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Laws of large numbers in von Neumann algebras and related results

by

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Abstract. Noncommutative weak and strong laws of large numbers and the Glivenko–Cantelli theorem are proved. Two cases are considered: a von Neumann algebra with a normal faithful state on it and the space of operators integrable with respect to a normal faithful finite trace.

1. Preliminaries and notation. One of the problems occurring in noncommutative probability theory concerns the extension of various results centred around laws of large numbers to the noncommutative context. Due to the existence of the nonabelian counterparts of the notions of convergence in measure and convergence almost everywhere, it is possible to consider the noncommutative setting of the well-known classical theorems such as e.g. Kolmogorov's law of large numbers or the law of the iterated logarithm. In this setting the role of a random variable is played by an element of a von Neumann algebra $\mathcal{M}$, and a probability measure is replaced by a normal faithful state on $\mathcal{M}$. If this state is tracial, the von Neumann algebra $\mathcal{M}$ can be replaced by an algebra consisting of unbounded operators. Many results in this area have recently been obtained by Batty [1], Goldstein [3] and Jajte [4].

The purpose of this paper is to present some noncommutative weak and strong laws of large numbers for elements from the space $L^r(\mathcal{M}, \tau)$ (see below) and a strong law of large numbers together with the noncommutative version of the Glivenko–Cantelli theorem for elements from a von Neumann algebra with a normal faithful state.

It is worthwhile to observe that the case where the state considered is not tracial is usually much more difficult to handle due to the "nonsubadditivity" of a state. At the same time, the techniques employed in the two cases differ considerably.

For the general theory of von Neumann algebras, the reader is referred to [7], [9] or [10]. Here we only establish the notation used throughout the paper and recall some basic definitions.

Let $\mathcal{M}$ be a von Neumann algebra. If p is a projection in $\mathcal{M}$, $p^\perp$ denotes $\mathbf{1}-p$ where $\mathbf{1}$ is the identity operator. For two projections p, q in $\mathcal{M}$, we write $p \prec q$ if p is equivalent to some subprojection of q.

For each self-adjoint operator x affiliated with $\mathcal{M}$, we denote by $e_B(x)$ the spectral projection of x corresponding to the Borel subset B of the line.

If ϱ is a normal faithful state on $\mathcal{M}$ and x, y are two self-adjoint operators affiliated with $\mathcal{M}$, we say that x and y are *identically distributed* if $\varrho(e_B(x)) = \varrho(e_B(y))$ for every Borel B.

If τ is a normal faithful semifinite trace on $\mathcal{M}$, then $\tilde{\mathcal{M}}$ denotes the algebra of operators measurable in Nelson's sense (see [6]), and $L^r(\mathcal{M}, \tau)$ – the space of r-integrable operators from $\tilde{\mathcal{M}}$. Detailed descriptions of $\tilde{\mathcal{M}}$, $L^r(\mathcal{M}, \tau)$, as well as of the theory of noncommutative integration, are contained in [6], [8] and [11].

Various notions of independence for sequences of elements from $\mathcal{M}$ (or $\tilde{\mathcal{M}}$) were introduced in [1]. We shall use the following one only:

Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful state ϱ, and $\mathcal{N}_1, \mathcal{N}_2$ two von Neumann subalgebras of $\mathcal{M}$. We say that $\mathcal{N}_1$ and $\mathcal{N}_2$ are *independent* if

$$\varrho(xy) = \varrho(x)\varrho(y) \quad \text{for all } x \in \mathcal{N}_1,\ y \in \mathcal{N}_2.$$

Note that this assumption is weaker than that in [1], both being equivalent when ϱ is tracial.

Two elements x, y from $\mathcal{M}$ (or $\tilde{\mathcal{M}}$ if ϱ is tracial) are said to be *independent* if the von Neumann algebras $W^*(x)$ and $W^*(y)$ generated by x and y, respectively, are independent. A sequence $\{x_n\}$ of elements from $\mathcal{M}$ ($\tilde{\mathcal{M}}$) is said to be *successively independent* if, for every n, the von Neumann algebra $W^*(x_n)$ generated by x_n is independent of the von Neumann algebra $W^*(x_1, \ldots, x_m)$ generated by the elements $x_1, \ldots, x_m$ for $m < n$.

A fundamental role in our considerations is played by two notions of convergence: convergence in measure and almost uniform convergence. Now, we give a brief description of these notions.

Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful semifinite trace τ. The measure topology in $\mathcal{M}$ is given by the fundamental system of neighbourhoods of zero of the form

$$N(\varepsilon, \delta) = \{x \in \mathcal{M}\colon \text{there exists a projection } e \text{ in } \mathcal{M} \text{ such that } xe \in \mathcal{M},\ \|xe\| \leqslant \varepsilon \text{ and } \tau(e^\perp) \leqslant \delta\}.$$

It follows that $\tilde{\mathcal{M}}$, being the completion of $\mathcal{M}$ in the above topology, is a topological $*$-algebra (see [6]).

The following "technical" form of convergence in measure will best suit our purposes [11, Prop. 2.7]:

$x_n \to x$ in measure if and only if, for each $\varepsilon > 0$,

$$\tau(e_{[\varepsilon,\infty)}(|x - x_n|)) \to 0.$$

Now, let ϱ be a normal faithful state on $\mathcal{M}$, and x, x_n – elements from $\mathcal{M}$ (or $\tilde{\mathcal{M}}$ if ϱ is tracial). We say that $x_n \to x$ *almost uniformly* (*a.u.*) if, for each $\varepsilon > 0$, there is a projection e in $\mathcal{M}$ with $\varrho(e^\perp) < \varepsilon$ such that $(x_n - x)e \in \mathcal{M}$ for sufficiently large n and $\|(x_n - x)e\| \to 0$.

It is worth noting that, in fact, the above definition does not depend on the choice of ϱ.

We shall now establish some simple facts related to the above notions.

Lemma 1.1 (Tchebyshev's inequality). *Let $\mathcal{M}$ be a von Neumann algebra with a normal faithful tracial state τ, and x – an element from $\tilde{\mathcal{M}}$. Then, for each $\varepsilon > 0$,*

$$\tau(e_{[\varepsilon,\infty)}(|x|)) \leqslant \varepsilon^{-2}\tau(|x|^2).$$

Proof. From the spectral decomposition $|x|^2 = \int_0^\infty \lambda^2 e_{d\lambda}(|x|)$ we find that

$$|x|^2 \geqslant \int_\varepsilon^\infty \lambda^2 e_{d\lambda}(|x|) \geqslant \varepsilon^2 e_{[\varepsilon,\infty)}(|x|)$$

and the conclusion follows.

The following lemma is a slight generalization of Lemma 3.2 from [1] (with $b_n = n$) and can be proved in virtually the same way.

Lemma 1.2 (Kronecker's lemma). *Let $\{x_n\}$ be a sequence in $\tilde{\mathcal{M}}$ and $b_n \uparrow \infty$. Put $S_n = \sum_{k=1}^n x_k$. If S_n converges almost uniformly, then $b_n^{-1}\sum_{k=1}^n b_k x_k$ converges almost uniformly to zero.*

The proposition below shows that, as in the classical case, almost uniform convergence can be described by the Cauchy condition.

Proposition 1.3. *Let $\{x_n\}$ be a sequence in $\tilde{\mathcal{M}}$ and τ a normal faithful tracial state on $\mathcal{M}$. If $\{x_n\}$ is Cauchy almost uniformly, i.e., for each $\varepsilon > 0$, there are a projection e in $\mathcal{M}$ with $\tau(e^\perp) < \varepsilon$ and a positive integer N such that $\|(x_n - x_m)e\| < \varepsilon$ for $n, m \geqslant N$, then $\{x_n\}$ converges almost uniformly.*

Proof. It follows that $\{x_n\}$ converges in measure (because it is Cauchy in measure); let x be its limit. It is known (see e.g. [11, p. 94]) that some subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x almost uniformly. For any $\varepsilon > 0$, choose projections p and q fulfilling the conditions

(i) $\tau(p^\perp), \tau(q^\perp) < \varepsilon/2$,

(ii) $\|(x_n - x_m)p\| < \varepsilon$ for sufficiently large n, m,

(iii) $\|(x_{n_k} - x)q\| \to 0$ as $k \to \infty$.

Put $e = p \wedge q$. Then $\tau(e^\perp) < \varepsilon$ and $(x_m - x)e = (x_m - x_{n_k})e + (x_{n_k} - x)e$ belongs to $\mathcal{M}$. Moreover, $\|(x_m - x)e\| \leqslant \|(x_m - x_{n_k})p\| + \|(x_{n_k} - x)q\| \to 0$ as $m, k \to \infty$, which ends the proof.

2. A multidimensional strong law of large numbers and the Glivenko–Cantelli theorem for states. Let $\mathscr{M}$ be a von Neumann algebra with a faithful normal state ϱ. For further reference we formulate the following immediate corollary to Theorem 1.2 in [3]:

THEOREM 2.1. *Let $\{x_n\}$ be a sequence of positive operators from $\mathscr{M}$ and $\{\varepsilon_n\}$ a sequence of positive numbers. If $\sum_{n=1}^{\infty}\varepsilon_n^{-1}\varrho(x_n)<\frac{1}{2}$, then there exists a projection e in $\mathscr{M}$ such that*

$$\varrho(e)\geqslant 1-2\sum_{n=1}^{\infty}\varepsilon_n^{-1}\varrho(x_n)\quad\text{and}\quad \|ex_ne\|\leqslant 2\varepsilon_n \text{ for } n=1,2,\ldots$$

In what follows we shall need the following elementary lemma.

LEMMA 2.2. *If $a_n\geqslant 0$ are such that $\sum_{n=1}^{\infty}a_n<\infty$, then there exists a sequence $\{\varepsilon_n\}$ of positive numbers such that $\varepsilon_n\downarrow 0$ and $\sum_{n=1}^{\infty}\varepsilon_n^{-1}a_n<\infty$.*

We say that x and y from $\mathscr{M}$ are *uncorrelated* if

$$\varrho(x^*y)=\varrho(x^*)\varrho(y).$$

This definition is consistent with the notion of orthogonality with respect to the natural scalar product $(x,y)=\varrho(y^*x)$ in $\mathscr{M}$ (when $\varrho(x)=\varrho(y)=0$); moreover, the relation defined in this way is symmetric regarding x and y.

Now, we shall prove a "multidimensional" law of large numbers in von Neumann algebras.

THEOREM 2.3. *Let $\{x_n^{(i)}\}$ for $i=1,\ldots,r$ be a sequence of pairwise uncorrelated, uniformly bounded (i.e., $\sup_n\|x_n^{(i)}\|<\infty$) operators from $\mathscr{M}$. Then*

$$\frac{1}{N}\sum_{n=1}^{N}[x_n^{(i)}-\varrho(x_n^{(i)})\mathbf{1}]\to 0\quad\text{a.u., uniformly in } i=1,\ldots,r,$$

i.e., for each $\varepsilon>0$, there exists a projection e in $\mathscr{M}$ with $\varrho(e)\geqslant 1-\varepsilon$ such that

$$\max_{1\leqslant i\leqslant r}\left\|\left(\frac{1}{N}\sum_{n=1}^{N}[x_n^{(i)}-\varrho(x_n^{(i)})\mathbf{1}]\right)e\right\|\to 0.$$

Remark. Let us observe that, unlike the tracial case, the multi-dimensional version is essentially stronger than that for $r=1$.

Proof. We can assume that $\|x_n^{(i)}\|\leqslant 1$ and $\varrho(x_n^{(i)})=0$, $i=1,\ldots,r$; $n=1,2,\ldots$ Put $S_N^{(i)}=\frac{1}{N}\sum_{n=1}^{N}x_n^{(i)}$. For any i, we have the following estimations:

$$\varrho(|S_{N^2}^{(i)}|^2)=\frac{1}{N^4}\sum_{n,m=1}^{N^2}\varrho((x_n^{(i)})^*x_m^{(i)})=\frac{1}{N^4}\sum_{n=1}^{N^2}\varrho(|x_n^{(i)}|^2)\leqslant\frac{1}{N^2}. \tag{2.1}$$

For K chosen so that $K^2\leqslant N<(K+1)^2$,

$$\||S_N^{(i)}-S_{K^2}^{(i)}|^2\|=\left\|\left(\frac{1}{N}-\frac{1}{K^2}\right)\sum_{n=1}^{K^2}x_n^{(i)}+\frac{1}{N}\sum_{n=K^2+1}^{N}x_n^{(i)}\right\|^2$$
$$\leqslant\left(\frac{N-K^2}{NK^2}K^2+\frac{N-K^2}{N}\right)^2\leqslant\frac{36}{K^2}.$$

Thus, by putting $T_K=(6/K)\mathbf{1}$, we obtain

$$|S_N^{(i)}-S_{K^2}^{(i)}|^2\leqslant T_K^2\quad\text{for each } i=1,\ldots,r. \tag{2.2}$$

Let

$$S_N=\Big(\sum_{i=1}^{r}|S_N^{(i)}|^2\Big)^{1/2}. \tag{2.3}$$

From (2.1) and the definition of T_K it follows that $\sum_{N=1}^{\infty}[\varrho(S_{N^2}^2)+\varrho(T_N^2)]<\infty$, thus, on account of Lemma 2.2, there exists a sequence of positive numbers $\{\varepsilon_N\}$ such that $\varepsilon_N\downarrow 0$ and

$$\sum_{N=1}^{\infty}\varepsilon_N^{-1}[\varrho(S_{N^2}^2)+\varrho(T_N^2)]<\infty.$$

For given $\varepsilon>0$, choose N_0 such that

$$\sum_{N=N_0}^{\infty}\varepsilon_N^{-1}[\varrho(S_{N^2}^2)+\varrho(T_N^2)]<\varepsilon/2.$$

From Theorem 2.1 applied to the sequence $S_{N_0^2}^2, T_{N_0}^2,\ldots,S_{N^2}^2, T_N^2,\ldots$ it follows that there is a projection e in $\mathscr{M}$ with

$$\varrho(e)\geqslant 1-2\sum_{N=N_0}^{\infty}\varepsilon_N^{-1}[\varrho(S_{N^2}^2)+\varrho(T_N^2)]\geqslant 1-\varepsilon$$

and

$$\|eS_{N^2}^2e\|\leqslant 2\varepsilon_N,\quad \|eT_N^2e\|\leqslant 2\varepsilon_N\quad\text{for } N\geqslant N_0. \tag{2.4}$$

Take an arbitrary $\eta>0$. Let $N_1\geqslant N_0$ be such that $\varepsilon_N<\eta^2/8$ for $N\geqslant N_1$. For any $N\geqslant N_1^2$, we find $K\geqslant N_1$ for which $K^2\leqslant N<(K+1)^2$. According to (2.2), (2.3) and (2.4), we have

$$\|S_N^{(i)}e\|^2\leqslant(\|(S_N^{(i)}-S_{K^2}^{(i)})e\|+\|S_{K^2}^{(i)}e\|)^2$$
$$\leqslant 2(\|(S_N^{(i)}-S_{K^2}^{(i)})e\|^2+\|S_{K^2}^{(i)}e\|^2)$$
$$=2(\|e|S_N^{(i)}-S_{K^2}^{(i)}|^2e\|+\|e|S_{K^2}^{(i)}|^2e\|)$$
$$\leqslant 2(\|eT_K^2e\|+\|eS_{K^2}^2e\|)\leqslant 8\varepsilon_K<\eta^2,\qquad i=1,\ldots,r.$$

Thus, for $N\geqslant N_1^2$, $\max_{1\leqslant i\leqslant r}\|S_N^{(i)}e\|<\eta$, which concludes the proof.

For $r = 1$, the above theorem extends Theorem 4.1 from [1], where $\{x_n\}$ are assumed to be independent and $\{S_N\}$ is convergent in a weaker sense.

We shall now prove the following extension of the well-known Glivenko–Cantelli theorem:

THEOREM 2.4. *Let $\{x_n\}$ be a sequence of closed, self-adjoint, pairwise independent and identically distributed operators affiliated with $\mathcal{M}$. Then, for each $\varepsilon > 0$, there is a projection e in $\mathcal{M}$ such that $\varrho(e) \geq 1-\varepsilon$ and*

$$\sup_{-\infty<\lambda<\infty} \left\| e\left[\frac{1}{N}\sum_{n=1}^{N} e_{(-\infty,\lambda)}(x_n) - \varrho\big(e_{(-\infty,\lambda)}(x_1)\big)\mathbf{1}\right] e \right\| \to 0 \quad \text{as } N\to\infty.$$

Remark. The type of convergence considered in the above theorem, a little weaker than a.u. convergence, was used in [2] and [12] in proving noncommutative martingale and ergodic theorems. One easily observes that it reduces to the usual notion in the commutative case.

Proof. For every real λ, put

$$S_N(\lambda+0) = \frac{1}{N}\sum_{n=1}^{N} e_{(-\infty,\lambda]}(x_n), \qquad S_N(\lambda-0) = \frac{1}{N}\sum_{n=1}^{N} e_{(-\infty,\lambda)}(x_n),$$
$$S(\lambda+0) = \varrho\big(e_{(-\infty,\lambda]}(x_1)\big)\mathbf{1}, \qquad S(\lambda-0) = \varrho\big(e_{(-\infty,\lambda)}(x_1)\big)\mathbf{1},$$

and let λ_{ir} be defined as

$$\lambda_{ir} = \inf\{\lambda:\ \varrho\big(e_{(-\infty,\lambda)}(x_1)\big) \leq i/r \leq \varrho\big(e_{(-\infty,\lambda]}(x_1)\big)\}.$$

for $i = 1, \ldots, r-1$; $r = 1, 2, \ldots$

If $\lambda_{ir} < \lambda \leq \lambda_{i+1,r}$ for some i and any r, then

$$S(\lambda_{ir}+0) \leq S(\lambda-0) \leq S(\lambda_{i+1,r}-0),$$
$$S_N(\lambda_{ir}+0) \leq S_N(\lambda-0) \leq S_N(\lambda_{i+1,r}-0);$$

thus

$$S_N(\lambda_{ir}+0) - S(\lambda_{i+1,r}-0) \leq S_N(\lambda-0) - S(\lambda-0) \leq S_N(\lambda_{i+1,r}-0) - S(\lambda_{ir}+0).$$

But, from the definition of λ_{ir}, we have

$$S(\lambda_{i+1,r}-0) - S(\lambda_{ir}+0) \leq (1/r)\mathbf{1}$$

and, consequently,

$$S_N(\lambda_{ir}+0) - S(\lambda_{ir}+0) - (1/r)\mathbf{1} \leq S_N(\lambda-0) - S(\lambda-0) \leq S_N(\lambda_{i+1,r}-0) - S(\lambda_{i+1,r}-0) + (1/r)\mathbf{1}. \tag{2.5}$$

If $\lambda \leq \lambda_{1r}$, then

$$-(1/r)\mathbf{1} \leq S_N(\lambda-0) - S(\lambda-0) \leq S_N(\lambda_{1r}-0) - S(\lambda_{1r}-0) + (1/r)\mathbf{1} \tag{2.6}$$

and, if $\lambda > \lambda_{rr}$, then

$$S_N(\lambda-0) - S(\lambda-0) = 0. \tag{2.7}$$

Putting, for convenience,

$$S_N(\lambda_{0r}+0) - S(\lambda_{0r}+0) = S_N(\lambda_{rr}-0) - S(\lambda_{rr}-0) = 0,$$

we conclude in virtue of (2.5), (2.6) and (2.7) that, for every real λ and each $r = 1, 2, \ldots$, there is an i between 0 and $r-1$ such that

$$S_N(\lambda_{ir}+0) - S(\lambda_{ir}+0) - (1/r)\mathbf{1} \leq S_N(\lambda-0) - S(\lambda-0) \leq S_N(\lambda_{i+1,r}-0) - S(\lambda_{i+1,r}-0) + (1/r)\mathbf{1};$$

thus, for an arbitrary projection p from $\mathcal{M}$, we have

$$p[S_N(\lambda_{ir}+0) - S(\lambda_{ir}+0)]p - (1/r)p \leq p[S_N(\lambda-0) - S(\lambda-0)]p \leq p[S_N(\lambda_{i+1,r}-0) - S(\lambda_{i+1,r}-0)]p + (1/r)p$$

and, as a consequence of the above inequality, we obtain

$$\|p[S_N(\lambda-0) - S(\lambda-0)]p\| \leq \max_{\substack{1\leq i\leq r\\ \theta=\pm 0}} \|p[S_N(\lambda_{ir}+\theta) - S(\lambda_{ir}+\theta)]p\| + 1/r,$$

which gives

$$\sup_{-\infty<\lambda<\infty} \|p[S_N(\lambda-0) - S(\lambda-0)]p\| \leq \max_{\substack{1\leq i\leq r\\ \theta=\pm 0}} \|p[S_N(\lambda_{ir}+\theta) - S(\lambda_{ir}+\theta)]p\| + 1/r \tag{2.8}$$

for each $r = 1, 2, \ldots$ and each projection p from $\mathcal{M}$.

But, according to Theorem 2.3, for each $\varepsilon > 0$, there is a projection e in $\mathcal{M}$ with $\varrho(e) \geq 1-\varepsilon$ such that

$$\max_{\substack{1\leq i\leq r\\ \theta=\pm 0}} \|[S_N(\lambda_{ir}+\theta) - S(\lambda_{ir}+\theta)]e\| \to 0 \quad \text{as } N\to\infty$$

for every fixed r. Thus, from (2.8) we infer that

$$\overline{\lim_{N\to\infty}} \sup_{-\infty<\lambda<\infty} \|e[S_N(\lambda-0) - S(\lambda-0)]e\| \leq 1/r$$

for every r, which completes the proof.

Remark. The theorem just proved is a generalization of Theorem 4.1 from [4], where ϱ is assumed to be tracial and $\{x_n\}$ – successively independent.

3. Laws of large numbers for traces. Now, we assume that $\mathcal{M}$ is a von Neumann algebra with a normal faithful finite trace τ. For an arbitrary positive number c, we define

$$x^{(c)} = xe_{[0,c)}(|x|).$$

THEOREM 3.1 (Weak law of large numbers). *Let $\{x_n\}$ be a successively independent sequence of self-adjoint, identically distributed elements from $\mathcal{M}$. If*

$$\lim_{n\to\infty} n\tau(e_{[n,\infty)}(|x_1|)) = 0,$$

then

$$\frac{1}{n}\sum_{k=1}^{n} x_k - \alpha_n \mathbf{1} \to 0 \text{ in measure}, \quad \text{where } \alpha_n = \tau(x_1 e_{[0,n)}(|x_1|)).$$

Proof. Assume that $\mathcal{M}$ acts in $\mathcal{H}$ and put $S_n = x_1 + \ldots + x_n$,

$$\tilde{S}_n = x_1^{(n)} + \ldots + x_n^{(n)}, \qquad \tilde{m}_n = \tau(\tilde{S}_n) = n\tau(x_1^{(n)}).$$

For an arbitrary $\gamma > 0$, we have

$$p = e_{[2\gamma,\infty)}(|S_n - \tilde{m}_n \mathbf{1}|) \wedge e_{[0,\gamma)}(|\tilde{S}_n - \tilde{m}_n \mathbf{1}|) \wedge \bigwedge_{k=1}^{n} e_{[0,n)}(|x_k|) = 0.$$

Indeed, if, for some ξ of norm one, $\xi \in p\mathcal{H}$, then $\xi \in e_{[0,n)}(|x_k|)\mathcal{H}$ and, consequently, $x_k\xi = x_k^{(n)}\xi$, $k = 1, \ldots, n$, which yields $S_n\xi = \tilde{S}_n\xi$. Thus, from the elementary properties of the spectral decomposition we obtain

$$2\gamma \leqslant \| |S_n - \tilde{m}_n \mathbf{1}| e_{[2\gamma,\infty)}(|S_n - \tilde{m}_n \mathbf{1}|)\xi\| = \|(S_n - \tilde{m}_n \mathbf{1})\xi\|$$
$$\leqslant \|(S_n - \tilde{S}_n)\xi\| + \|(\tilde{S}_n - \tilde{m}_n \mathbf{1})\xi\|$$
$$= \| |\tilde{S}_n - \tilde{m}_n \mathbf{1}| e_{[0,\gamma)}(|\tilde{S}_n - \tilde{m}_n \mathbf{1}|)\xi\| \leqslant \gamma,$$

which is impossible; so $p = 0$ and this implies (see [7], p. 80)

$$e_{[2\gamma,\infty)}(|S_n - \tilde{m}_n \mathbf{1}|) \prec e_{[\gamma,\infty)}(|\tilde{S}_n - \tilde{m}_n \mathbf{1}|) \vee \bigvee_{k=1}^{n} e_{[n,\infty)}(|x_k|). \tag{3.1}$$

From the properties of a trace, Lemma 1.1 and the inequality $\tau(|x - \tau(x)\mathbf{1}|^2) \leqslant \tau(|x|^2)$ we obtain the following consequence of (3.1):

$$\tau(e_{[2\gamma,\infty)}(|S_n - \tilde{m}_n \mathbf{1}|)) \leqslant \gamma^{-2}\tau(|\tilde{S}_n - \tilde{m}_n \mathbf{1}|^2) + \tau\Big(\bigvee_{k=1}^{n} e_{[n,\infty)}(|x_k|)\Big) \tag{3.2}$$
$$\leqslant \gamma^{-2}\sum_{k=1}^{n}\tau(|x_k^{(n)}|^2) + \sum_{k=1}^{n}\tau(e_{[n,\infty)}(|x_k|))$$
$$= \gamma^{-2} n\tau(|x_1^{(n)}|^2) + n\tau(e_{[n,\infty)}(|x_1|)).$$

Now, take any $\varepsilon > 0$. From (3.2) with $\gamma = n\varepsilon/2$ we get

$$\tau\left(e_{[\varepsilon,\infty)}\left(\left|\frac{S_n}{n} - \alpha_n \mathbf{1}\right|\right)\right) = \tau\left(e_{[\varepsilon,\infty)}\left(\left|\frac{S_n - \tilde{m}_n \mathbf{1}}{n}\right|\right)\right)$$
$$= \tau(e_{[n\varepsilon,\infty)}(|S_n - \tilde{m}_n \mathbf{1}|)) \leqslant \frac{4\tau(|x_1^{(n)}|^2)}{\varepsilon^2 n} + n\tau(e_{[n,\infty)}(|x_1|))$$
$$= 4\varepsilon^{-2} n^{-1}\int_{[0,n)} \lambda^2 \tau(e_{d\lambda}(|x_1|)) + n\tau(e_{[n,\infty)}(|x_1|)).$$

The integral can be estimated as follows:

$$\int_{[0,n)} \lambda^2 \tau(e_{d\lambda}(|x_1|)) \leqslant \sum_{k=0}^{n-1}(k+1)^2\tau(e_{[k,k+1)}(|x_1|))$$
$$= \sum_{k=0}^{n-1}(2k+1)\tau(e_{[k,n)}(|x_1|)) \leqslant \tau(e_{[0,n)}(|x_1|)) +$$
$$+ 3\sum_{k=1}^{n-1} k\tau(e_{[k,n)}(|x_1|)) \leqslant 1 + 3\sum_{k=1}^{n-1} k\tau(e_{[k,\infty)}(|x_1|)).$$

Thus, finally,

$$\tau\left(e_{[\varepsilon,\infty)}\left(\left|\frac{S_n}{n} - \alpha_n \mathbf{1}\right|\right)\right) \leqslant 4\varepsilon^{-2} n^{-1}\Big[1 + 3\sum_{k=1}^{n-1} k\tau(e_{[k,\infty)}(|x_1|))\Big] +$$
$$+ n\tau(e_{[n,\infty)}(|x_1|)) \to 0 \quad \text{as } n \to \infty,$$

which ends the proof.

Using a similar technique, one can prove the following version of the weak law of large numbers:

THEOREM 3.2. *Let $\{x_n\}$ be a successively independent sequence of self-adjoint elements from $L^1(\mathcal{M}, \tau)$. If*

(i) $\displaystyle\sum_{k=1}^{n} \tau(e_{[n,\infty)}(|x_k - \alpha_k \mathbf{1}|)) \to 0$ *as* $n \to \infty$,

(ii) $\displaystyle\frac{1}{n}\sum_{k=1}^{n} \tau(|x_k - \alpha_k \mathbf{1}|^{(n)}) \to 0$ *as* $n \to \infty$,

(iii) $\displaystyle\frac{1}{n^2}\sum_{k=1}^{n} \tau([|x_k - \alpha_k \mathbf{1}|^2]^{(n)}) \to 0$ *as* $n \to \infty$,

where $\alpha_k = \tau(x_k)$, then

$$\frac{1}{n}\sum_{k=1}^{n}(x_k - \alpha_k \mathbf{1}) \to 0 \quad \textit{in measure}.$$

Our next goal is to obtain a noncommutative version of the well-known Marcinkiewicz law of large numbers which is a generalization of the celebrated Kolmogorov strong law of large numbers for independent, identically distributed random variables. We begin with a lemma on the "equivalence of convergence".

LEMMA 3.3. *Let $\{x_n\}$, $\{y_n\}$ be sequences of elements from $\tilde{\mathscr{M}}$ and $\{c_n\}$ a sequence of positive numbers. If*

$$\sum_{n=1}^{\infty} \tau\left(e_{[c_n,\infty)}(|x_n|)\right) < \infty,$$

then the series $\sum_{n=1}^{\infty}(x_n+y_n)$ converges almost uniformly if and only if the series $\sum_{n=1}^{\infty}(x_n^{(c_n)}+y_n)$ converges almost uniformly.

Proof. From the inequality

$$\tau\Big(\bigvee_{m=n}^{\infty} e_{[c_m,\infty)}(|x_m|)\Big) \leqslant \sum_{m=n}^{\infty} \tau\left(e_{[c_m,\infty)}(|x_m|)\right) \to 0 \quad \text{as } n\to\infty$$

we have

$$\bigvee_{m=n}^{\infty} e_{[c_m,\infty)}(|x_m|) \to 0 \quad \text{strongly as } n\to\infty;$$

thus

$$\bigvee_{n=1}^{\infty} \bigwedge_{m=n}^{\infty} e_{[0,c_m)}(|x_m|) = \mathbf{1}. \tag{3.3}$$

Put

$$S_n = \sum_{m=1}^{n}(x_m+y_m), \qquad \bar{S}_n = \sum_{m=1}^{n}(x_m^{(c_m)}+y_m)$$

and assume that $\bar{S}_n$ converges almost uniformly.

For each $\varepsilon>0$, there are a projection q in $\mathscr{M}$ with $\tau(q^{\perp})<\varepsilon$ and a number N_1 such that $\|(\bar{S}_n-\bar{S}_k)q\|<\varepsilon$ for $n, k\geqslant N_1$. From equality (3.3) it follows that we can find a number N_2 with

$$\tau\Big(\bigwedge_{m=N_2}^{\infty} e_{[0,c_m)}(|x_m|)\Big) \geqslant 1-\varepsilon.$$

Put

$$N=\max(N_1, N_2) \quad \text{and} \quad p=\bigwedge_{m=N}^{\infty} e_{[0,c_m)}(|x_m|).$$

Then $\tau((p\wedge q)^{\perp})\leqslant \tau(p^{\perp})+\tau(q^{\perp})<2\varepsilon$ and, for $n\geqslant k\geqslant N$, we have

$$\|(S_n-S_k)(p\wedge q)\| \leqslant \|(x_{k+1}-x_{k+1}^{(c_{k+1})}+\ldots+x_n-x_n^{(c_n)})(p\wedge q)\|+$$
$$+\|(\bar{S}_n-\bar{S}_k)(p\wedge q)\| = \|(\bar{S}_n-\bar{S}_k)(p\wedge q)\| \leqslant \|(\bar{S}_n-\bar{S}_k)q\| \leqslant \varepsilon.$$

Thus $\{S_n\}$ is Cauchy almost uniformly and, on account of Proposition 1.3, we infer that $\{S_n\}$ converges almost uniformly. Analogously, we prove that the convergence of $\{S_n\}$ implies that of $\{\bar{S}_n\}$.

LEMMA 3.4. *Let $\{x_n\}$ be a successively independent sequence in $\tilde{\mathscr{M}}$ and let*

$$\sum_{n=1}^{\infty}\int_0^1 \lambda\tau\left(e_{[\lambda,\infty)}(|x_n|)\right)d\lambda<\infty.$$

Then $\sum_{n=1}^{\infty}(x_n-\tau(x_n^{(1)})\mathbf{1})$ converges almost uniformly.

Proof. Integrating by parts, we obtain

$$\int_0^1 \lambda\tau\left(e_{[\lambda,\infty)}(|x_n|)\right)d\lambda = \tfrac{1}{2}\tau\left(e_{[1,\infty)}(|x_n|)\right)+\tfrac{1}{2}\int_0^1 \lambda^2\tau\left(e_{d\lambda}(|x_n|)\right)$$
$$=\tfrac{1}{2}\tau\left(e_{[1,\infty)}(|x_n|)\right)+\tfrac{1}{2}\tau(|x_n^{(1)}|^2).$$

By assumption, we have

$$\sum_{n=1}^{\infty}\tau(|x_n^{(1)}|^2)<\infty \quad\text{and}\quad \sum_{n=1}^{\infty}\tau\left(e_{[1,\infty)}(|x_n|)\right)<\infty.$$

The first inequality together with Proposition 5.2 from [1] gives the almost uniform convergence of the series $\sum_{n=1}^{\infty}(x_n^{(1)}-\tau(x_n^{(1)})\mathbf{1})$, which, along with the second inequality and Lemma 3.3, proves the almost uniform convergence of the series $\sum_{n=1}^{\infty}(x_n-\tau(x_n^{(1)})\mathbf{1})$.

LEMMA 3.5. *Let $\{x_n\}$ be as above and*

$$\sum_{n=1}^{\infty}\int_0^1 \lambda\tau\left(e_{[b_n\lambda,\infty)}(|x_n|)\right)d\lambda<\infty$$

for some positive constants b_n. Then

(i) $\sum_{n=1}^{\infty}\frac{1}{b_n}(x_n-\tau(x_n^{(b_n)})\mathbf{1})$ *converges almost uniformly,*

(ii) $\frac{1}{b_n}\sum_{k=1}^{\infty}(x_k-\tau(x_k^{(b_k)})\mathbf{1})\to 0$ *almost uniformly.*

Proof. We have

$$\int_0^1 \lambda\tau\left(e_{[b_n\lambda,\infty)}(|x_n|)\right)d\lambda=\int_0^1 \lambda\tau\left(e_{[\lambda,\infty)}(|x_n/b_n|)\right)d\lambda$$

and, by Lemma 3.4, the series $\sum_{n=1}^{\infty}(x_n/b_n - \tau([x_n/b_n]^{(1)})\mathbf{1})$ converges almost uniformly. But, for every positive c,

$$\left(\frac{x}{c}\right)^{(1)} = \frac{x}{c} e_{[0,1)}\left(\left|\frac{x}{c}\right|\right) = \frac{x}{c} e_{[0,c)}(|x|) = \frac{x^{(c)}}{c};$$

thus we get (i). (ii) is a consequence of (i) and Lemma 1.2.

Now, let η be a nonnegative random variable with distribution $\operatorname{Prob}(\eta < \lambda) = \tau(e_{[0,\lambda)}(|x|))$, $\lambda \in \boldsymbol{R}$, for an x from $\tilde{\mathscr{M}}$. We have $E\eta^r = \tau(|x|^r)$ for each $r > 0$ and, from the classical moments lemma (see e.g. [5]) we obtain the estimation

$$\lambda^r \sum_{n=1}^{\infty} \tau(e_{[\lambda n^{1/r},\infty)}(|x|)) \leqslant \tau(|x|^r) \leqslant 1 + \lambda^r \sum_{n=1}^{\infty} \tau(e_{[\lambda n^{1/r},\infty)}(|x|)). \tag{3.4}$$

THEOREM 3.6. *Let $\{x_n\}$ be a sequence of successively independent, identically distributed elements from $L^r(\mathscr{M}, \tau)$, $0 < r < 2$. Then*

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} (x_k - \alpha_k \mathbf{1}) \to 0 \quad \textit{almost uniformly}$$

where $\alpha_k = 0$ for $r < 1$ and $\alpha_k = \tau(x_1)$ for $1 \leqslant r < 2$.

Proof. Using (3.4), we get

$$\sum_{n=1}^{\infty} \int_0^1 \lambda \tau(e_{[\lambda n^{1/r},\infty)}(|x_1|))\, d\lambda \leqslant \tau(|x|^r) \int_0^1 \lambda^{1-r}\, d\lambda < \infty \quad \text{if } r < 2.$$

The above inequality together with Lemma 3.5 yields

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} (x_k - \tau(x_k^{(n^{1/r})})\mathbf{1}) \to 0 \quad \text{almost uniformly}.$$

The classical measure-theoretic arguments show that

$$\frac{1}{n^{1/r}} \sum_{k=1}^{n} (\tau(x_k^{(n^{1/r})}) - \alpha_k) \to 0 \quad \text{as } n \to \infty,$$

which completes the proof.

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