STUDIA MATHEMATICA, T. LXXXI. (1985)

Existence of a multiplicative functional and joint spectra

by

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Abstract. We show that, among other results, a unital Banach algebra has a nonzero multiplicative linear functional if and only if the joint spectrum $\sigma(a_1, \ldots, a_n)$ is non-empty for every finite set of elements $a_1, \ldots, a_n$ in the algebra.

Acknowledgement. The first author would like to thank the Natural Sciences and Engineering Research Council of Canada for the financial support. The second author would like to thank Professor Peter Rosenthal and the first author for their hospitality during his visit to the University of Toronto where this paper was prepared.

§ 1. Introduction. Let A be a unital, complex Banach algebra. The unit of A will be denoted by 1_A or simply by 1. The *left joint spectrum* of an n-tuple $(a_1, \ldots, a_n)$ of elements in A, denoted by $\sigma_l^A(a_1, \ldots, a_n)$ or simply by $\sigma_l(a_1, \ldots, a_n)$ if there is no confusion, is defined to be the subset of $\boldsymbol{C}^n$ consisting of those $(\lambda_1, \ldots, \lambda_n)$ which safisfy

$$A(a_1-\lambda_1)+A(a_2-\lambda_2)+ \ldots +A(a_n-\lambda_n) \neq A.$$

(Here, $a_j-\lambda_j$ stands for $a_j-\lambda_j 1_A$.) The right joint spectrum $\sigma_r(a_1, \ldots, a_n)$ is defined in a similar manner. The *joint spectrum* $\sigma^A(a_1, \ldots, a_n)$, or simply written as $\sigma(a_1, \ldots, a_n)$, is defined to be their union:

$$\sigma(a_1, \ldots, a_n) = \sigma_l(a_1, \ldots, a_n) \cup \sigma_r(a_1, \ldots, a_n).$$

If the algebra A is commutative, then $\sigma(a_1, \ldots, a_n)$ is always non-empty (see [6], p. 47 and p. 77). However, easy examples show that, in general, $\sigma(a_1, \ldots, a_n)$ may be void: see, for example, Harte [2], p. 93. We observe that, if A has a (nonzero) multiplicative (linear) functional φ, then $\sigma(a_1, \ldots, a_n)$ is non-empty; in fact, in that case we have

$$(\varphi(a_1), \ldots, \varphi(a_n)) \in \sigma(a_1, \ldots, a_n).$$

The main purpose of the present paper is to show the converse of this fact:

THEOREM. *If* $\sigma(a_1, \ldots, a_n)$ *is non-empty for an arbitrary n-tuple*

$(a_1, \ldots, a_n)$ *of elements in the Banach algebra* A *with* $n = 1, 2, \ldots,$ *then* A *has a multiplicative functional.*

§ **2. Main results.** We introduce the following concept to facilitate our argument.

DEFINITION. For an n-tuple $(a_1, \ldots, a_n)$ of elements in A, we write $\sigma_s^A(a_1, \ldots, a_n)$, or simply $\sigma_s(a_1, \ldots, a_n)$ for the set of all those $(\lambda_1, \ldots, \lambda_n)$ in C^n which safisfy

$$A(a_1-\lambda_1)A+A(a_2-\lambda_2)A+ \ldots +A(a_n-\lambda_n)A \neq A.$$

PROPOSITION 1. *A unital Banach algebra* A *has a multiplicative functional if and only if* $\sigma_s^A(a_1, \ldots, a_n)$ *is non-empty for an arbitrary n-tuple* $(a_1, \ldots, a_n)$ *of elements in* A *with an arbitrary* n.

Proof. The "only if" part is easy to show: if φ is a multiplicative functional on A, then $(\varphi(a_1), \ldots, \varphi(a_n))$ is in $\sigma_s(a_1, \ldots, a_n)$ since

$$\sum_{j=1}^{n} A(a_j-\varphi(a_j))A \subset \text{kernel of } \varphi.$$

Now we prove the "if" part and henceforth assume that $\sigma_s(a_1, \ldots, a_n)$ is always non-empty. Let I be the commutator ideal of A: the elements of I are exactly those which can be expressed as finite sums of terms of the form $c(ab-ba)d$ with $a, b, c, d \in A$. If I is proper, then, since $\|1-x\| \geqslant 1$ for all $x \in I$, its closure $\bar{I}$ is also proper. In that case, $A/\bar{I}$ is a nontrivial unital commutative Banach algebra and each multiplicative functional on $A/\bar{I}$ composed with the quotient map from A onto $A/\bar{I}$ gives rise to a multiplicative functional on A. Thus it suffices to show that I is proper.

Suppose to the contrary that I is not proper: we have

$$(*) \qquad \sum_{k=1}^{n} c_k(a_k b_k - b_k a_k) d_k = 1$$

for some a_k, b_k, c_k, d_k in A. By the assumption, $\sigma_s(a_1, \ldots, a_n, b_1, \ldots, b_n)$ is non-empty. Let

$$(**) \qquad (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \in \sigma_s(a_1, \ldots, a_n, b_1, \ldots, b_n).$$

From $(*)$ we have

$$\sum_{k=1}^{n} s_k(a_k-\alpha_k)t_k + \sum_{k=1}^{n} u_k(b_k-\beta_k)v_k = 1$$

where $s_k = c_k$, $t_k = (b_k-\beta_k)d_k$, $u_k = c_k$ and $v_k = -(a_k-\alpha_k)d_k$ $(k = 1, \ldots, n)$. This contradicts $(**)$. ■

Remarks. 1. If A is commutative then we have $\sigma_l(a_1, \ldots, a_n) = \sigma(a_1, \ldots, a_n) = \sigma_s(a_1, \ldots, a_n)$ for $a_1, \ldots, a_n$ in A.

2. In the general situation, $\sigma_s(a_1, \ldots, a_n) \subset \sigma_l(a_1, \ldots, a_n) \cap \sigma_r(a_1, \ldots, a_n)$.

3. Even for $n = 1$, $\sigma_s(a_1, \ldots, a_n)$ may be empty. For example, if $A = B(H)$ where H is a separable infinite-dimensional Hilbert space, then, via polar decomposition, one can see that, for $a \in A$, $\sigma_s(a)$ is empty if and only if a is not of the form $\lambda+k$ where λ is a scalar and k is a compact operator on H.

4. In the case where $\sigma_l(a) = \sigma_r(a)$ for all a in A, we have $\sigma_s(a) = \sigma_l(a) = \sigma(a)$. For example, such is the case if $A = K(X)+C1$, where $K(X)$ is the set of all compact operators on an infinite-dimensional Banach space X, in view of Fredholm's alternative.

5. $\sigma_s(a_1, \ldots, a_n)$ has the following "similarity properties" which are lacking in $\sigma(a_1, \ldots, a_n)$.

(a) *If* $a_1, \ldots, a_n$ *are similar to* $b_1, \ldots, b_n$, *respectively* (i.e., $b_1 = c_1^{-1} a_1 c_1, \ldots, b_n = c_n^{-1} a_n c_n$ for some invertible $c_1, \ldots, c_n$) *then* $\sigma_s(a_1, \ldots, a_n) = \sigma_s(b_1, \ldots, b_n)$.

(b) *If* a *and* b *are similar, then* $(\lambda, \mu) \in \sigma_s(a, b)$ *implies* $\lambda = \mu$.

PROPOSITION 2. *If the Banach algebra* A *is generated by finitely many elements* $a_1, \ldots, a_n$ *and* $\lambda_1, \ldots, \lambda_n \in C$ *then the following three conditions are equivalent*:

(i) $(\lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n)$;

(ii) *there exists a multiplicative functional* φ *on the algebra* A *such that* $\varphi(a_j) = \lambda_j$ $(j = 1, \ldots, n)$;

(iii) $p(\lambda_1, \ldots, \lambda_n) \in \sigma(p(a_1, \ldots, a_n))$ *for all* $p(x_1, \ldots, x_n) \in \boldsymbol{P}(x_1, \ldots, x_n)$, *where* $\boldsymbol{P}(x_1, \ldots, x_n)$ *is the algebra of all polynomials over* C *with "noncommutative indeterminates"* $x_1, \ldots, x_n$, *in other words,* $\boldsymbol{P}(x_1, \ldots, x_n)$ *is the free associative algebra generated by the symbols* $x_1, \ldots, x_n$.

Proof. That (i) implies (iii) follows immediately from the following special case of the result of Harte [2]: If $b_1, \ldots, b_n$ are elements in a Banach algebra B, $(\lambda_1, \ldots, \lambda_n) \in \sigma^B(b_1, \ldots, b_n)$ and $p(x_1, \ldots, x_n) \in \boldsymbol{P}(x_1, \ldots, x_n)$, then $p(\lambda_1, \ldots, \lambda_n) \in \sigma^B(p(b_1, \ldots, b_n))$.

It is obvious that (ii) implies (i).

Now we assume (iii) and deduce (ii). The algebra generated by $a_1, \ldots, a_n$ is

$$A_0 = \{p(a_1, \ldots, a_n) \in A\colon\ p(x_1, \ldots, x_n) \in \boldsymbol{P}(x_1, \ldots, x_n)\},$$

and hence A_0 is dense in A. We define a map $\varphi\colon A_0 \to C$ by putting

$$\varphi(p(a_1, \ldots, a_n)) = p(\lambda_1, \ldots, \lambda_n).$$

We have to show that φ is well defined. Once this is done, by the way φ is defined we see that φ is linear, multiplicative, bounded, $\varphi(a_j) = \lambda_j$ and $\varphi(1) = 1$, and hence φ can be extended to a multiplicative functional on A. Thus it remains to show that φ is well defined.

Suppose that $p(a_1, \ldots, a_n) = q(a_1, \ldots, a_n)$. We have to show that $p(\lambda_1, \ldots, \lambda_n) = q(\lambda_1, \ldots, \lambda_n)$. Let $r(x_1, \ldots, x_n) = p(x_1, \ldots, x_n) - q(x_1, \ldots, x_n)$. Then $r(a_1, \ldots, a_n) = 0$, and hence $p(\lambda_1, \ldots, \lambda_n) - q(\lambda_1, \ldots, \lambda_n) = r(\lambda_1, \ldots \ldots, \lambda_n) \in \sigma(r(a_1, \ldots, a_n)) = \{0\}$. Therefore $p(\lambda_1, \ldots, \lambda_n) = q(\lambda_1, \ldots, \lambda_n)$. ■

Remarks. 1. Recall that the joint numerical range of an n-tuple $(a_1, \ldots, a_n)$ of elements in A is

$$V(a_1, \ldots, a_n) = \{(f(a_1), \ldots, f(a_n)) \in C^n \colon f \in A', \ \|f\| = f(1) = 1\}.$$

The well-known fact that $\sigma(a_1, \ldots, a_n)$ is contained in $V(a_1, \ldots, a_n)$ (see [1]; p. 24) follows from Proposition 2 in a straightforward manner. Indeed, if $(\lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n)$, then $(\lambda_1, \ldots, \lambda_n) \in \sigma^B(a_1, \ldots, a_n)$ where B is the closed (unital) subalgebra generated by $a_1, \ldots, a_n$, and hence, by Proposition 2, there exists a multiplicative functional φ on B such that $\varphi(a_j) = \lambda_j$ $(j = 1, \ldots, n)$. By the Hahn–Banach extension theorem, φ can be extended to a linear functional f on A such that $\|f\| = f(1) = 1$.

2. Harte's result mentioned in the proof of Proposition 2 has the following noteworthy consequence: if $(\lambda_1, \ldots, \lambda_n) \in \sigma^A(a_1, \ldots, a_n)$ and if $p(a_1, \ldots, a_n) = 0$ for some $p(x_1, \ldots, x_n) \in \boldsymbol{P}(x_1, \ldots, x_n)$, then $p(\lambda_1, \ldots, \lambda_n) = 0$. In fact, from the assumption $p(a_1, \ldots, a_n) = 0$ we obtain $p(\lambda_1, \ldots, \lambda_n) \in \sigma(p(a_1, \ldots, a_n)) = \{0\}$.

It is known that if A is a commutative Banach algebra, then the joint spectrum of elements in A has the following so-called "projection property":

$$P_n^{n+m} \sigma(a_1, \ldots, a_n, b_1, \ldots, b_m) = \sigma(a_1, \ldots, a_n)$$

where P_n^{n+m} is the canonical projection from C^{n+m} onto C^n which sends $(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \ldots, \lambda_{n+m})$ to $(\lambda_1, \ldots, \lambda_n)$. (See [5].) From this result we see that if $A/\mathrm{Rad}(A)$ is commutative (where $\mathrm{Rad}(A)$ stands for the radical of A), then the joint spectrum of elements in A also has the projection property, because, in the general situation, we have

$$\sigma^A(a_1, \ldots, a_n) = \sigma^{A/\mathrm{Rad}(A)}(a_1 + \mathrm{Rad}(A), \ldots, a_n + \mathrm{Rad}(A)),$$

a fact easy to verify.

Now we use Proposition 2 to prove the converse of this statement.

Corollary. *If the joint spectrum of elements in A has the projection property, then $A/\mathrm{Rad}(A)$ is commutative.*

Proof. We suppose to the contrary that there exist a, b in A such that $ab - ba \notin \mathrm{Rad}(A)$. Then there exists an element c in A such that $(ab-ba)c$ is not quasi-nilpotent and hence $\sigma((ab-ba)c)$ contains a nonzero number λ. By the projection property of spectrum, $(\lambda, \alpha, \beta, \gamma) \in \sigma((ab-ba)c, a, b, c)$ for some α, β, γ. Let B be the closed subalgebra generated by a, b, c. Then, a fortiori, $(\lambda, \alpha, \beta, \gamma) \in \sigma^B((ab-ba)c, a, b, c)$. By Proposition 2, there is a multiplicative functional φ on B such that $\varphi((ab-ba)c) = \lambda$. But $\varphi((ab-ba)c) = (\varphi(a)\varphi(b) - \varphi(b)\varphi(a))\varphi(c) = 0$, contradicting the fact that $\lambda \neq 0$. ■

Proof of the Theorem. We assume that $\sigma(a_1, \ldots, a_n)$ is always non-empty for finitely many elements $a_1, \ldots, a_n$ in A. To show that A has a multiplicative functional, in view of Proposition 1, it suffices to show that $\sigma_s(a_1, \ldots, a_n)$ is non-empty for an arbitrary n-tuple $(a_1, \ldots, a_n)$ which is fixed from now on.

For each m-tuple $(b_1, \ldots, b_m)$ of elements in A, we write $\pi(b_1, \ldots, b_m)$ for the set of n-tuples $(\lambda_1, \ldots, \lambda_n)$ in C^n such that

$$(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m) \in \sigma(a_1, \ldots, a_n, b_1, \ldots, b_m)$$

for some $(\mu_1, \ldots, \mu_m)$ in C^m. By our assumption, $\pi(b_1, \ldots, b_m)$ is a non-empty compact subset of C^n. It is obvious that

$$\pi(b_1, \ldots, b_m, c_1, \ldots, c_l) \subset \pi(b_1, \ldots, b_m) \cap \pi(c_1, \ldots, c_l).$$

Therefore $\{\pi(b_1, \ldots, b_m)\}$, where $(b_1, \ldots, b_m)$ runs through all m-tuples in A^m and m runs through all positive integers, is a family of compact sets with the finite intersection property, and hence its intersection is non-empty. Let $(\alpha_1, \ldots, \alpha_n)$ be an n-tuple in this intersection. We claim $(\alpha_1, \ldots, \alpha_n) \in \sigma_s(a_1, \ldots, a_n)$. Suppose to the contrary that there exist $u_1, \ldots, u_n, v_1, \ldots, v_n$ in A such that $\sum_{k=1}^{n} u_k(a_k - \alpha_k)v_k = 1$. Since

$$(\alpha_1, \ldots, \alpha_n) \in \pi(u_1, \ldots, u_n, v_1, \ldots, v_n),$$

there exist $\mu_1, \ldots, \mu_n, \lambda_1, \ldots, \lambda_n$ in C such that

$$(\alpha_1, \ldots, \alpha_n, \mu_1, \ldots, \mu_n, \lambda_1, \ldots, \lambda_n) \in \sigma(a_1, \ldots, a_n, u_1, \ldots, u_n, v_1, \ldots, v_n).$$

By Proposition 2, there exists a multiplicative functional φ on the algebra generated by $a_1, \ldots, a_n, u_1, \ldots, u_n, v_1, \ldots, v_n$ such that $\varphi(a_1) = \alpha_1, \ldots, \varphi(a_n) = \alpha_n$. However, we then have

$$1 = \varphi(1) = \varphi\Big(\sum_{k=1}^{n} u_k(a_k - \alpha_k)v_k\Big) = 0,$$

which is absurd. Therefore $(\alpha_1, \ldots, \alpha_n) \in \sigma_s(a_1, \ldots, a_n)$. ■

Remarks. 1. From the Theorem the following two facts follow which are not a priori obvious.

(i) If all $\sigma(a_1, \ldots, a_n)$ $(a_1, \ldots, a_n \in A)$ are non-empty, then so are all $\sigma_s(a_1, \ldots, a_n)$.

(ii) If all $\sigma(a_1, \ldots, a_n)$ are non-empty, then there is a family $\{\lambda_a \colon a \in A\}$ of complex numbers indexed by A such that $(\lambda_{a_1}, \ldots, \lambda_{a_n}) \in \sigma(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n$ in A.

2. From the main theorem we can deduce the following stronger

result: if A has a generating set S such that $\sigma^A(a_1, \ldots, a_n) \neq \emptyset$ for all n-tuples $(a_1, \ldots, a_n)$ of elements in S ($n = 1, 2, \ldots$), then A has a multiplicative functional.

3. For an n-tuple $(a_1, \ldots, a_n)$ of elements in A, we let $\sigma_\mu^A(a_1, \ldots, a_n)$ to be the set of all $(\lambda_1, \ldots, \lambda_n)$ in $\boldsymbol{C}^n$ such that whenever $\{b_1, \ldots, b_m\}$ is a finite set in A, $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m) \in \sigma(a_1, \ldots, a_n, b_1, \ldots, b_m)$ for some $\mu_1, \ldots, \mu_m$ in $\boldsymbol{C}$. It follows from the proof of the main theorem that if A has a multiplicative functional then $\sigma_\mu(a_1, \ldots, a_n)$ is always non-empty and has the projection property. With a little extra effort, one can show that

$$\sigma_\mu^A(a_1, \ldots, a_n) = \{(\varphi(a_1), \ldots, \varphi(a_n)) \colon \varphi \text{ is a multiplicative functional on } A\}.$$

Hence if we let C to be the closure of the commutator ideal of A, then

$$\sigma_\mu^A(a_1, \ldots, a_n) = \sigma^{A/C}(a_1 + C, \ldots, a_n + C).$$

§ 3. **Examples.** In this final section we consider the existence of multiplicative functionals in two examples of noncommutative Banach algebras.

Example 1. Let H be a complex finite-dimensional Hilbert space and let A be a unital subalgebra of $B(H)$. A subspace N of H is said to be *semi-invariant* for A if there are subspaces M_1 and M_2, both invariant for all operators in A, such that $M_1 \subset M_2$ and $N = M_2 \ominus M_1$. (See [4].) It is easy to see that if A has a one-dimensional semi-invariant subspace, then A has a multiplicative functional. Now we show that the converse is also true. We are indebted to M.-D. Choi who supplied the following proof.

Assume that there exists a multiplicative functional $\varphi\colon A \to \boldsymbol{C}$ but A has no one-dimensional semi-invariant subspace. Since $A \neq B(H)$ the Burnside theorem (see, e.g. [3], p. 142) tells us that A is intransitive : there exists a nontrivial subspace M which is invariant for all operators in A. Let p be the projection of H onto M. We get two algebras pAp and $(1-p)A(1-p)$. If pAp or $(1-p)A(1-p)$ is not isomorphic to the algebra of full matrices then we shall repeat the previous argument to produce further decompositions. After a finite number of steps we obtain orthogonal projections p_j $(j = 1, \ldots, n)$ with $p_1 + p_2 + \ldots + p_n = 1$ and $p_k \neq p_l$ for $k \neq l$, such that each element a of A can be written in the upper triangular form

$$a = \begin{bmatrix} a_1 & & * \\ & a_2 & \\ 0 & & \ddots & \\ & & & a_n \end{bmatrix}$$

(where $a_j = p_j a p_j$ for $j = 1, \ldots, n$) and $p_j A p_j$ is isomorphic to the algebra of all $m_j \times m_j$ matrices. By our assumption that A has no one-dimensional semi-invariant subspace, we have rank $p_j = m_j > 1$ for $j = 1, \ldots, n$. Fix an arbitrary j in $\{1, \ldots, n\}$. We can find in $p_j A p_j$ two linearly independent elements b_j and c_j. Taking b_j' and c_j' in A such that $b_j = p_j b_j' p_j$ and $c_j = p_j c_j' p_j$ we can choose scalars λ_j and μ_j (not both equal to zero) with the property $\varphi(\lambda_j b_j' + \mu_j c_j') = 0$. By the linear independence of b_j and c_j, $d_j = \lambda_j b_j + \mu_j c_j \neq 0$. Let $d_j' = \lambda_j b_j' + \mu_j c_j'$. Since the algebra $p_j A p_j$ is simple, the two-sided ideal generated by d_j is trivial. Therefore $1_j = \sum_k u_{jk} d_j v_{jk}$ (a finite sum; here 1_j stands for the unit of the algebra $p_j A p_j$) for some u_{jk} and v_{jk} in $p_j A p_j$. Let u_{jk}' and v_{jk}' in A be such that $u_{jk} = p_j u_{jk}' p_j$ and $v_{jk} = p_j v_{jk}' p_j$ and let $e_j = 1 - \sum_k u_{jk}' d_j' v_{jk}'$. Then $p_j e_j p_j = 0$ and $\varphi(e_j) = 1$. The product $e_1 e_2 \ldots e_n$ is strictly upper triangular, and hence nilpotent. Thus we have $\varphi(e_1 e_2 \ldots e_n) = 0$. But on the other hand $\varphi(e_1 e_2 \ldots e_n) = \varphi(e_1)\varphi(e_2) \ldots \varphi(e_n) = 1$. This contradiction concludes the proof.

Example 2. Let X be a compact metric space and B be a Banach algebra. We write $C(X, B)$ for the Banach algebra consisting of continuous functions from X into B. Let A be a closed subalgebra of $C(X, B)$ which satisfies the following condition:

(C) *A contains all mappings of the form $\xi \mapsto \alpha(\xi) 1_B$, where α is a scalar-valued continuous function and 1_B is the identity of B.*

For each $\xi \in X$, we write A_ξ for the closure of $\{f(\xi)\colon f \in A\}$. If, for some $\xi_0 \in X$, A_{ξ_0} has a multiplicative functional φ, then $\Phi\colon A \to \boldsymbol{C}$ defined by $\Phi(f) = \varphi(f(\xi_0))$ is a multiplicative functional on A. Now we show the converse: that if A has a multiplicative functional Φ, then, for some ξ_0 in X, A_{ξ_0} has a multiplicative functional φ such that $\Phi(f) = \varphi(f(\xi_0))$.

The algebra $C(X)$ of all scalar-valued continuous functions on X can be regarded as a subalgebra of A. Let ψ be the restriction of Φ to $C(X)$. Then ψ is a multiplicative functional, and hence there exists $\xi_0 \in X$ such that $\psi(\alpha) = \alpha(\xi_0)$ for $\alpha \in C(X)$.

We claim that, if $f, g \in A$ and $f(\xi_0) = g(\xi_0)$, then $\Phi(f) = \Phi(g)$. Let ε be an arbitrary positive number. Then there is a neighbourhood U of ξ_0 such that $\|f(\xi) - g(\xi)\| < \varepsilon$ for $\xi \in U$. Let $\alpha \in C(X)$ be a function such that $\alpha(\xi_0) = 1$, $|\alpha(\xi)| \leqslant 1$ for all $\xi \in X$ and $\alpha(\xi) = 0$ for $\xi \notin U$. Then $\|\alpha f - \alpha g\| < \varepsilon$, and hence $|\Phi(\alpha f) - \Phi(\alpha g)| < \varepsilon$. However, $\Phi(\alpha f) = \psi(\alpha)\Phi(f) = \Phi(f)$ and similarly $\Phi(\alpha g) = \Phi(g)$. Thus $|\Phi(f) - \Phi(g)| < \varepsilon$ for an arbitrary $\varepsilon > 0$ and therefore $\Phi(f) = \Phi(g)$.

Now we can define a multiplicative functional φ on $\{f(\xi_0)\colon f \in A\}$ by putting $\varphi(f(\xi_0)) = \Phi(f)$. Using the argument similar to the previous one of truncating f by some α in $C(X)$, we can show that φ is bounded, and hence we can extend φ to A_{ξ_0}.

We remark that assumption (C) is essential. For example, if $X = \partial D$, the unit circle in $\boldsymbol{C}$, if $B = M_2$, the algebra of all 2×2 matrices and if A is the algebra of those functions f in $C(X, M_2)$ which have analytic extensions $\tilde{f}$ to the unit disc D such that $\tilde{f}(0)$ are scalar multiples of the identity matrix, then A has a multiplicative functional; but, for each $\xi \in X$, $A_\xi = M_2$, and hence there is no multiplicative functional on A_ξ.

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Received January 6, 1984 (1948)

STUDIA MATHEMATICA, T. LXXXI. (1985)

Sur les espaces stables universels

par

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Abstract. We consider the class $\mathscr{A}$ of Banach spaces X such that the set of types on X is separable for the topology of uniform convergence on bounded sets of X. This class contains the class $\mathscr{B}$ of separable stable spaces. We construct an ordinal index for $\mathscr{A}$ and we prove that there is no element Y of $\mathscr{A}$, 1-universal for $\mathscr{B}$ in the sense that every element of $\mathscr{B}$ is at Banach–Mazur distance 1 of subspaces of Y.

Introduction. L'existence d'espace universel ou isométriquement universel pour une classe d'espaces $\mathscr{A}$ a été étudiée dans de nombreux cadres. Notamment dans [1], il est prouvé que l'espace des fonctions continues réelles sur l'intervalle [0, 1] est isométriquement universel pour les espaces de Banach séparables. Dans [6], il apparaît un résultat négatif: il n'a y pas d'espace de Banach à dual séparable, universel pour les espaces de Banach réflexifs et séparables. Ce dernier résultat a été étendu à la classe des espaces à duaux séparables dans [7].

Nous nous inspirons ici des méthodes de [6] pour prouver un résultat analogue pour les espaces de Banach dont l'espace des types est séparable pour la topologie uniforme. (Ces espaces sont nécessairement séparables). Cette classe d'espaces contient les espaces de Banach stables et séparables. Dans toute la suite nous utiliserons les techniques de stabilité qui figurent dans [4]. Nous définissons une notion intermédiaire entre espace universel et espace isométriquement universel, adaptée à ce problème:

Soient X et Y deux espaces de Banach. On notera $d(X, Y)$ la distance de Banach–Mazur de X à Y. Rappelons que $d(X, Y) = 1$ si et seulement si pour tout $\eta > 0$ il existe un isomorphisme T de X sur Y tel que:

$$\forall x \in X, \quad \|x\| \leqslant \|Tx\| \leqslant (1+\eta)\|x\|.$$

On dira qu'un espace de Banach X est 1-*universel* pour une classe d'espaces de Banach $\mathscr{A}$ si et seulement si tout espace Z de $\mathscr{A}$ est à distance 1 des sous-espaces de X, c'est-à-dire: $\operatorname{Inf}\{d(Z, Y) \mid Y \subset X\} = 1$.

Dans la première partie, nous construisons un indice pour les espaces ayant un espace de types séparable pour la topologie uniforme en suivant les méthodes de [6] et nous examinons ses propriétés.