does not imply DC since even (topological) completeness does not imply DC in general. When sequences replace nets, one obtains σ-properties: σ-BOB, σ-POB, etc. ...; trivially σ-BOB implies σ-POB.

The following easy propositions shed some light on the significance of the properties under consideration. $(\hat{L}, \hat{\tau})$ is the topological completion of (L, τ).

1.2. Proposition. *An F-normed lattice (L, ϱ) has σ-POB iff $(\hat{L}, \hat{\varrho})$ embeds in (L^δ, τ^δ).*

Proof. Since L is full in L^δ, the "if" part is clear. Moreover, the metrizability assumption is not needed in this case. "Only if": by [A&B], 15.3, any $\hat{x} \in \hat{L}$ is a difference of upper elements $\hat{y}$ and $\hat{z}$. It is clear that $\hat{y}$ and $\hat{z}$ are in L^δ if σ-POB is assumed.

1.3. Proposition. *A Hausdorff locally solid tRs (L, τ) has BOB iff $L^\# = L^\delta$.*

Proof. By a maximality argument ([L], 3.3), the inclusion $L^\delta \subset L^\#$ is always true. By [L], 2.5, $L^\# = \{y \in L^u : L_y \text{ is bounded}\}$. Since $L_y \uparrow y$ in L^u, by BOB there exists an $x \in L$ such that $y \leqslant x$. It follows that $y \in L^\delta$, i.e., $L^\# \subset L^\delta$.

1.4. Corollary. *If (L, ϱ) is an F-normed lattice having BOB, then it has*

(3) This is the Levi property of Fremlin [F].

the unique F-lattice enlargement $(L^\delta, \varrho^\delta)$. In particular, $(L^{\delta\#}, \varrho^{\delta\#}) = (L^\delta, \varrho^\delta)$, i.e., $(L^\delta, \varrho^\delta)$ is enlarged.

Proof. Suppose (M, μ) is an F-lattice enlarging (L, ϱ). Then $(L^\delta, \varrho^\delta) \subset (M, \mu)$ continuously ([L], 3.2). On the other hand, $M \subset L^\#$ ([L], 3.3) and $L^\# \subset L^\delta$ by the above. Hence $M = L^\delta$ and consequently $\mu = \varrho^\delta$.

Finally, in order to have a more complete picture of the situation, recall the following facts.

1.5. By [L], 1.6, POC+locally pseudo-s-o-c = locally pseudo-S-O-C = MCP and the σ-version of this result is also valid.

1.6. By [L], 1.5, BOC+locally s-o-c = locally boundedly S-O-C and the σ-version of this result is also valid.

1.7. By a theorem of Amemiya (cf. [K&A], p. 378, Th. 2 and [L], § 5 for some comments concerning this result), in metrizable spaces σ-POC implies completeness or, equivalently, σ-POC implies locally σ-pseudo-S-O-C.

It is therefore natural to ask for an analogue of this result, replacing "pseudo" by "bounded". Such an analogue exists and will be treated in some detail in § 3.

1.8. In nonmetrizable spaces even DC+MCP $\Rightarrow$ completeness is an open problem and, by the Nakano theorem, locally boundedly S-O-C spaces are complete.

§ 2. **Universally complete spaces have BOC.** Let L be a universally complete Riesz space and let τ be a Hausdorff locally solid vector topology on L. In view of the results of Fremlin [4], § 3, there is little hope that it will be possible to decide whether such τ is locally boundedly order-complete, since it would imply in particular that the real-valued measurable cardinals do not exist. However, quite remarkably, it is possible to decide that the whole space (L, τ) has BOC. The remaining part of this section will be devoted to the proof of this result.

The proposition which follows may be found e.g. in [A&B]. Since it will be used below, a simple proof is provided for the convenience of the reader.

2.1. Proposition. *Let L be a σ-laterally complete Archimedean Riesz space and τ a locally solid topology on L. Then*

(i) *For any disjoint sequence $(x_n) \subset L_+$, $x_n \to 0$ (τ).*

(ii) *For any $x_n \downarrow 0$, $x_n \to 0$ (τ), i.e., τ is σ-order-continuous.*

Proof. (i) Indeed, let $u = (o)\text{-}\sum_{n=1}^{\infty} n x_n = \sup\{n x_n\}$. Then $x_n \leqslant (1/n) u \to 0$ (τ).

(ii) Consider L as a subspace of $C^\infty(\Omega_L)$ and take $x_n \downarrow 0$. It will be sufficient to show that for each $\varepsilon > 0$ there exists $u \in L_+$ such that $0 \leqslant x_n \leqslant (1/n) u + \varepsilon x_1$ for $n \in N$. To this end put $E_n = \text{cl}\{t \in \Omega_L : x_n(t) > \varepsilon x_1(t)\}$

where "cl" = closure in Ω_L. One has

(a) $$x_n \leqslant x_n \chi_{E_n} + x_n \chi_{\Omega_L \setminus E_n} \leqslant x_1 \chi_{E_n} + \varepsilon x_1$$

Consider $v_n := x_1 \chi_{E_n}$, $n \in N$; $v_n \downarrow 0$ since $\varepsilon v_n = \varepsilon x_1 \chi_{E_n} \leqslant x_n \downarrow 0$.

(b) $$x_1 \chi_{E_n} = \sum_{i=n}^{\infty} x_1 \chi_{E_i \setminus E_{i+1}} = \sum_{i=n}^{\infty} (v_i - v_{i+1}),$$

and $(v_i - v_{i+1})$, $i = 1, 2, \ldots$, is a disjoint sequence in L_+. By σ-lateral completeness $\sup\{i(v_i - v_{i+1})\} = (o)\text{-}\sum_{i=1}^{\infty} i(v_i - v_{i+1}) = u$ exists in L. Now, for each $n \in N$,

(c) $$x_1 \chi_{E_n} = \sum_{i=n}^{\infty} (v_i - v_{i+1}) \leqslant (1/n) \sum_{i=n}^{\infty} i(v_i - v_{i+1}) \leqslant (1/n) u,$$

and by (a), $x_n \leqslant (1/n) u + \varepsilon x_1$, which ends the proof.

2.2. Remarks. (1) Those familiar with Amemiya's theorem mentioned in the preceding section will note that its proof as well as the one above are variations of the same idea.

(2) A vector topology τ on a Riesz space L is said to be *exhaustive* (= pre-Lebesgue [A&B]) if $x_n \to 0$ (τ) for any disjoint order-bounded (x_n) in L. In view of the σ-lateral completeness of L, (i) above means precisely that τ is exhaustive.

(3) The elements $\{x\chi_{E_n}: x \in L, n \in N\}$ appearing in the proof of (ii) are in L since L has the principal projection property ([A&B], 23.4) and $x\chi_{E_n}$ corresponds to $P_n(x)$ where P_n is the projection defined by the element $(x_n - \varepsilon x_1)_+$.

In what follows (L, τ) is a Hausdorff locally solid tRs. Additional assumptions will be specified whenever needed.

Recall that the topological completion $(\hat{L}, \hat{\tau})$ is a Hausdorff locally solid Riesz space containing (L, τ) as a $\hat{\tau}$-dense Riesz subspace (sup and inf are extended by continuity on $\hat{L}$, cf. [A&B], 7.1). However, L need not be order dense in $\hat{L}$ or even regular in $\hat{L}$. Let $\operatorname{sol} L$ be the solid hull of L in $\hat{L}$.

Recall that $0 < e$ is a weak unit of L if $e \wedge x = 0 \Rightarrow x = 0$, and that this happens iff $x = \sup\{x \wedge ne\}$ for each $x \in L$.

2.3. Proposition. *Suppose L is σ-regular in $\hat{L}$. Let e be a weak unit of L. Then e is a weak unit of $\hat{L}$.*

Proof. It will first be shown that e is a weak unit of $\operatorname{sol} L$. Observe that, by the assumption, it makes no difference whether we take countable suprema in L or $\hat{L}$. Let $0 < w \in \operatorname{sol} L$ and let $y \in L$ be such that $w < y$. Then

$$\sup\{w \wedge ne\} = \sup\{w \wedge y \wedge ne\} = w \wedge \sup\{y \wedge ne\} = w \wedge y = w.$$

Hence e is a weak unit of $\operatorname{sol} L$. Now, $\operatorname{sol} L$ is clearly order dense in $\hat{L}$. If $u \wedge e = 0$ for some $0 < u \in \hat{L}$, the same would hold for some $0 < z < u$ in $\operatorname{sol} L$, which is impossible. Thus e is a weak unit in $\hat{L}$.

Assume that L is universally complete. Then τ is σ-order-continuous on L, whence L is σ-regular in $\hat{L}$. Furthermore, since τ is exhaustive, $(\hat{L}, \hat{\tau})$ is order-continuous and therefore, in particular, Dedekind complete ([A&B], 10.3 and 10.5). In virtue of 2.3 above, representations may be chosen in such a way that

(1) $$C(\Omega_L) \subset C^{\infty}(\Omega_L) = L \xrightarrow{R} \hat{L} \subset C^{\infty}(\Omega_{\hat{L}}),$$

(2) $$L \ni \chi_{\Omega_L} \xleftrightarrow{R} \chi_{\Omega_{\hat{L}}} \in \hat{L},$$

where R is a Riesz isomorphism into and $\hat{L}$ is a solid order dense vector subspace of $C^{\infty}(\Omega_{\hat{L}})$.

In particular, the restriction $R|C(\Omega_L)$ injects $C(\Omega_L)$ Riesz isomorphically into $C(\Omega_{\hat{L}})$ in view of (2). By using some ideas connected with the Weierstrass–Stone theorem, a more precise information about R and the connection between Ω_L and $\Omega_{\hat{L}}$ will be derived.

To simplify notation put $\Omega = \Omega_L$, $\hat{\Omega} = \Omega_{\hat{L}}$ and define an equivalence relation in $\hat{\Omega}$ by putting

$$t_1 \sim t_2 \Leftrightarrow \forall x \in C(\Omega)\ Rx(t_1) = Rx(t_2).$$

Let $q: \hat{\Omega} \to \hat{\Omega}/\sim$ be the corresponding quotient map. Let $\Phi = \mathscr{C}(q): C(\hat{\Omega}/\sim) \to C(\hat{\Omega})$ be defined by

$$\Phi(f) = f \circ q.$$

Put

$C(\hat{\Omega}/q) :=$ the image of Φ in $C(\hat{\Omega})$,

$C^{\text{const}}(\hat{\Omega}) := \{f \in C(\hat{\Omega}): f \text{ is constant on } \sim \text{ equivalence classes}\}$.

The following facts are to be noted.

(3) *$C(\hat{\Omega}/\sim)$ is Riesz isomorphic to $C(\hat{\Omega}/q)$. The isomorphism is given by Φ, which is moreover a multiplicative isometry such that $\Phi(\chi_{\hat{\Omega}/\sim}) = \chi_{\hat{\Omega}}$* (see [S], 4.2.2).

(4) $C(\hat{\Omega}/q) = C^{\text{const}}(\hat{\Omega})$ ([S], 5.2.7).

(5) *$\hat{\Omega}/\sim$ is a compact space.*

The proof is well known. Here is the argument. As q is continuous, $\hat{\Omega}/\sim$ is quasicompact. It has to be shown that it is Hausdorff. Let X_f = the image of $\hat{\Omega}$ by f; it is compact. Therefore the product $X = (\prod X_f: f \in C^{\text{const}}(\hat{\Omega}))$ is compact. Define $\varphi = (f: f \in C^{\text{const}}(\hat{\Omega}))$. For any projection pr_f on the f-axis, $\text{pr}_f \circ \varphi = f$ is continuous, whence φ is continuous from $\hat{\Omega} \to X$. Let $Z = \varphi(\hat{\Omega})$. The decompositions $\{\varphi^{-1}(z): z \in Z\}$ and $\{q^{-1}(s): s \in \hat{\Omega}/\sim\}$ are identical. Therefore $\hat{\Omega}/\sim$ admits a continuous one-one map onto a Hausdorff space Z, and hence must be Hausdorff.

Now, denoting still by R its restriction to $C(\Omega)$, by (3) and (4) we have the following situation:

$$C(\hat{\Omega}/\sim) \xrightarrow[\text{onto}]{\Phi} C^{\text{const}}(\hat{\Omega}) \xleftarrow{R} C(\Omega),$$

$$\chi_\Omega \xrightarrow{R} \chi_{\hat{\Omega}} \xrightarrow{\Phi^{-1}} \chi_{\hat{\Omega}/\sim};$$

moreover, in view of the definition of $\sim$, it is clear that the elements of the Riesz subspace $\Phi^{-1}\circ R(C(\Omega)) = W$ separate the points in $\hat{\Omega}/\sim$. By [S], 7.3.7, W is dense in $C(\hat{\Omega}/\sim)$ (with the uniform norm). Furthermore, a unit of W, $\Phi^{-1}\circ R(\chi_\Omega)$, equals $\chi_{\hat{\Omega}/\sim}$, a unit of $C(\hat{\Omega}/\sim)$, and so the uniform convergence transported onto W from $C(\Omega)$ and the uniform convergence induced in W from $C(\hat{\Omega}/\sim)$ are equivalent. So W is a dense and complete subspace of $C(\hat{\Omega}/\sim)$, i.e., $W = C(\hat{\Omega}/\sim)$. In other words, $C(\Omega)$ and $C(\hat{\Omega}/\sim)$ are Riesz isomorphic. Consequently

(6) $R\colon C(\Omega)\to C^{\text{const}}(\hat{\Omega})$ *is an isomorphism onto*

and, by a theorem of Kaplansky ([S], 7.8.1),

(7) Ω *is homeomorphic to* $\hat{\Omega}/\sim$.

As Ω is homeomorphic to $\hat{\Omega}/\sim$, $C^\infty(\Omega)$ is Riesz isomorphic to $C^\infty(\hat{\Omega}/\sim)$, and so it may be assumed right away that $\Omega = \hat{\Omega}/\sim$ and $C^\infty(\hat{\Omega}/\sim) = C^\infty(\Omega)$. Under this identification $q\colon \hat{\Omega}\to\Omega$ is the quotient map and

(8) $$R|C(\Omega) = \Phi.$$

Now, given any quotient map $q\colon \hat{\Omega}\to\Omega$, the operator $\mathscr{C}(q) := x\mapsto x\circ q$ from $C^{\overline{\infty}}(\Omega)$ into $C^{\overline{\infty}}(\hat{\Omega})$ may be considered. There is no reason, in general, to have

$$\mathscr{C}(q)\colon\ C^\infty(\Omega)\to C^\infty(\hat{\Omega}),$$

since it may happen that to a point $t\in\Omega$ corresponds a nonmeager equivalence class $t/\sim$ in $\hat{\Omega}$. However, in the case considered here we have

2.4. Proposition. *Suppose* $\Omega = \hat{\Omega}/\sim$. *Then* $\mathscr{C}(q)|C^\infty(\Omega) = R$, *i.e., under the identification accepted above* R *is of the form* $\mathscr{C}(q)$ *on the whole* $C^\infty(\Omega)$ *and* $\mathscr{C}(q) = R\colon C^\infty(\Omega)\to\hat{L}\subset C^\infty(\hat{\Omega})$.

Proof. Since $\Phi = R$ on $C(\Omega)$, $\Phi\colon (C(\Omega),\tau)\to(\hat{L},\hat{\tau})$ is continuous where $\tau = \tau|C(\Omega)$. Note that given $0 < f\in C^\infty(\Omega)$, one can define $f^n = f\wedge n\chi_\Omega$, $n = 1, 2, \ldots$ Then $f_n\uparrow f$ pointwise and so $f^n\to f$ (τ) by the σ-order continuity of τ.

In particular, $C(\Omega)$ is τ-dense in $C^\infty(\Omega)$ and therefore Φ may be extended, say to $\Phi^\sim$, by continuity. Obviously, we must have $\Phi^\sim = R$. Hence $\Phi(f^n) = f^n\circ q\to Rf\ (\hat{\tau})$. It follows that (i) $f^n\circ q\uparrow Rf$ in $C^\infty(\hat{\Omega})$. On the other hand, clearly, $f^n\circ q\uparrow f\circ q$ pointwise, whence (ii) $f^n\circ q\uparrow f\circ q$ in $C^{\overline{\infty}}(\hat{\Omega})$. Finally, (i) and (ii) imply $Rf = f\circ q = \mathscr{C}(q)f$ (C^∞ is "regular" in $C^{\overline{\infty}}$).

2.5. Corollary. *Let E be a closed nowhere dense G_δ (in Ω). Then $q^{-1}(E)$ is so too (in $\hat{\Omega}$).*

Proof. Let E be a closed nowhere dense G_δ; then by [E], 1.5.11 there exists $g\in C(\Omega)$ such that $g^{-1}\{0\} = E$. Then $f = 1/g\in C^\infty(\Omega)$ and $E = \{t\colon f(t) = \infty\}$. Then $q^{-1}(E) = \{t\colon \mathscr{C}(q)f(t) = \infty\}$ and $\mathscr{C}(q)f = Rf\in\hat{L}\subset C^\infty(\hat{\Omega})$, i.e., $q^{-1}(E)$ is nowhere dense.

2.6. Theorem. *Let L be a universally complete Riesz space and suppose that L admits a Hausdorff locally solid vector topology τ. Then (L,τ) has the Bounded Order-Completeness property.*

Proof. It may be assumed that $L = C^{\overline{\infty}}(\Omega_L)$. Let $x_\alpha\uparrow$ be a τ-bounded net therein. As $(\hat{L},\hat{\tau})$ is order-continuous, it is in particular locally s-o-c. Consequently $(L^{\wedge\#},\tau^{\wedge\#})$ is a locally boundedly S-O-C ([L], Th. 4.2) order dense subspace of $C^\infty(\Omega_{\hat{L}})$. A fortiori, $(L^{\wedge\#},\tau^{\wedge\#})$ has BOC. Now, in view of 2.4 and of what has been shown before, we may assume that there is a quotient map $q\colon \Omega_{\hat{L}}\to\Omega_L$ such that $R = \mathscr{C}(q)\colon C^\infty(\Omega_L)\to\hat{L}\subset C^\infty(\Omega_{\hat{L}})$, where R is the embedding of L into $\hat{L}$. In particular,

$$\bar{x}_\alpha := \mathscr{C}(q)x_\alpha\uparrow x^{\wedge\#} \quad\text{in } L^{\wedge\#}.$$

Consider moreover the following two functions in $C^{\overline{\infty}}(\Omega_L)$:

$$s := t\mapsto s(t) = \lim x_\alpha(t),$$

and

$$x := \sup x_\alpha \ \text{ in } C^{\overline{\infty}}(\Omega_L).$$

Denote $\bar{s} = \mathscr{C}(q)s$, $\bar{x} = \mathscr{C}(q)x$, and let

$$D_{sx} = \{t\in\Omega_L\colon s(t)\neq x(t)\}.$$

Claim: *D_{sx} is a countable union of closed nowhere dense sets, i.e., it is a meager subset of Ω_L.*

To see this, consider the function $y_\alpha = x - x_\alpha$; $y_\alpha > 0$ and $y_\alpha\downarrow$. Let $y(t) = \lim y_\alpha(t) = \inf y_\alpha(t) = x(t) - s(t)$. Let $E_n = \{t\in\Omega_L\colon y(t)\geqslant 1/n\} = \bigcap\{t\colon y_\alpha(t)\geqslant 1/n\}$; it is closed and so either nowhere dense or containing a closed-open set, say A. Then $z = (1/n)\chi_A\in C(\Omega_L)$ and $y_\alpha - z = (x-z) - z_\alpha\geqslant 0$ for each α, i.e., $x - z\geqslant\sup x_\alpha = x$: a contradiction. Consequently, E_n is nowhere dense and $D_{sx} = \bigcup E_n$ is as claimed.

Now, it is clear that

$$\bar{s}\leqslant x^{\wedge\#}\leqslant\bar{x}.$$

Hence

$$A^\infty_{\bar{s}} := \{t\in\Omega_{\hat{L}}\colon \bar{s}(t) = \infty\} = A^\infty_{\bar{s}\bar{x}} := \{t\in\Omega_{\hat{L}}\colon \bar{s}(t) = \bar{x}(t) = \infty\}$$

is contained in a closed nowhere dense set

$$A^\infty_{x^{\wedge\#}} = \{t\in\Omega_{\hat{L}}\colon x^{\wedge\#}(t) = \infty\}.$$

Consequently,

$$A^{\infty}_{sx} := \{t \in \Omega_L:\ s(t) = x(t) = \infty\}$$

is nowhere dense in Ω_L since $A^{\infty}_{sx} = q(A^{\infty}_{\tilde{s}\tilde{x}}) \subset q(A^{\infty}_{\tilde{x}\wedge\#})$, and the latter set is nowhere dense. Finally, $A^{\infty}_{x} := \{t \in \Omega_L:\ x(t) = \infty\} \subset A^{\infty}_{sx} \cup D_{sx}$, a meager set in Ω_L. As A^{∞}_{x} is closed, it must be nowhere dense, i.e., $x \in C^{\infty}(\Omega_L)$, which ends the proof.

2.7. Corollary. *Let L be laterally complete and suppose that τ is a Hausdorff locally solid topology on L. Then (L, τ) has the BOB property.*

Proof. It may be assumed that $S^{\infty}(\Omega_L) \subset L \subset C^{\infty}(\Omega_L)$ where $S^{\infty}(\Omega_L)$ is the collection of step functions (see [A&B], 23.29). Now, it is easy to see that, for each $x \in C^{\infty}$, there exists an s belonging to $S^{\infty}(\Omega_L)$ (and even to $S^{\infty}_{\sigma}(\Omega_L)$) exceeding x. Hence L is full in $C^{\infty}(\Omega_L)$ and the result follows.

§ 3. Spaces in which BOC implies local order-completeness. This section is devoted to some "bounded" analogues of a theorem of Amemiya, as indicated in § 1. (L, τ) is a locally solid tRs.

3.1. Theorem. *Let (L, τ) be locally bounded and have the σ-BOC property. If L admits a metrizable σ-order-continuous locally solid topology λ, then (L, τ) is locally solid-order-complete.*

Proof. The existence of λ implies that any order-bounded family of positive disjoint elements in L is at most countable. By σ-BOC, L is σ-DC. In view of ([L&Z], 29.3) L is DC and has the countable sup property. In particular, λ is order-continuous. Embed L as a solid order dense vector subspace of $C^{\infty}(\Omega)$, with $\Omega = \Omega_L$. By the representation theorem of [5], on the σ-algebra $\mathrm{BP}(\Omega)$ of subsets of Ω which have the Baire property there exists a finite Lebesgue complete σ-order-continuous submeasure $\mu(:)$ such that $C^{\infty}(\Omega)$ is Riesz isomorphic to $L^0 = L^0(\mathrm{BP}(\Omega), \mu(:))$, the latter being the (universally complete Riesz) space of equivalence (mod the difference on a meager set) classes of $\mathrm{BP}(\Omega)$-measurable functions = the space of μ-classes of μ-measurable functions.

Denote by μ the topology of convergence in $\mu(:)$-submeasure. Then $(L, \tau) \subset (L^0, \mu)$ continuously by [3], 3.6. Another, more conventional proof of this fact is as follows. The topology τ, being locally bounded, is metrizable and therefore (L, τ) is an F-lattice by Amemiya's theorem. Hence $\tau \geqslant \mu | L$ by [A&B], 16.7.

Now, let $\{n^{-1}B\}$, $n \in N$, be a base of neighbourhoods at 0 for τ, where B is a solid, pseudo-convex([4]) and bounded τ-neighbourhood of 0. The following lemma will be needed.

3.2. Lemma. *Let V be a solid subset in L^0. Denote $\underline{V} = \{x \in L^0:\ \exists (x_n) \subset V,\ x_n \xrightarrow{(o)} x\}$, and $\bar{V}^{\mu}$ the closure of V in (L^0, μ). Then $\underline{V} = \bar{V}^{\mu}$.*

([4]) i.e., there exists a constant $k \geqslant 2$ such that $B+B \subset kB$.

Proof of 3.2. Indeed, $x_n \xrightarrow{(o)} x$ means that there exist $y_n \in L^0_+$ such that $|x_n - x| \leqslant y_n \downarrow 0$. By the Egorov Theorem, $y_n \to 0$ $\mu(:)$-uniformly. It follows that $x_n \to x$ $\mu(:)$-uniformly, whence $\underline{V} \subset \bar{V}^{\mu}$. Conversely, suppose $V \ni x_n \to x$ in (L^0, μ). By passing to a subsequence it may be assumed that $x_n \to x$ $\mu(:)$-almost everywhere, whence again $\mu(:)$-uniformly. Moreover, it may also be assumed that $|x_n| \leqslant |x|$ (replacing x_n by $(x_n \wedge |x|) \vee -|x|$). Let $E_n \uparrow$ be measurable subsets such that $\mu(E_n) \to \mu(\Omega)$ and $x_n \chi_{E_i} \to x \chi_{E_i}$ uniformly in n for each $i \in N$. By the diagonal process find (x_{n_i}) such that

$$|x_{n_j} - x| \leqslant 1/i \quad \text{on } E_i, \text{ for } j \geqslant i,\ i \in N.$$

Clearly $|x_{n_j} - x| \leqslant 2|x|$. Consequently,

$$|x_{n_i} - x| \leqslant (1/i)\chi_{E_i} + 2|x|\chi_{\Omega \setminus E_i} \downarrow 0$$

and therefore $x \in \underline{V}$, i.e., $\bar{V}^{\mu} \subset \underline{V}$.

Remark. The reader not familiar with submeasures may treat $\mu(:)$ as a (positive finite countably additive) measure and then check that the standard results from measure theory, used in the proof above, generalize immediately to the case of $\mu(:)$.

We return to the proof of Theorem 3.1. Applying Lemma 3.2 to the neighbourhood B, one concludes the following facts.

(1) *$\bar{B}^{\mu}$ is μ-complete, pseudo-convex and solid* (by its very definition).

(2) $\bar{B}^{\mu} \subset L$.

Indeed, $\bar{V}^{\mu} = \underline{V}$ and it is well known that $\underline{V} = \{x \in L^0:\ \exists x_n \in V \text{ with } x_n \uparrow x\}$. As B is τ-bounded, so is (x_n), whence $x \in L$ by σ-BOC.

Now, by (1) and (2) the base $\{(1/n)\bar{B}^{\mu}\}$ defines a metrizable locally solid vector topology $\bar{\tau}^{\mu}$ on L such that $\mu | L \leqslant \bar{\tau}^{\mu} \leqslant \tau$. However, as $\bar{B}^{\mu}$ is μ-complete, $\bar{\tau}^{\mu}$ is an F-space topology and therefore $\bar{\tau}^{\mu} = \tau$. In particular, $\bar{B}^{\mu}$ is τ-bounded. It remains to show that $\bar{B}^{\mu}$ is order-complete. To this end take $x_{\alpha} \uparrow \subset \bar{B}^{\mu}$. Then (x_{α}), being bounded in (L^0, μ), has its sup, say x, in L^0. As μ is order-continuous, $x_{\alpha} \to x$ (μ), i.e., $x \in \bar{B}^{\mu}$ as the latter set is μ-complete. Hence $\bar{B}^{\mu}$ is order-complete. This ends the proof of the theorem.

It has seemed more compatible with the character of this paper and the previous one ([L]) to formulate the above theorem as it is, i.e., by stating the additional condition on (L, τ) in terms of the existence of the topology λ. However, as indicated in the proof, one can equivalently speak about Riesz spaces of measurable functions over appropriate submeasure spaces (see [5]). Then the condition of the metrizability of λ corresponds to the assumption that $L \subset L^0$ over a σ-decomposable submeasure space (which can be replaced by an equivalent finite one). On the other hand, in many concrete function spaces of measurable functions a (positive countably additive) measure space $(T, \Sigma, \mu(:))$ is a starting point of their construction. Then often even if

$(T, \Sigma, \mu(:))$ is quite arbitrary, $L \subset L^0(T, \Sigma, \mu(:))$ has the property that any $x \in L$ is supported on a σ-finite measurable subset of T. This condition could be translated back to the abstract setting by imposing e.g. that λ is Hausdorff order-continuous and, for any $x \in L$, the principal band generated by x in L^0 has the countable sup property. But such a formulation seems to be somewhat far-fetched.

For the measure theoretic notions used in the next theorem see [5] or [F].

3.3. Theorem. *Let $(T, \Sigma, \mu(:))$ be a measure space, and L a solid vector subspace of $L^0(T, \Sigma, \mu(:))$ such that any $x \in L$ is supported on a σ-finite subset in Σ. Suppose (L, τ) is locally bounded and (L, τ) has σ-BOC. Then (L, τ) is locally s-o-c (and therefore locally σ-S-O-C).*

Proof. The parenthetical statement is clear in view of 1.6. Now, let μ be the topology of convergence in measure on sets of finite measure. Consider $0 \leqslant x_n \uparrow x$ in L. It is clear that the support of x, say E, is contained in the union of the supports of x_n's and therefore is σ-finite. Consider $L(E) = \{x\chi_E\colon x \in L\}$. Then $\mu | L(E)$, i.e., the topology of convergence in measure on sets of finite measure for the σ-finite measure $\mu_E(:) = \mu(:) | \Sigma \cap E$, is metrizable σ-order-continuous. Hence by the preceding theorem $(L(E), \tau | L(E))$ is locally S-O-C. In particular ([F], 23 B), $\tau | L(E)$ can be given by a $\tau | L(E)$-continuous, semi-order-continuous Riesz F-norm, say ϱ_E. Let $\bar{\varrho}_E$ be the corresponding soc Riesz F-seminorm on L defined by

$$\bar{\varrho}_E(x) = \varrho_E(x\chi_E).$$

Define $\varrho(x) = \sup\{\bar{\varrho}_E(x)\colon E \in \Sigma^\sigma\}$ where Σ^σ denotes the σ-finite subsets in Σ. It is easy to see that ϱ is a soc Riesz F-norm on L. Since (L, τ) is an F-lattice, $\tau \geqslant \varrho$ (cf. the proof of 3.1). Now, let $x_n \to 0(\varrho)$ and find the σ-finite set E such that the supports of x_n's are contained in E. Then $x_n \to 0$ (ϱ_E), whence $x_n \to 0$ $(\tau | L(E))$ and it follows that $x_n \to 0$ (τ). Consequently, $\varrho \geqslant \tau$.

3.4. Corollary. *If (L, τ) in 3.3 above has BOC then it is locally S-O-C.*

Remarks. (1) Theorem 3.1 for Banach function spaces is due to Luxemburg and Zaanen [6], 7.7, who have shown that the so-called Lorentz seminorm is in fact equivalent to the original norm in the presence of the weak Fatou property (= σ-BOC). The proof here is more topological in nature, and perhaps can be adopted to solve the following

Problem. Is the theorem true for solid F-sublattices in L^0 (over a finite measure space for simplicity)?

Then of course one seeks the implication σ-BOC $\Rightarrow$ "locally boundedly S-O-C" as τ is no longer locally bounded. Let me mention in this context that Costé ([2], Th. 1) claims this result to be true, but his proof is erroneous.

(2) The restriction to the Riesz subspaces of L^0 over a σ-decomposable submeasure space is apparently not accidental as the use of the Egorov Theorem seems to be essential (cf. [L&Z], § 10).

(3) Let $|\sigma|$ be the absolute weak topology of l_∞, i.e., the one defined by the seminorms

$$l_\infty \ni x \mapsto |x'|(|x|)$$

where x' is a continuous linear functional (and thus a member of the order dual $(l_\infty)^\sim$ of l_∞). Then $|\sigma|$ is a (nonmetrizable) Hausdorff locally solid-convex topology such that $\sigma(l_\infty, l'_\infty) \leqslant |\sigma| \leqslant \|\cdot\|_\infty$. In particular, given $x_\alpha \uparrow \subset l_\infty$ such that (x_α) is $|\sigma|$-bounded, it is norm bounded and (since l_∞ has BOC) $x_\alpha \uparrow x \in l_\infty$. Thus $(l_\infty, |\sigma|)$ has BOC. Now, let (e_n) be the unit vectors, i.e., $e_n = (0, \ldots, 0, 1, \ldots)$ with the one in the nth position. It is easy to see that, if $|\sigma|$ were locally s-o-c, $\sum_{n=1}^{\infty} e_n$ would be $|\sigma|$-subseries convergent, whence $\| \ \|_\infty$-subseries convergent by the Orlicz–Pettis Theorem. However, $\|e_n\|_\infty = 1$ and so $|\sigma|$ is not locally s-o-c. This shows that the assumption of metrizability in 3.1 is essential.

§ **4. Applications.** In this section some characterizations of the properties considered above, in terms of more or less "lateral" nets, will be given. (L, τ) is a Hausdorff locally solid Riesz space.

Agree to say that a net $(x_\alpha) \subset L_+$ is *laterally increasing* (denoted $x_\alpha \lrcorner\!\uparrow$) if

$$\forall \beta > \alpha, \quad x_\beta - x_\alpha \wedge x_\alpha = 0.$$

Suppose that P is a property of L, τ, or (L, τ) stated in terms of an increasing net. Then L, τ, or (L, τ) has the *lateral property* P if it has P with respect to the laterally increasing nets.

For instance, (L, τ) has the lateral BOC property if:

$$x_\alpha \lrcorner\!\uparrow \text{ in } L, (x_\alpha) \ \tau\text{-bounded} \Rightarrow x_\alpha \lrcorner\!\uparrow x \in L.$$

Similarly, L, τ, or (L, τ) has the *disjoint property* P if $\forall$ disjoint $(x_\alpha) \subset L_+$, the corresponding (laterally increasing) net of finite sums of x_α's has P.

For instance, (L, τ) has the disjoint BOC property if:

$$(x_\alpha) \text{ disjoint and perfectly bounded} \Rightarrow \text{(o)-}\textstyle\sum x_\alpha \text{ exists in } L.$$

Here obviously, (o)-$\sum x_\alpha$ means the sup of the net of finite sums of x_α's. A family (x_α) in a topological vector space is said (after Orlicz) to be *perfectly bounded* if the net of finite sums of x_α's is bounded.

4.1. Proposition. *Let (L, τ) be a Hausdorff locally solid DC Riesz space. (L, τ) has the BOC property iff it has the lateral BOC property.*

The proof proceeds as in Abramovich [1], where he has the result for normed lattices, and is based on the following simple

4.2. Lemma. *Suppose that (L, τ) is as above and has the lateral BOC* *property. Given any* $(x_\alpha) \subset L_+$ *such that* $x_\alpha \uparrow x \in L^u$ *and* (x_α) *is* τ*-bounded, we have* $x \in L$.

Proof. One can assume that $L^u = C^\infty(\Omega_L)$ and L is a solid order dense subspace of $C^\infty(\Omega_L)$. Let

$$E_\alpha = \text{closure}\{t \in \Omega_L:\ 2x_\alpha(t) > x(t)\}.$$

Then $x\chi_{E_\alpha} \uparrow x$ and $x\chi_{E_\alpha} \leqslant 2x_\alpha$. Thus $(x\chi_{E_\alpha})$ is τ-bounded and consequently $x \in L$.

Proof of 4.1. With the same notation as above, suppose that $x_\alpha \uparrow$ is not order-bounded in $C^\infty(\Omega_L)$. Then there exists an open-closed set $E \in \Omega_L$ such that $x_\alpha \chi_E \uparrow +\infty\chi_E$. Consider $C^\infty(E) = \{x\chi_E:\ x \in C^\infty(\Omega_L)\}$ and $L(E) = \{x\chi_E:\ x \in L\}$. Observe that $C^\infty(E)$ is a universal completion of $L(E)$, $(x_\alpha \chi_E)$ is $\tau | L(E)$ bounded, and moreover

$$\forall y \in C^\infty(E) \qquad L(E) \ni x_\alpha \chi_E \wedge y \uparrow y.$$

Hence, by 4.2, $y \in L(E)$, i.e., $L(E) = C^\infty(E)$. Now, by 2.6, $\tau | L(E) = C^\infty(E)$ has BOC, which contradicts the fact that $x_\alpha \chi_E \uparrow \infty\chi_E$. Consequently, (x_α) is order-bounded in $C^\infty(E)$, i.e., has sup therein. Applying again 4.2, we find that this sup is in L, which ends the proof.

4.3. Corollary. *Let* (L, τ) *be a Hausdorff locally solid DC Riesz space. Then* (L, τ) *has* σ*-BOC iff it has disjoint* σ*-BOC (i.e., iff for any perfectly bounded disjoint sequence* $(x_n) \subset L_+$, (o)-$\sum_{n=1}^{\infty} x_n \in L$).

Proof. It is essentially the same as that of 4.1, noting that the lateral sequence $x_n \uparrow$ defines a disjoint sequence $y_n := x_n - x_{n-1}$ $(x_0 = 0)$, and (o)-$\sum_{n=1}^{\infty} y_n = \sup x_n$.

4.4. Lemma. *Let* (x_γ), $\gamma \in \Gamma$, *be a disjoint family in* L_+. *Suppose that for any subsequence* $(\gamma_n) \subset \Gamma$, (o)-$\sum_{n=1}^{\infty} x_{\gamma_n}$ *exists in L. Then* (x_γ) *is perfectly bounded in* (L, τ).

Proof. Denote by $\mathscr{F}(\Gamma)$ the family of finite subsets of Γ. Suppose that for some sequence $(e_n) \subset \mathscr{F}(\Gamma)$ the corresponding sequence $\big(\sum_{\gamma \in e_n} x_\gamma\big)_{n=1}^{\infty}$ is not bounded. Then, passing to a subsequence if needed, we may assume that for some positive sequence of reals tending to 0, say (a_n), $a_n \sum_{\gamma \in e_n} x_\gamma \notin V$ for some closed neighbourhood V of 0 in (L, τ). Now, it is easy to see that, by truncating e_n's properly, one may obtain a disjoint subsequence (e'_{n_k}) such that

$$a_{n_k} \sum_{\gamma \in e'_{n_k}} x_\gamma \notin V.$$

In other words, (x_γ) is perfectly bounded if, for any disjoint sequence $(e_n) \subset \mathscr{F}(\Gamma)$, the corresponding sequence of sums is bounded. Let $e = \sum_{n=1}^{\infty} e_n$. By assumption $x = $ (o)-$\sum_{\gamma \in e} x_\gamma \in L$. Hence $\big(\sum_{\gamma \in e_n} x_\gamma\big)_{n=1}^{\infty}$ is order-bounded and the result follows.

4.5. Proposition. *Suppose that* (L, τ) *is DC and admits a Hausdorff order-continuous topology. Then* (L, τ) *has the BOC property iff it has the disjoint BOC property.*

Proof. By the representation theorem in [5], L may be treated as a solid vector subspace of $C^\infty(\Omega)$, with $\Omega = \Omega_L$, such that $C^\infty(\Omega)$ admits a Hausdorff order-continuous topology μ and decomposes into (the product of) disjoint bands $C^\infty(\Omega_\gamma) = \{x\chi_{\Omega_\gamma}:\ x \in C^\infty(\Omega)\}$, $\gamma \in \Gamma$, with $\mu | C^\infty(\Omega_\gamma)$ metrizable. Consequently if we put, for each γ, $L_\gamma = L \cap C^\infty(\Omega_\gamma)$ and $\tau_\gamma = \tau | L_\gamma$, then (L_γ, τ_γ) has the property that

(1) BOC $\Leftrightarrow$ σ-BOC $\Leftrightarrow$ σ-lateral BOC $\Leftrightarrow$ disjoint σ-BOC.

Take $x_\alpha \subset L_+$, (x_α) τ-bounded. With the obvious notation, $x_{\alpha\gamma} \uparrow x_\gamma$ and by (1) $x_\gamma \in L_\gamma$. Now, given any sequence $(\gamma_n) \subset \Gamma$, let $E = \text{closure}\big(\bigcup_{n=1}^{\infty} \Omega_{\gamma_n}\big)$. Then (cf. [5], *loc. cit.*) the topology $\mu | C^\infty(E) = \{x\chi_E:\ x \in C^\infty(\Omega)\}$ is metrizable, whence again $\big(L(E) = \{x\chi_E:\ x \in L\}, \gamma | L(E)\big)$ has the property (1) above. It follows that $x_\alpha \chi_E \uparrow x\chi_E \in L$ and, since $x\chi_E = $ (o)-$\sum_{n=1}^{\infty} x_{\gamma_n}$, the family (x_γ) is perfectly bounded in (L, τ) by 4.4. Hence $x = $ (o)-$\sum x_\gamma \in L$, by the disjoint BOC property. Moreover, as $\Omega \setminus \bigcup \Omega_\gamma$ is nowhere dense ([5]), $x_\alpha \uparrow x$, which ends the proof.

4.6. Remark. One could consider the σ-perfect boundedness of (x_γ) as the stronger property meaning the boundedness of countable sums of x_γ's, i.e.,

$$\Big\{\sum_{\gamma \in F} x_\gamma:\ F \subset \Gamma \text{ and } F \text{ is at most countable}\Big\} \text{ bounded}.$$

Of course one has to be able to specify in L the meaning of the series considered (in our case above $\sum_{\gamma \in F} x_\gamma = $ (o)-$\sum_{\gamma \in F} x_\gamma \in L$). Then, in order to infer the σ-perfect boundedness of (x_γ) it is not sufficient in general to check the corresponding "disjoint boundedness". However, if τ is e.g. locally convex, such a criterion is still valid.

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Added in proof. The proof of Theorem 2.6 is not correct. In its final part it is asserted that the set $q(A_x^{\infty} \wedge \#)$ is nowhere dense. It is not clear why this should be so since the function $x^{\wedge \#}$ need not be, a priori, constant on equivalence classes. In fact, it can be shown that Theorem 2.6 implies the non-existence of measurable cardinals (cf. the beginning of § 2).

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The Lie structure of C^* and Poisson algebras

by

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Abstract. Associative algebras with a Lie structure are considered. In particular, we describe the form of maximal Lie ideals of C^* algebras, maximal Lie ideals and maximal finite-codimensional Lie subalgebras of Poisson algebras of functions on symplectic manifolds.

1. Notation and preliminaries. There are many natural algebraic objects which carry both an associative and a Lie ring structure. For example, every associative ring A can also be regarded as a Lie ring with the Lie bracket $[X, Y] := XY - YX$.

It is easy to see that in this case ad_X is a derivation of the associative ring A for all $X \in A$, i.e.,

$$(1.1)\qquad [X, YZ] = [X, Y]Z + Y[X, Z].$$

We also have the identity

$$(1.2)\qquad [X, YZ] + [Y, ZX] + [Z, XY] = 0.$$

Another example is the associative ring $C^\infty(M)$ of all smooth functions on a symplectic manifold M with a Lie ring structure given by the Poisson bracket. In this case also ad_X is a derivation of $C^\infty(M)$ for all $X \in C^\infty(M)$.

More generally, by a *Poisson ring* we shall understand an associative commutative ring A equipped with a Lie bracket which makes A a Lie ring and is such that ad_X is a derivation of the associative ring A for all $X \in A$.

One can check that (1.2) is then also satisfied.

Our aim in this note is to propose a general approach to investigations of such structures (close to the methods used in [1] and [3]), which gives us various results (partially well-known) concerning the relations between the Lie and the associative structures.

The above examples lead to the following definition:

(1.3) Definition. An associative ring (algebra) A equipped with a Lie bracket which makes A a Lie ring (algebra) and satisfies (1.1) and (1.2) will be called an AL-*ring* (*algebra*).

A *topological* AL-*ring* (*algebra*) is defined in the natural way.

(1.4) Definition. An associative ideal K of an AL-ring (algebra) A which is also a Lie ideal of A will be called an AL-*ideal* of A. An AL-*homomorphism*