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Spline approximation in $H_p(T)$, $p \leqslant 1$

by

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Abstract. The paper deals with direct and inverse inequalities for H_p-approximation ($p \leqslant 1$) by polynomial splines defined with respect to the dyadic partitions of the torus and complements previous work of several authors concerning spline bases in real Hardy spaces.

0. Introduction. The aim of this paper is to develop a theory of direct and inverse inequalities for spline approximation in the real Hardy spaces $H_p(T)$, $p \leqslant 1$.(¹)

First the background of this topic will briefly be illuminated. Investigations concerning inequalities of Jackson type in the classical Hardy spaces

$$(0.1)\quad H_p(D) = \Big\{F(z) \text{ analytic in } D = \{z \in C\colon |z| < 1\} \text{ and } \|F\|_{H_p(D)} = \sup_{r<1} \Big\{\frac{1}{2\pi}\int_0^{2\pi} |F(re^{it})|^p\,dt\Big\}^{1/p} < \infty\Big\}$$

were started for $p \leqslant 1$ by Storozhenko [13], [14]. Her main results consist in direct inequalities for polynomial approximation in $H_p(D)$ such as

$$(0.2)\quad E_n(F)_{H_p(D)} = \inf_{P_n(z)=\sum_{m=0}^{n} c_m z^m} \|F-P_n\|_{H_p(D)} \leqslant C\cdot(n+1)^{-1}\cdot\omega_k\big(\pi/(n+1),\, F^{(1)}\big)_{H_p(D)}, \qquad F^{(1)}(z)\in H_p(D),$$

where $k = 1, 2, \ldots, l$, $n = 0, 1, \ldots$ (the positive constant is independent of n and $F(z)$), and in applications to various summation methods of the power series of $F(z) \in H_p(D)$. However, the methods used by Storozhenko explicitly explored the analyticity of the functions and approximating polynomials in D and thus do not directly apply to such approximations as splines, which are based on non-analytic, purely real constructions.

On the other hand, real techniques do play an important role in the

(¹) Detailed definitions will be given in the next section.

proofs that spline systems are (unconditional) Schauder bases in Hardy spaces for $p<1$, which were independently obtained by P. Sjölin, J.-O. Strömberg [12], [11], P. Wojtaszczyk [16], and the present author [7]. In order to estimate the degree of approximation of the basis expansions with respect to the spline systems under consideration the author [8] has recently given a characterization of the H_p-moduli of continuity in terms of a corresponding K'-functional of Peetre. Thus, Jackson type estimates in H_p, $p \leqslant 1$, can be obtained in a straightforward manner from the usual quasi-norm estimates for the related operators which can be checked by well-developed real methods, e.g., by using atomic and molecular decompositions. On these lines it was established in [8] that for the partial sums $P_n^{(m)} f(t)$ of the basis expansion with respect to the periodic orthonormal spline system $F^{(m)}$ of degree $m = 0, 1, \ldots$ of a function $f(t)$ belonging to the real space $H_p(T)$ on the periodic interval $T = [0, 1)$ we have the estimate

$$\|f - P_n^{(m)} f\|_{H_p} \leqslant C \cdot \omega_{m+1}(1/n, f)_{H_p}, \quad n = 1, 2, \ldots, \tag{0.3}$$

if $(m+1)^{-1} \leqslant p \leqslant 1$. This inequality is equivalent to

$$E_n^{(m)}(f)_{H_p} = \inf_{g \in S_n^{(m)}} \|f-g\|_{H_p} \leqslant C \cdot \omega_{m+1}(1/n, f)_{H_p}, \quad n = 1, 2, \ldots, \tag{0.4}$$

where $S_n^{(m)}$ denotes the corresponding spline spaces and again $(m+1)^{-1} \leqslant p \leqslant 1$ has to be assumed (here the constants depend only on m). In the particular case $m = 0$ some extension of (0.4) to other values of p was provided in [8]; the general case, however, remained unsolved.

In the present paper we cover the remaining gaps for the direct as well as for the inverse inequalities. The main result is the following

THEOREM. *Let $0 < p < 1$, $m = 0, 1, \ldots$, and $f \in H_p(T)$. Then the following inequalities hold for $n = 1, 2, \ldots$:*

$$E_n^{(m)}(f)_{H_p} \leqslant C \cdot \omega_{m+1}(1/n, f)_{H_p}, \tag{0.5}$$

$$\omega_{m+1}(1/n, f)_{H_p} \leqslant C \cdot n^{-m-1} \Big\{ \sum_{k=1}^{n} k^{(m+1)p-1} \cdot E_k^{(m)}(f)_{H_p}^p \Big\}^{1/p} \tag{0.6}$$

with positive constants C depending on m and p.

After the preliminary Section 1 the proof of the direct estimate will be given in Section 2. Section 3 deals with the inverse inequalities and the corresponding Bernstein inequalities. The in some sense exceptional case $p = 1$ will also be considered there. As in [7], [8], it turns out that the atomic decomposition methods are well-situated for handling H_p-estimates ($p \leqslant 1$) for splines.

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1. Preliminaries.

Atomic decompositions in $H_p(T)$. Let $0 < p \leqslant 1$ and $T = [0, 1)$ be the periodic unit interval. Then, as usual, the real Hardy space $H_p(T)$ is the quasi-Banach space of all real-valued distributions $f = f(t) \in \mathscr{D}'(T)$ arising as limits

$$f(t) = \lim_{r \to 1-0} \operatorname{Re} F(r \cdot e^{i \cdot 2\pi t}) \quad (\text{in the sense of } \mathscr{D}'(T))$$

for some $F(z) \in H_p(D)$ satisfying $\operatorname{Im} F(0) = 0$. The quasi-norm is given by $\|f\|_{H_p} = \|F\|_{H_p(D)}$ and thus $H_p(D)$ and $H_p(T)$ are the same spaces in a certain sense.

However, for our purpose it is more favourable to work with the atomic real description of $H_p(T)$ due to R. Coifman [2]. A function $a(t) \in L_q(T)$, $1 \leqslant q \leqslant \infty$, is called a (p, q, s)-*atom* with respect to $t_0 \in T$ if $p < q$, the integer s satisfies $s \geqslant [1/p-1]$, and

$$\begin{aligned} &\operatorname{supp} a(t) \subset J = [t_0, t_0+|J|), \quad |J| \leqslant 1, \\ &\|a\|_{L_q} \leqslant |J|^{1/q-1/p}, \\ &\int_0^{|J|} a(t+t_0) \cdot t^r \, dt = 0, \quad r = 0, \ldots, s, \end{aligned} \tag{1.1}$$

where intervals and translations have to be understood in the periodic sense.

PROPOSITION 1. (cf. [2], [3], [15]). *Let $0 < p \leqslant 1$, $1 \leqslant q \leqslant \infty$, $p < q$, and $s \geqslant [1/p-1]$. Then $f(t) \in H_p(T)$ if and only if there exists a decomposition*

$$f(t) = \sum_{j=0}^{\infty} \lambda_j \cdot a_j(t) \quad (\text{convergence in } \mathscr{D}'(T)) \tag{1.2}$$

with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$ where the functions $a_j(t)$, $j \geqslant 1$, are (p, q, s)-atoms and $a_0(t) \in L_q(T)$ satisfies $\|a_0\|_{L_q} \leqslant 1$. Moreover, the quantity $\inf \{\sum_{j=0}^{\infty} |\lambda_j|^p\}^{1/p}$ where the infimum is taken over all decompositions (1.2) *is an equivalent quasi-norm in $H_p(T)$.*

Splines and spline systems. Let $n = 1, 2, \ldots$ The dyadic partitions of T

$$\pi_n = \{0 = s_{n,0} < s_{n,1} < \ldots < s_{n,n-1} < 1\}$$

are defined by setting

$$s_{n,i} = \begin{cases} 2^{-k-1} \cdot i, & i = 0, \ldots, 2l, \\ 1 - 2^{-k}(n-i), & i = 2l, \ldots, n-1 \end{cases}$$

for $n = 2^k + l > 1$ where $l = 1, \ldots, 2^k$ and $k = 0, 1, \ldots$ are uniquely determined by n. Furthermore, we put

$$s_{n,jn+i} = s_{n,i}, \quad s'_{n,jn+i} = s_{n,i} + j, \quad j = 0, \pm 1, \ldots,$$

for the corresponding expanded partitions on T and $\boldsymbol{R} = (-\infty, \infty)$.

For $m = 0, 1, \ldots$ we denote by $S_n^{(m)}(T)$ the n-dimensional subspace of $C^{(m-1)}(T)$ (of $L_\infty(T)$ if $m = 0$) consisting of all periodic spline functions of degree m with respect to π_n. The corresponding B-splines

$$N_{n,i}^{(m)}(t) = \sum_{j=-\infty}^{\infty} N'^{(m)}_{n,jn+i}(t), \quad t \in [0, 1),$$

where

$$N'^{(m)}_{n,j}(t) = (s'_{n,j+m+1} - s'_{n,j}) \cdot [s'_{n,j}, \ldots, s'_{n,j+m+1}; (s-t)_+^m]$$

($t \in \boldsymbol{R}$) have the properties (cf. [10], [4], [16])

(1.3) $$\operatorname{supp} N_{n,i}^{(m)}(t) = [s_{n,i}, s_{n,i+m+1}), \quad n > m+1,$$

(1.4) $$\sum_{i=0}^{n-1} N_{n,i}^{(m)}(t) = 1, \quad N_{n,i}^{(m)}(t) \geqslant 0, \quad t \in T,$$

(1.5) $\{N_{n,i}^{(m)}(t)\}_{i=0}^{n-1}$ forms an algebraic basis in $S_n^{(m)}(T)$,

(1.6) $$\frac{d}{dt} N_{n,i}^{(m)}(t) = m \cdot \left\{ \frac{N_{n,i}^{(m-1)}(t)}{(s'_{n,i+m} - s'_{n,i})} - \frac{N_{n,i+1}^{(m-1)}(t)}{(s'_{n,i+m+1} - s'_{n,i+1})} \right\},$$

(1.7) if $g(t) = \sum_{i=0}^{n-1} \beta_i \cdot N_{n,i}^{(m)}(t) \in S_n^{(m)}(T)$ then for $0 < p < \infty$

$$C_1 \cdot \{\sum_{i=0}^{n-1} n^{-1} |\beta_i|^p\}^{1/p} \leqslant \|g\|_{L_p} \leqslant C_2 \cdot \{\sum_{i=0}^{n-1} n^{-1} |\beta_i|^p\}^{1/p}$$

with constants independent of $g(t)$ and n,

(1.8) if $n = 2^l$ then $N_{n,i}^{(m)}(t) = N_{n,0}^{(m)}(t - i \cdot 2^{-l})$.

The periodic orthonormal spline systems $F^{(m)} = \{f_n^{(m)}(t)\}$ are uniquely determined by the conditions

(1.9) $$f_n^{(m)}(t) \in S_n^{(m)}(T), \quad n = 1, 2, \ldots, \qquad F^{(m)} \text{ is orthonormal}, \qquad f_n^{(m)}(s_{n,2l-1}) > 0, \quad f_1^{(m)}(t) = 1.$$

These systems were introduced by Z. Ciesielski ([1], [4]). We used them in [7] to prove the following

PROPOSITION 2. *The systems $F^{(m)}$, $m = 0, 1, \ldots$, form Schauder bases in $H_p(T)$ if $(m+1)^{-1} \leqslant p \leqslant 1$.*

An analogous statement has been given by P. Wojtaszczyk [16] for a slightly modified spline system on T; moreover, his results also include unconditional convergence of the corresponding expansions if $p > (m+1)^{-1}$ (cf. also [12], [11]).

Moduli of continuity. Let $k = 1, 2, \ldots$ By

(1.10) $$\Delta_h^k f(t) = \sum_{l=0}^{k} \binom{k}{l} \cdot (-1)^{k-l} \cdot f(t + lh), \quad h \in \boldsymbol{R},$$

we denote the usual differences of order k of a function (or distribution) $f(t)$ on T. We define by

(1.11) $$\omega_k(\delta, f)_{L_p} = \sup_{0 \leqslant h \leqslant \delta} \|\Delta_h^k f\|_{L_p}, \quad 0 < \delta \leqslant 1,$$

and

(1.12) $$\omega_k(\delta, f)_{H_p} = \sup_{0 \leqslant h \leqslant \delta} \|\Delta_h^k f\|_{H_p}, \quad 0 < \delta \leqslant 1,$$

the moduli of continuity of order k of $f \in L_p(T)$ and $f \in H_p(T)$, resp. In the case $0 < p \leqslant 1$ the following properties hold (cf. e.g. [5], [9]) for $f, g \in L_p(T)$:

(1.13) $$\omega_k(\delta, f+g)_{L_p}^p \leqslant \omega_k(\delta, f)_{L_p}^p + \omega_k(\delta, g)_{L_p}^p,$$

(1.14) $$\omega_k(\lambda, f)_{L_p} \leqslant C \cdot (\lambda/\delta + 1)^{k+1/p-1} \omega_k(\delta, f)_{L_p}, \quad \delta < \lambda,$$

(1.15) $$\omega_k(\delta, f)_{L_p} \leqslant C \cdot \omega_l(\delta, f)_{L_p} \leqslant C \cdot \|f\|_{L_p}, \quad l < k,$$

with constants independent of $f(t)$, λ, and δ. The same relations remain generally valid in the H_p case, too. But in this case there is another useful approach to the moduli of continuity via a certain K'-functional. In [8] we have obtained

PROPOSITION 3. *Let $0 < p \leqslant 1$, $k = 1, 2, \ldots$, and $f \in H_p(T)$ be given. Then for $0 < \delta \leqslant 1$ we have*

(1.16) $$C_1 \cdot \omega_k(\delta, f)_{H_p} \leqslant \inf_{g \in H_p^k(T)} \{\|f - g\|_{H_p} + \delta^k \cdot \|g^{(k)}\|_{H_p}\} \leqslant C_2 \cdot \omega_k(\delta, f)_{H_p}$$

where $H_p^k(T)$ is the space of all distributions $g(t) \in \mathscr{D}'(T)$ with $g(t), g^{(k)}(t) \in H_p(T)$, and the constants C_1, C_2 are independent of δ and $f(t)$.

In [8], (1.16) has been used in the proof of the Jackson type estimates (0.3), (0.4) for spline approximation in $H_p(T)$. However, in the present paper we only need the elementary properties (1.13)–(1.15) of the moduli of continuity.

Finally, by $\Delta^k \beta_i = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \cdot \beta_{i+l}$, $\Delta \beta_i = \Delta^1 \beta_i$, we denote the differences of order k of a sequence $\{\beta_i\}$.

2. Direct inequality. In this section we prove inequality (0.5) of Jackson type stated in the main theorem in the Introduction. The idea is to reduce (0.5) to (0.4) by purely "atomic" considerations. In order to do this we begin with

LEMMA 1. *Let $m = 1, 2, \ldots$, $s = 0, 1, \ldots$, and $n = 2^l > s+m+1$ be given. Then for any spline*

$$g(t) = \sum_{i=0}^{n-1} \beta_i \cdot N_{n,i}^{(m)}(t) \in S_n^{(m)}(T)$$

there exists a step function $\bar{g}(t) \in S_n^{(0)}(T)$ such that

$$g(t) - \bar{g}(t) = \sum_{j=0}^{n-1} b_j(t) \tag{2.1}$$

where the functions $b_j(t)$, $j = 0, \ldots, n-1$, satisfy

$$\operatorname{supp} b_j(t) \subset [s_{n,j}, s_{n,j+s+1}) = I_{n,j}, \tag{2.2}$$

$$\|b_j\|_{L_\infty} \leqslant C \cdot \sum_{i=j-m}^{j-1} |\Delta\beta_i| \tag{2.3}$$

where $\beta_{nj+i} = \beta_i$ for $i = 0, \ldots, n-1$, $j = 0, \pm 1, \ldots$, and

$$\int_{I_{0,j}} b_j(t+s_{n,j}) \cdot t^r \, dt = 0, \quad r = 0, \ldots, s \tag{2.4}$$

(the positive constant C in (2.3) is independent of $n = 2^l$ and $g(t)$).

Proof. We fix an arbitrary $m = 1, 2, \ldots$ and $n = 2^l > m+s+1$ (for the sake of briefness we omit the indices m and n in the notations, e.g., $s_j = s_{n,j}$, $N_{n,i}^{(m)}(t) = N_i(t)$, and so on). As usual, we denote by $f(t)|_\Delta$ the restriction of $f(t)$ to an interval $\Delta \subset T$. Setting $\Delta_j = [s_j, s_{j+1})$ we obtain (cf. (1.3), (1.4))

$$\begin{aligned} g(t) &= \sum_{j=0}^{n-1} g(t)|_{\Delta_j} = \sum_{j=0}^{n-1} \Big(\sum_{i=j-m}^{j} \beta_i \cdot N_i(t)|_{\Delta_j} \Big) \\ &= \sum_{j=0}^{n-1} \Big\{ \sum_{i=j-m}^{j-1} (-\Delta\beta_i) \cdot \Big(\sum_{k=j-m}^{i} N_k(t)|_{\Delta_j} \Big) + \beta_j \cdot 1|_{\Delta_j} \Big\}. \end{aligned} \tag{2.5}$$

Obviously, the functions

$$A_{i,j}(t) = \sum_{k=j-m}^{i} N_k(t)|_{\Delta_j}, \quad i = j-m, \ldots, j-1, \quad j = 0, \ldots, n-1,$$

satisfy (cf. (1.4), (1.8))

$$\operatorname{supp} A_{i,j}(t) \subset \Delta_j, \quad 0 \leqslant A_{i,j}(t) \leqslant 1, \quad t \in \Delta_j, \tag{2.6}$$

$$A_{i,j}(t) = A_{i-j,0}(t - j \cdot 2^{-l}). \tag{2.7}$$

Now we define the step functions

$$h_k(t) = \sum_{r=0}^{s} c_{k,r} \cdot 1|_{\Delta_r}, \quad k = -m, \ldots, -1,$$

by the conditions

$$\int_{I_0} (A_{k,0}(t) - h_k(t)) \cdot t^r \, dt = 0, \quad r = 0, \ldots, s, \tag{2.8}$$

and observe that the reals $c_{k,r}$ are uniquely determined by (2.8) and independent of $n = 2^l$ (make a change of variable $t' = 2^l \cdot t$ in (2.8) and consider the corresponding system of linear equations).

If we put $h_{i,j}(t) = h_{i-j}(t - j \cdot 2^{-l})$, $i = j-m, \ldots, j-1$, and

$$\bar{g}(t) = \sum_{j=0}^{n-1} \Big\{ \sum_{i=j-m}^{j-1} (-\Delta\beta_i) \cdot h_{i,j}(t) + \beta_j \cdot 1|_{\Delta_j} \Big\} \in S_n^{(0)}(T),$$

then by (2.5)–(2.8) and the obvious properties of $h_k(t)$ we obtain the assertions of the lemma (with

$$b_j(t) = \sum_{i=j-m}^{j-1} \Delta\beta_i \cdot (h_{i,j}(t) - A_{i,j}(t)), \quad j = 0, \ldots, n-1). \tag{2.9}$$

Remark. In the case $s = 0$ this lemma was essentially proved in [8], Remark 3.

Now, by Lemma 1 we can obtain

LEMMA 2. *Let $m, m^* = 0, 1, \ldots$, $m > m^*$, $0 < p \leqslant 1$, and $n = 2^l > m + 1 + [1/p]$ be given. Then for any $g(t) \in S_n^{(m)}(T)$ there exists a spline $g^*(t) \in S_n^{(m^*)}(T)$ satisfying*

$$\|g - g^*\|_{H_p} \leqslant C \cdot \omega_{m^*+1}(2^{-l}, g)_{L_p} \tag{2.10}$$

where C does not depend on $g(t)$ and $n = 2^l$.

Proof. As in the proof of Lemma 1 we fix the parameters and omit m and $n = 2^l$ in the notations. Let $g(t) = \sum_{i=0}^{n-1} \beta_i \cdot N_i(t)$ be the B-spline representation of $g(t)$. From (1.6) we obtain

$$g^{(m^*)}(t) = \frac{d^{m^*}}{dt^{m^*}} g(t) = 2^{lm^*} \cdot \sum_{i=0}^{n-1} \Delta^{m^*} \beta_{i-m} \cdot N_i^{(m-m^*)}(t) \in S_n^{(m-m^*)}(T).$$

Let $s = [1/p - 1] + m^*$. By Lemma 1 there exists a step function $\overline{g^{(m^*)}}(t)$ such that $g^{(m^*)}(t) - \overline{g^{(m^*)}}(t) = \sum_{j=0}^{n-1} b_j(t)$ where $b_j(t)$ satisfies (2.2), (2.4), and (instead of (2.3))

$$\|b_j\|_{L_\infty} \leqslant C \cdot 2^{lm^*} \cdot \sum_{i=j-m}^{j-m^*-1} |\Delta^{m^*+1} \beta_i|. \tag{2.11}$$

Now we consider the functions

(2.12) $$B_j(t) = \int_{s_j}^{t} \dots \int_{s_j}^{\xi_{m^*-1}} b_j(\xi_{m^*})\, d\xi_{m^*} \dots d\xi_1 .$$

According to (2.2), (2.4), (2.11), and (2.12) we have

(2.13) $$\operatorname{supp} B_j(t) \subset I_j,$$

(2.14) $$\|B_j\|_{L_\infty} \leqslant C\cdot \Big\{\sum_{i=j-m}^{j-m^*-1} |\Delta^{m^*+1}\beta_i|\cdot 2^{-l/p}\Big\}\cdot |I_j|^{-1/p},$$

and

(2.15) $$\int_{I_0} B_j(t+s_j)\cdot t^r\, dt = 0, \qquad r = 0, \dots, [1/p-1].$$

Thus, the functions $B_j(t)$ are multiples of $(p, \infty, [1/p-1])$-atoms and by Proposition 1, (1.7) and (1.8), we obtain

(2.16) $$\Big\|\sum_{j=0}^{n-1} B_j\Big\|_{H_p} \leqslant C\cdot\Big\{\sum_{j=0}^{n-1}\Big(\sum_{i=j-m}^{j-m^*-1} |\Delta^{m^*+1}\beta_i|\cdot 2^{-l/p}\Big)^p\Big\}^{1/p}$$
$$\leqslant C\cdot\Big\{\sum_{i=0}^{n-1} |\Delta^{m^*+1}\beta_i|^p\cdot 2^{-l}\Big\}^{1/p} \leqslant C\cdot\Big\|\sum_{i=0}^{n-1} \Delta^{m^*+1}\beta_i\cdot N_i\Big\|_{L_p}$$
$$= C\cdot\|\Delta_{2^{-l}}^{m^*+1} g(t)\|_{L_p} \leqslant C\cdot\omega_{m^*+1}(2^{-l}, g)_{L_p}.$$

To complete the proof of Lemma 2 it remains to observe that

$$g^*(t) = g(t) - \sum_{j=0}^{n-1} B_j(t)$$

belongs to $S_n^{(m^*)}(T)$. This, however, becomes clear if we consider

$$g^{*(m^*)}(t) = g^{(m^*)}(t) - \sum_{j=0}^{n-1} b_j(t) = \overline{g^{(m^*)}}(t) \in S_n^{(0)}(T)$$

and use the simple fact that a function $f(t)$ with $f'(t) \in S_n^{(k)}(T)$ obviously belongs to $S_n^{(k+1)}(T)$.

Proof of the direct inequality (0.5). For arbitrarily given $m^* = 0, 1, \dots$ and $0 < p \leqslant 1$ we pick any integer $m > m^*$ such that (0.4) holds, i.e., if $f \in H_p(T)$ then there exists a spline $g_n(t) \in S_n^{(m)}(T)$ satisfying the inequality

(2.17) $$\|f - g_n\|_{H_p} \leqslant C\cdot\omega_{m+1}(n^{-1}, f)_{H_p} \leqslant C\cdot\omega_{m^*+1}(n^{-1}, f)_{H_p}$$

where $n = 1, 2, \dots$ (cf. (0.4), (1.15)).

On the other hand, for $n = 2^l > m + [1/p]$ by Lemma 2 there exists a corresponding spline $g_n^*(t) \in S_n^{(m^*)}(T)$ with

(2.18) $$\|g_n - g_n^*\|_{H_p} \leqslant C\cdot\omega_{m^*+1}(n^{-1}, g_n)_{L_p} \leqslant C\cdot\omega_{m^*+1}(n^{-1}, g_n)_{H_p}.$$

Now the desired result can be obtained by standard considerations from (2.17), (2.18), and the properties of the moduli of continuity (1.13)–(1.15): If $m + [1/p] < 2^l \leqslant n < 2^{l+1}$ then

$$E_n^{(m^*)}(f)_{H_p} \leqslant E_{2^l}^{(m^*)}(f)_{H_p} \leqslant \|f - g_{2^l}^*\|_{H_p}$$
$$\leqslant C\cdot(\|f - g_{2^l}\|_{H_p} + \|g_{2^l} - g_{2^l}^*\|_{H_p})$$
$$\leqslant C\cdot(\|f - g_{2^l}\|_{H_p} + \omega_{m^*+1}(2^{-l}, f)_{H_p})$$
$$\leqslant C\cdot\omega_{m^*+1}(2^{-l}, f)_{H_p} \leqslant C\cdot\omega_{m^*+1}(n^{-1}, f)_{H_p}.$$

But for small $n = 1, \dots, 2\cdot(m + [1/p])$ we have

$$E_n^{(m^*)}(f)_{H_p} \leqslant E_1^{(m^*)}(f)_{H_p} = E_1^{(m)}(f)_{H_p} \leqslant C\cdot\omega_{m^*+1}(1, f)_{H_p}$$
$$\leqslant C\cdot\omega_{m^*+1}(n^{-1}, f)_{H_p}.$$

It can easily be seen that the constants are independent of n and $f(t)$ (they might depend on p and m), and thus (0.5) (with m^* instead of m) is completely proved.

Remark. In [16] P. Wojtaszczyk has proved that several spline systems form (unconditional) basic sequences in $H_p(T)$ (cf. Theorems 2 and 2′ in [16]). Now the inequality (0.5) of Jackson type allows us to establish in all cases considered by P. Wojtaszczyk the corresponding estimates for the rate of convergence of the basis expansions with respect to these systems.

3. Inverse inequalities. As usual, inverse inequalities easily follow from appropriate inequalities of Bernstein type. For spline approximation in $H_p(T)$, $p \leqslant 1$, only a very special case of such an inequality was considered in [8] Remark 3. Now the general case will be considered.

PROPOSITION 4. *Let* $m = 0, 1, \dots$, $n = 2, 3, \dots$, $0 < p < 1$, *and* $g(t) \in S_n^{(m)}(T)$ *be given. Then we have*

(3.1) $$\|\Delta_h^{m+1} g\|_{H_p} \leqslant C\cdot(hn)^{m+1}\cdot\|g\|_{H_p}, \qquad 0 \leqslant h \leqslant C/n$$

where the positive constants C *are independent of* n, h, *and* $g(t)$.

Proof. First we show a somewhat sharper inequality but in a special case (the idea of this part of the proof was essentially contained in [8]): Let m, n, and $g(t)$ be as above, $k = 1, 2, \dots$, and $(k+1)^{-1} < p < 1$. Then

(3.2) $$\|\Delta_h^k g\|_{H_p} \leqslant C\cdot(hn)^{\min(k, m+1)}\cdot\|g\|_{L_p}, \qquad 0 \leqslant h \leqslant C/n.$$

To prove (3.2), we consider the B-spline representation of $g(t)$ as in the preceding section and the corresponding formula for the difference of order k

$$\Delta_h^k g(t) = \sum_{i=0}^{n-1} \beta_i \cdot \Delta_h^k N_i(t).$$

We observe that for sufficiently large $n \geqslant n(m, k)$ and small $0 \leqslant h \leqslant C(m, k) \cdot n^{-1}$ the functions $\Delta_h^k N_i(t)$ have the properties of multiples of $(p, 1, k-1)$-atoms:

$$\operatorname{supp} \Delta_h^k N_i(t) \subset J_i = [s_{i-1}, s_{i+m+1}), \tag{3.3}$$

$$\int_0^{|J_i|} \Delta_h^k N_i(t+s_{i-1}) \cdot t^r \, dt = (-1)^k \int_0^{|J_i|} N_i(t+s_{i-1}) \cdot \Delta_{-h}^k t^r \, dt = 0, \tag{3.4}$$

$r = 0, \ldots, k-1$ (cf (1.3)), and careful computations give

$$\begin{aligned} \|\Delta_h^k N_i\|_{L_1} &\leqslant C \cdot (hn)^{\min(k,m+1)} \cdot n^{-1} \\ &\leqslant C \cdot \{(hn)^{\min(k,m+1)} \cdot n^{-1/p}\} \cdot |J_i|^{1-1/p}. \end{aligned} \tag{3.5}$$

Therefore, by Proposition 1 and (1.7) we obtain for the indicated values of h, m, k, p, and n

$$\begin{aligned} \|\Delta_h^k g\|_{H_p} &\leqslant C \{\sum_{i=0}^{n-1} |\beta_i|^p (hn)^{p \cdot \min(k,m+1)} \cdot n^{-1}\}^{1/p} \\ &\leqslant C \cdot (hn)^{\min(k,m+1)} \cdot \|g\|_{L_p}. \end{aligned}$$

This yields (3.2), and thus (3.1) in the case $(m+2)^{-1} < p < 1$ (for small $n = 1, \ldots, n(m, k)$ we still have the estimate (3.5) and the obvious relation $\|\Delta_h^k N_i\|_{H_p} \leqslant C \cdot \|\Delta_h^k N_i\|_{L_1}$, so that (3.2) remains valid for these n, too).

It should be mentioned that (3.2) does not hold if $k = m+1$ and $0 < p \leqslant (m+2)^{-1}$. A slight modification of the above considerations, however, gives (3.1) for all $0 < p < 1$. Taking for given p some integer $s > \max([1/p-1], m)$ and denoting $h_j = j \cdot (n \cdot (s-m))^{-1}$, we consider the appropriately normed divided difference

$$C(h) \cdot [h, h_1, \ldots, h_{s-m}; f(\xi)] = f(h) + \sum_{j=1}^{s-m} c_j \cdot f(h_j)$$

where $0 < h < 1/2 \cdot h_1$ and the constant $C(h)$ is chosen in such a way that the coefficient of $f(h)$, as indicated in the representation of the divided difference, is 1 for all h. It can easily be seen that the other coefficients c_j are uniformly bounded with respect to h because the knot points $h, h_1, \ldots, h_{s-m}$ are "almost" equally spaced if $0 < h < 1/2 \cdot h_1$. Using these coefficients we define

the function

$$\begin{aligned} g_0(t) &= \Delta_h^{m+1} g(t) + \sum_{j=1}^{s-m} c_j \cdot \Delta_{h_j}^{m+1} g(t) \cdot (h/h_j)^{m+1} \\ &= \sum_{i=0}^{n-1} \beta_i \cdot (\Delta_h^{m+1} N_i(t) + \sum_{j=1}^{s-m} c_j \cdot \Delta_{h_j}^{m+1} N_i(t) \cdot (h/h_j)^{m+1}) \\ &= \sum_{i=0}^{n-1} \beta_i \cdot d_i(t) \end{aligned} \tag{3.6}$$

and check the functions $d_i(t)$ to be multiples of $(p, 1, s)$-atoms if $n > 4(m+1)$. Again it is clear that

$$\operatorname{supp} d_i(t) \subset J_i^* = [s_i^*, s_{i+m+1}), \quad s_i^* = s_i - (m+1)/n$$

and

$$\|d_i\|_{L_1} \leqslant C \cdot (nh)^{m+1} \cdot n^{-1} \leqslant C \cdot (nh)^{m+1} \cdot n^{-1/p} \cdot |J_i^*|^{1-1/p}.$$

But by the definition of the coefficients c_j we also have zero moments up to the order s:

$$\begin{aligned} \int_0^{|J_i^*|} d_i(t+s_i^*) \cdot t^r \, dt &= (-1)^{m+1} \int_0^{|J_i^*|} N_i(t+s_i^*) \times \\ &\qquad \times \{\Delta_{-h}^{m+1} t^r + \sum_{j=1}^{s-m} c_j \cdot (h/h_j)^{m+1} \cdot \Delta_{-h_j}^{m+1} t^r\} \, dt \\ &= (-1)^{m+1} \cdot C(h) \cdot h^{m+1} \int_0^{|J_i^*|} N_i(t+s_i^*) \cdot [h, h_1, \ldots, h_{s-m}, \xi^{-m-1} \Delta_{-\xi}^{m+1} t^r] \, dt = 0 \end{aligned}$$

because the function $f(\xi) = \xi^{-m-1} \cdot \Delta_{-\xi}^{m+1} t^r$ is a polynomial in ξ of degree less than $s-m$ for all $r = 0, \ldots, s$.

Thus, by Proposition 1 we obtain (cf. (1.7), the modifications for small n are obvious)

$$\|g_0\|_{H_p} \leqslant C \cdot (hn)^{m+1} \{\sum_{i=0}^{n-1} |\beta_i|^p \cdot n^{-1}\}^{1/p} \leqslant C \cdot (hn)^{m+1} \cdot \|g\|_{L_p}, \tag{3.7}$$

and this yields, together with the construction of $g_0(t)$ in (3.6), the desired result:

$$\begin{aligned} \|\Delta_h^{m+1} g\|_{H_p}^p &\leqslant \|g_0\|_{H_p}^p + \max_{j=1,\ldots,s-m} (|c_j| \cdot (h/h_j)^{m+1})^p \cdot \sum_{j=1}^{s-m} \|\Delta_{h_j}^{m+1} g\|_{H_p}^p \\ &\leqslant C \cdot (hn)^{p(m+1)} \{\|g\|_{L_p}^p + \|g\|_{H_p}^p\} \leqslant C \cdot (hn)^{p(m+1)} \cdot \|g\|_{H_p}^p. \end{aligned}$$

This finally proves Proposition 4.

Proof of the inverse inequality (0.6). Let $f(t)\in H_p(T)$, $0<p<1$, and $m=0,1,\ldots$ be given. By $g_l(t)$ we denote the best approximating splines in $S_{2^l}^{(m)}(T)$ for $f(t)$, i.e.,

$$\|f-g_l\|_{H_p}=E_{2^l}^{(m)}(f)_{H_p},\qquad l=0,1,\ldots$$

Then (3.1) yields the inequality (cf. (1.12), (1.14))

$$\text{(3.8)}\qquad \omega_{m+1}(n^{-1},g_l-g_{l-1})_{H_p}\leqslant C\cdot(n^{-1}\cdot 2^l)^{m+1}\cdot\|g_l-g_{l-1}\|_{H_p}$$
$$\leqslant C\cdot(n^{-1}\cdot 2^l)^{m+1}\cdot E_{2^{l-1}}^{(m)}(f)_{H_p},\qquad n\geqslant 2^l.$$

Thus, according to (1.13), (1.15), and (3.8), we obtain for $2^k\leqslant n<2^{k+1}$, $k=0,1,\ldots,$

$$\omega_{m+1}(1/n,f)_{H_p}^p\leqslant\omega_{m+1}(1/n,f-g_k)_{H_p}^p+\sum_{l=1}^{k}\omega_{m+1}(1/n,g_l-g_{l-1})_{H_p}^p$$
$$\leqslant C\cdot\{\|f-g_k\|_{H_p}^p+\sum_{l=1}^{k}(2^l\cdot n^{-1})^{p(m+1)}\cdot E_{2^{l-1}}^{(m)}(f)_{H_p}^p\}$$
$$\leqslant C\cdot n^{-p(m+1)}\cdot\{\sum_{l=0}^{k}2^{l(m+1)p}\cdot E_{2^l}^{(m)}(f)_{H_p}^p\}.$$

Since the sequence of best approximations is decreasing with respect to n, (0.6) is a direct consequence of the latter inequality.

Remark. The more general statement

$$\text{(3.9)}\qquad \omega_k(1/n,f)_{H_p}\leqslant C\cdot n^{-l}\{\sum_{r=1}^{k}r^{pl-1}\cdot E_r^{(m)}(f)_{H_p}^p\}^{1/p}$$

where $l=\min(k,m+1)$, $k=1,2,\ldots$, and the constant is independent of $n=1,2,\ldots$ and $f\in H_p(T)$, $0<p<1$, immediately follows from (0.6) and the inequality of Marchaud type

$$\text{(3.10)}\qquad \omega_k(2^{-l},f)_{H_p}^p\leqslant C\begin{cases}\omega_{m+1}(2^{-l},f)_{H_p}^p, & k\geqslant m+1\\ 2^{-lkp}\{\sum_{r=0}^{l}2^{rkp}\omega_{m+1}(2^{-r},f)_{H_p}^p\}, & k=1,\ldots,m,\end{cases}$$

where $l=0,1,\ldots$ The nontrivial part of (3.10) (i.e., the case $k=1,\ldots,m$) results from the Jackson type inequality (0.2) for $H_p(D)$ and the corresponding inverse inequalities for trigonometric approximation in $L_p(T)$ (cf. [5], p. 89).

In the case $p=1$ the above proof of the inverse inequality (0.6) does not formally work because $(1,1,s)$-atoms are not allowed in Proposition 1. Moreover, (0.6) cannot be true in this form for $p=1$. Suppose, on the contrary, that (0.6) also holds with $p=1$. Considering any function $g(t)\neq\text{const}$ belonging to some $S_n^{(m)}(T)$ $(n>1)$, it follows that

$$\omega_{m+1}(\delta,g)_{H_1}=O(\delta^{m+1}),\qquad \delta\to 0.$$

It can be proved that this property is equivalent to $g(t)\in H_1^{m+1}(T)$ or $g^{(m)}(t)\in H_1^1(T)$. But by a well-known theorem of F. and M. Riesz this means that $g^{(m)}(t)$ should be absolutely continuous, which gives the desired contradiction.

A correct version of the inverse theorem for spline approximation in $H_1(T)$ (the direct estimate is contained in (0.3)–(0.4) for arbitrary $m=0,1,\ldots$) can be stated as

PROPOSITION 5. *Let $m=0,1,\ldots$, $n=2,3,\ldots$, and $g(t)\in S_n^{(m)}(T)$ be given. Then for $0<h\leqslant C\cdot n^{-1}$ we have*

$$\text{(3.11)}\qquad \|\Delta_h^k g\|_{H_1}\leqslant C\cdot(hn)^k\cdot\|g\|_{L_1}\cdot\begin{cases}1, & k=1,\ldots,m\ (m>0),\\ |\ln(hn)|, & k=m+1,\end{cases}$$

with constants C independent of n, $g(t)$, and h.

Thus, if $f\in H_1(T)$ then

$$\text{(3.12)}\qquad \omega_k(1/n,f)_{H_1}\leqslant C\cdot n^{-k}\begin{cases}\sum_{r=1}^{n}r^{k-1}\cdot E_r^{(m)}(f)_{H_1}, & k=1,\ldots,m,\\ \sum_{r=1}^{n}r^{m}\cdot\ln(n/r)\cdot E_r^{(m)}(f)_{H_1}, & k=m+1\end{cases}$$

with a constant independent of n and $f(t)$.

Proof. It suffices to consider (3.11) (the other assertion is again a standard corollary to the Bernstein type estimate). The case $k=1,\ldots,m$ can be handled as above (cf. the proof of (3.2) but use $(1,\infty,0)$-atoms). The details are left to the reader.

The idea in the case $k=m+1$ will be demonstrated if $m=0$; the technical modifications for $m>0$ are obvious. For estimating the H_1 norm of $\Delta_h g(t)$ the crucial point is to obtain an appropriate atomic decomposition of

$$\Delta_h N_l^{(0)}(t)=1|_{[s_l-h,s_l)}-1|_{[s_{l+1}-h,s_{l+1})}.$$

For this purpose we introduce special functions

$$A^+(t,t_0,h)=2|_{[t_0-h,t_0)}-1|_{[t_0,t_0+2h)},$$
$$A^-(t,t_0,h)=1|_{[t_0-2h,t_0)}-2|_{[t_0,t_0+h)},$$

which are multiples of (1, ∞, 0)-atoms (with H_1 norm $\leqslant C\cdot h$) for arbitrary $t_0\in T$ and $h<1/3$. Clearly, we have

$$\Delta_h N_i^{(0)}(t)=\sum_{j=1}^{k}2^{-j}\cdot A^{+}\big(t,\, s_i+(2^j-2)\,h,\, 2^{j-1}\cdot h\big)+$$
$$+\sum_{j=1}^{k}2^{-j}\cdot A^{-}\big(t,\, s_{i+1}-(2^j-1)\,h,\, 2^{j-1}\cdot h\big)+$$
$$+2^{-k}\cdot\big(1|_{[s_i+(2^k-2)h,\,s_i+(2^{k+1}-2)h)}-1|_{[s_{i+1}-(2^{k+1}-1)h,\,s_{i+1}-(2^k-1)h)}\big)$$

where for $k=[|\ln(nh)|]+1$ the last term is again a multiple of (1, ∞, 0)-atom with support length $\leqslant C\cdot n^{-1}$. Thus, by Proposition 1,

$$\|\Delta_h N_i^{(0)}\|_{H_1}\leqslant C\cdot\sum_{j=1}^{k}2^{-j}\cdot(2^{j-1}\cdot h)+2^{-k}\cdot n^{-1}$$
$$\leqslant C\cdot h\cdot(k+1)\leqslant C\cdot h\cdot|\ln(nh)|,\qquad 0<h\leqslant 1/2n.$$

This estimate immediately yields (3.11) (cf. (1.7)).

In the concluding part of this paper we state (without detailed proof) two standard corollaries to the direct and inverse inequalities. Let

$$B^s_{p,q}(T)=\{f\in H_p(T):\quad \|f\|_{B^s_{p,q}}$$
$$=\|f\|_{H_p}+\Big(\int_0^1\omega_k(t,f)^q_{H_p}\cdot t^{-sq-1}\,dt\Big)^{1/q}<\infty\},$$

where $0<p\leqslant 1$, $0<s<k$, $0<q<\infty$, be the spaces of Besov type defined with respect to the scale of Hardy spaces (for $q=\infty$ the usual modified definition will be used). By Proposition 3 it is also clear (cf. [8]) that $B^s_{p,q}(T)$ is an intermediate space for real interpolation between two Hardy–Sobolev spaces, i.e.,

$$B^s_{p,q}(T)=\big(H_p(T),\, H^k_p(T)\big)_{1-s/k,q}.$$

From (0.5), (0.6) we immediately obtain

PROPOSITION 6. *Let* $m=0,1,\ldots,$ $0<p\leqslant 1$, *and* $0<q\leqslant\infty$. *Then* $f(t)\in H_p(T)$ *belongs to* $B^s_{p,q}(T)$, $0<s<m+1$, *if and only if*

$$\|f\|_{B^s_{p,q}}=\|f\|_{H_p}+\Big\{\sum_{r=0}^{\infty}2^{rsq}\cdot E^{(m)}_{2^r}(f)^q_{H_p}\Big\}^{1/q}<\infty \tag{3.13}$$

where $\|\cdot\|_{B^s_{p,q}}$ *is an equivalent quasi-norm in* $B^s_{p,q}(T)$ (*modification if* $q=\infty$).

For the proof of a similar statement in the L_p case $(0<p<1)$ see [5], p. 90/91.

The case $q=\infty$ of Proposition 6 is essentially contained in the somewhat stronger

PROPOSITION 7. *Let the function* $w(t)$ *satisfy the conditions* $w(t)\searrow 0$, $t^{-(m+1)}\cdot w(t)\nearrow$ *for* $t\searrow 0$ *and, in addition,*

$$\int_t^1 w(\lambda)\cdot\lambda^{-(m+2)}\,d\lambda=O\big(w(t)\cdot t^{(m+1)}\big),\qquad t\to 0. \tag{3.14}$$

Then for $f\in H_p(T)$, $0<p\leqslant 1$, *the following two properties are equivalent:*

$$\omega_{m+1}(t,f)_{H_p}=O\big(w(t)\big),\qquad t\to 0, \tag{3.15}$$

$$E^{(m)}_n(f)_{H_p}=O\big(w(1/n)\big),\qquad n\to\infty. \tag{3.16}$$

Remark. In the L_p case $(0<p\leqslant\infty)$ conditions of type (3.14) are known to be also necessary for equivalence statements such as in Proposition 7 (cf. [5], [6]). However, in the H_p case $(0<p\leqslant 1)$ it is not yet clear how to construct the corresponding examples.

References

[1] Z. Ciesielski, *Bases and approximation by splines*, Proc. Internat. Congr. Math. (Vancouver 1974), vol. 2, 47–51.
[2] R. R. Coifman, *A real variable characterization of* H^p, Studia Math. 51 (1974), 269–274.
[3] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.
[4] J. Domsta, *A theorem on B-splines. II. The periodic case*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), 1077–1084.
[5] P. Oswald, *Approximation by splines in* L_p *metric*, $0<p<1$ (in Russian), Math. Nachr. 94 (1980), 69–96.
[6] —, *On inequalities for spline approximation and spline systems in the spaces* L^p $(0<p<1)$, in: *Approximation and Function Spaces*, Proc. Internat. Conf. (Gdańsk 1979), PWN—North Holland, Warszawa 1981, 531–552.
[7] —, *On spline bases in periodic Hardy spaces* $(0<p\leqslant 1)$, Math. Nachr. 108 (1982), 219–229.
[8] —, *On some approximation properties in real Hardy spaces* $(0<p\leqslant 1)$, J. Approx. Theory 40 (1984), 45–65.
[9] T. V. Radoslavova, *Decrease orders of the* L^p*-moduli of continuity* $(0<p\leqslant\infty)$, Anal. Math. 5 (1979), 219–234.
[10] L. Schumaker, *Spline Functions: Basic Theory*, J. Wiley & Sons, N.Y. 1981.
[11] P. Sjölin and J.-O. Strömberg, *Spline systems as bases in Hardy spaces*, Dept. of Math., Univ. of Stockholm, Report N° 1, 1982.
[12] P. Sjölin and J.-O. Strömberg, *Basis properties of Hardy spaces*, Dept. of Math., Univ. of Stockholm, Report N° 19, 1981.
[13] E. A. Storozhenko, *Approximation of the functions of class* H^p, $0<p\leqslant 1$ (in Russian), Mat. Sb. 105 (147) (1978), 601–626.

[14] —, *On the theorems of Jackson type in H^p, $0 < p \leqslant 1$* (in Russian), Izv. AN SSSR Ser. Mat. 44 (1980), 946–962.
[15] M. H. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Asterisque 77 (1980), 68–149.
[16] P. Wojtaszczyk, *H_p-spaces, $p \leqslant 1$, and spline systems*, Studia Math. 77 (1984), 289–320.

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Note on differentiation of integrals and the halo conjecture

by

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Abstract. In this paper several results on differentiation of integrals are obtained from restricted weak type estimates of the maximal operator associated to certain differentiation bases in $\boldsymbol{R}^n$. The only tool used is a simple lemma in measure theory due to E. Stein and N. Weiss which explains how functions add up in weak-L^1 (Lemma 5). In the process, we construct for each index $m \geqslant 0$ a quasi-Banach function space which plays with respect to $L(\log^+ L)^m$ the same role as the Lorentz class $L(p, 1)$ does with respect to L^p, $1 < p < \infty$ (see Theorems 2 and 3). We follow here some ideas originated in Taibleson–Weiss [7].

The same methods are used to exhibit a weak type estimate for the maximal operator on the partial sums of Fourier series and, as a consequence, a.e. convergence a little bit beyond $L\log^+ L\log^+ \log^+ L$.

1. Introduction and statement of results. Let $\mathscr{B}$ be a differentiation basis in $\boldsymbol{R}^n$ and $\Phi(u)$ its halo function; that is, $\Phi(u) = u$ if $0 < u \leqslant 1$ and, if $u > 1$,

$$\Phi(u) = \sup \{|A|^{-1} |\{x\colon (M\chi_A)(x) > 1/u\}|\colon A \text{ a msble. subset, } |A| > 0\},$$

where $M = M_{\mathscr{B}}$ denotes the maximal operator associated to $\mathscr{B}$, $|A|$ the Lebesgue measure of the subset A and χ_A its characteristic function.

In the present work we give partial answers to the following question: Assuming certain knowledge on the growth of $\Phi(u)$ at infinity, what can be said about differentiation properties of the basis $\mathscr{B}$?. (For an introduction to the subject, including some basic definitions, the reader is referred to de Guzmán [1].) We will state now a first result in this direction.

THEOREM 1. *Suppose that $\Phi(u) \leqslant c_0 u(1+\log^+ u)^m$ for some non-negative constants m and c_0. Then $\mathscr{B}$ differentiates any function which is locally in $L(\log^+ L)^m \log^+ \log^+ L$.*

$L(\log^+ L)^m \log^+ \log^+ L$ is not, however, the appropriate class to fully exploit the information given by such a behavior of the halo function and our next step will be to introduce more adequate classes to deal with this kind of problem. We will also show that, at least in one case, our results are best possible (see Theorem 3).