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The decomposition of functions of bounded $\varkappa$-variation into differences of $\varkappa$-decreasing functions

by

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Abstract. The concept of variation of a function, first introduced by Camille Jordan a century ago, has been generalized in many ways since then. However, Jordan's characterization that functions of bounded variation are differences of two decreasing functions has, with one single exception, failed to generalize. In 1975 B. Korenblum distorted the measurement of intervals in the domain instead of the range space, as had been the common practice, to arrive at the concept of $\varkappa$-variation. In this case functions of bounded $\varkappa$-variation can in fact be characterized as the difference of two $\varkappa$-decreasing functions. We present in this paper a careful proof of this decomposition theorem, which in the piecewise linear case is constructive and maximally efficient. A Helley-type selection theorem is also provided.

1. Introduction. Just over a century ago Camille Jordan introduced the concept of *variation* of a function and characterized functions of *bounded* variation as differences of decreasing (increasing) functions. Some forty years later N. Wiener [13] generalized the concept of variation of a function to what he called its *quadratic variation*, in which he distorts the measurements of intervals in the range space by *squaring* their length. In 1936 L. C. Young [14] used arbitrary positive powers to distort intervals and eventually people began to fix a *distortion* function φ, which is continuous and nondecreasing, and proceeded to define the φ-variation of a function (see e.g. [9] and [8]). This idea was then generalized further by D. Waterman [11] and most recently by M. Schramm [10] who consider a countable *family* of distortion functions in order to generalize the concept of variation of a function. All of these efforts lead to results on existence of Riemann–Stieltjes integrals and to convergence criteria for Fourier series. The price one pays in all these generalizations is the loss of an effective decomposition of a function of bounded variation into, hopefully, simpler functions, such as one has in the case of Jordan's original concept.

In 1975 while studying Poisson integral representations of certain classes of harmonic functions in the unit disc of the complex plane, B. Korenblum [4] was led in a natural way to consider a new kind of variation for the boundary functions involved, called *ϰ-variation*. This concept differs from those mentioned above in that a distortion function $\varkappa$ is introduced for measuring intervals in the *domain* of the function, and not the *range*. One advantage of this alternate approach over those above is that a function of *bounded ϰ-variation* can be decomposed into the difference of two *ϰ-decreasing* functions. (Korenblum introduces the expression *ϰ-bounded above* for this concept, which is appropriate when considering the premeasures which generate these functions. For our purposes we feel that the term *ϰ-decreasing* is more graphic.)

The decomposition theorem for functions of bounded $\varkappa$-variation is stated in [4], p. 206, Theorem 5 (for a particular function $\varkappa$) along with a very brief outline of a proof. Because of the important connection between these functions and complex function theory as demonstrated in [4], (see also [5], [2] and [3]), we present in this paper a careful proof of the decomposition theorem (for a broad class of functions $\varkappa$). In the piecewise linear case the proof is constructive and "maximally efficient" in a sense that will become more clear later. The authors are indebted to Professor Korenblum who in a private communication suggested an outline for this project.

Definitions and preliminary results. Throughout this paper we only consider real functions defined on $[0, 1]$. All the results apply to functions f on an arbitrary interval $[a, b]$ by referring to the function $f \circ \alpha$ defined on $[0, 1]$, where $\alpha(x) = (b-a)x+a$. If $I = [x, y]$ is an interval, we write $f(I)$ for $f(y)-f(x)$, and $|I|$ for $y-x$. A partition P of $[0, 1]$ shall denote, as usual, a finite collection of intervals $I_i = [x_{i-1}, x_i]$, $i = 1, \ldots, n$, with $x_0 = 0$, $x_n = 1$. We will write either $P = \{I_i\}$ or $P = \{x_i\}$ as convenience dictates.

We begin by fixing a distortion function $\varkappa$: $[0, 1] \to [0, 1]$ which is *continuous, increasing, concave* (down), $\varkappa(0) = 0$, $\varkappa(1) = 1$, and having *infinite slope* at the origin:

(1) $$\lim_{x \to +0} \frac{\varkappa(x)}{x} = \infty.$$

We note that $\varkappa$ is *subadditive*:

(2) $$\varkappa(x+y) \leqslant \varkappa(x)+\varkappa(y).$$

Important special cases for choices of $\varkappa$ are $\varkappa_0(x) = x(1-\log x)$, used by Korenblum in [4], and $\varkappa_\alpha(x) = x^\alpha$, $0 < \alpha < 1$, referred to in the literature now as the Gevrey case.

DEFINITION 1. A real function f on $[0, 1]$ is said to be of *bounded ϰ-variation* if there is a $C > 0$ such that for every partition $P = \{I_i\}$ of $[0, 1]$,

(3) $$\sum_i |f(I_i)| \leqslant C \sum_i \varkappa(|I_i|).$$

$C_0 = \min C$ is called the *total ϰ-variation* of f: $C_0 = \varkappa V(f)$. The family of all functions of bounded $\varkappa$-variation is denoted by $\varkappa\,\mathrm{BV}$. As mentioned above, a function f defined on an arbitrary interval $[a, b]$ is of *bounded ϰ-variation* if the function $f((b-a)x+a)$ is of bounded $\varkappa$-variation on $[0, 1]$.

A number of observations follow immediately from (3). First of all, every function of bounded variation in the classical sense is of bounded $\varkappa$-variation, and the total variation

(4) $$\varkappa V(f) \leqslant V(f),$$

since the subadditivity of $\varkappa$ implies that the sum on the right-hand side of (3) is always $\geqslant 1$. Also if f is *monotone*,

(5) $$\varkappa V(f) = V(f) = |f(1)-f(0)|,$$

since the trivial partition $P = \{0, 1\}$ of $[0, 1]$ is optimum in (3) for monotone f.

Furthermore a function of bounded $\varkappa$-variation has *left* and *right-hand limits* at each point of its domain; hence these functions have at most countably many (jump) discontinuities, and are bounded. In fact,

(6) $$|f(x)| \leqslant |f(0)|+\tfrac{3}{2}\varkappa V(f),$$

which is easily shown from (3). To show the existence of these limits, suppose for example that $0 \leqslant a < 1$, and that

$$A = \liminf f(x) \leqslant \limsup f(x) = B$$

as $x \to a+$. Then for each (sufficiently large) positive integer n, one can choose points x_i, $i = 1, \ldots, n+1$, such that $a < x_1 < \ldots < x_{n+1} \leqslant a+1/n < 1$, and such that $|f(x_{i+1})-f(x_i)| \geqslant (B-A)/2$, $i = 1, \ldots, n$. Using the partition $\{0, x_1, x_2, \ldots, x_{n+1}, 1\}$ and the definition (3), one obtains

$$\begin{aligned} n(B-A)/2 &\leqslant \sum_{i=1}^{n} |f(x_{i+1})-f(x_i)|+|f(x_1)-f(0)|+|f(1)-f(x_{n+1})| \\ &\leqslant C\Big[\sum_{i=1}^{n} \varkappa(x_{i+1}-x_i)+\varkappa(x_1)+\varkappa(1-x_{n+1})\Big] \\ &\leqslant C[n\varkappa(1/n)+2]. \end{aligned}$$

Dividing by n and letting $n \to \infty$ yields $A = B$.

An example of a function of bounded $\varkappa$-variation which is *not* of bounded variation in the classical sense is given in Section 3.

DEFINITION 2. A function f (on $[0, 1]$) is said to be *ϰ-decreasing with constant* $C \geqslant 0$ if for each interval $I = [x, y]$, $0 \leqslant x < y \leqslant 1$,

(7) $$f(I) \leqslant C\varkappa(|I|).$$

Note that *every* decreasing function is $\varkappa$-decreasing (with $C = 0$). Intuitively a function is $\varkappa$-decreasing if it is either decreasing or, at least locally, not increasing any faster than some fixed multiple of $\varkappa$ itself.

As a simple example we note that every function Hölder continuous on $[0, 1]$ with exponent α, $0 < \alpha < 1$, is $\varkappa$-decreasing (and $\varkappa$-increasing) with $\varkappa(x) = x^\alpha$. Hence, for example, there exist continuous *nowhere differentiable* functions which are $\varkappa$-decreasing, and therefore, by Theorem 1 below, of bounded $\varkappa$-variation.

We observe that if f is $\varkappa$-decreasing, then f has at worst *downward* jump discontinuities:

$$f(a-) \geqslant f(a) \geqslant f(a+). \tag{8}$$

This follows by applying (7) to $I = [x, y]$ with $x \leqslant a \leqslant y$ and letting x and y approach a.

THEOREM 1. *If a function f is $\varkappa$-decreasing with constant C, then f is of bounded $\varkappa$-variation and*

$$\varkappa V(f) \leqslant 2C + |f(1) - f(0)|. \tag{9}$$

This theorem is proved in [4], p. 204 in the case $f(1) = f(0)$, but the simple proof there can be easily adapted.

Not every function of bounded $\varkappa$-variation is $\varkappa$-decreasing, as one sees with the function $f(x) = \varkappa(x)^{1/2}$. Here f is increasing and is therefore of bounded $\varkappa$-variation, but since $[f(x) - f(0)]/\varkappa(x-0) = \varkappa(x)^{-1/2} \to +\infty$ as $x \to 0+$, f cannot satisfy (7) for any $C > 0$.

In the proof of the decomposition theorem we make use of a Helly-type selection theorem for $\varkappa$-decreasing functions. Of course, in conjunction with the decomposition theorem, as in the classical case, one obtains a Helly-type selection theorem for functions of bounded $\varkappa$-variation as well.

THEOREM 2. *An arbitrary infinite family of functions defined on* $[0, 1]$ *which is both uniformly bounded and uniformly $\varkappa$-decreasing contains a subsequence which converges at every point of* $[0, 1]$ *to a $\varkappa$-decreasing function.*

Proof. Denote the family by $\mathscr{F}$. The hypotheses are taken to mean that there exists a constant $C > 0$ such that for every f in $\mathscr{F}$ and every pair $0 \leqslant x < y \leqslant 1$,

$$|f(x)| \leqslant C \tag{10}$$

and

$$f(y) - f(x) \leqslant C\varkappa(y-x). \tag{11}$$

Using (10) we can, by means of the standard Cantor diagonalization technique, find a sequence of functions f_k in $\mathscr{F}$ which converge pointwise at, say, each *rational* point of $[0, 1]$, to a function φ. Since each f_k satisfies (11), so does φ, at least for rational x and y.

Next define φ at irrational points x by

$$\varphi(x) = \lim_{y \to x-} \varphi(y), \quad y \text{ rational}. \tag{12}$$

That this limit exists can be seen as follows. If $A = \liminf \varphi(y) \leqslant \limsup \varphi(y) = B$ as $y \to x-$, y rational, let y_i and y'_i be two sequences of rational points converging to x, arranged so that $y_1 < y'_1 < y_2 < y'_2 < \ldots \ldots < x$, and such that $\varphi(y_i) \to A$ and $\varphi(y'_i) \to B$ as $i \to \infty$. Then $\varphi(y'_i) - \varphi(y_i) \leqslant C\varkappa(y'_i - y_i)$. Taking limits yields $B - A \leqslant 0$, and hence $A = B$.

Since φ satisfies (11) for pairs of rational points, it follows from (12), again by taking limits of rational points, that φ satisfies (11) for all pairs of points, i.e. φ is $\varkappa$-decreasing with constant C on $[0, 1]$. By Theorem 1 φ is of bounded $\varkappa$-variation, and therefore has at most countably many discontinuities, and hence by another Cantor diagonalization process a subsequence of the functions f_k can be found which converges at the points of discontinuity of φ.

The proof of the theorem will be complete if we show that $f_k(x) \to \varphi(x)$ at each point of continuity of φ. Assume $0 < a < 1$ is such a point and let $\varepsilon > 0$ be given. Fix two rational points y_1 and y_2, $y_1 < a < y_2$ such that

$$|\varphi(y_i) - \varphi(a)| < \varepsilon/3; \quad C\varkappa(|y_i - a|) < \varepsilon/3; \quad i = 1, 2. \tag{13}$$

Since $f_k \to \varphi$ at rational points, there exists $N > 0$ such that $k \geqslant N$ implies

$$|f_k(y_i) - \varphi(y_i)| < \varepsilon/3, \quad i = 1, 2. \tag{14}$$

Now one can show $\varphi(a) - f_k(a) < \varepsilon$ for $k \geqslant N$ by adding and substracting from the left-hand side $\varphi(y_2)$ and $f_k(y_2)$ and using (13), (14), and the fact that the f_k are uniformly $\varkappa$-decreasing. Similarly one obtains $f_k(a) - \varphi(a) < \varepsilon$ by adding and subtracting $\varphi(y_1)$ and $f_k(y_1)$. The proof of Theorem 2 is now complete.

3. Two examples. In this section we present two examples which give some indication as to the flavor of the subject. The first is an example of a function of bounded $\varkappa$-variation which is not of bounded variation. The second example shows that the total $\varkappa$-variation *function*:

$$\varkappa V_f(x) = \varkappa V(f, x) = \varkappa V(f \circ \alpha), \quad \alpha(t) = xt, \quad 0 \leqslant t \leqslant 1, \tag{15}$$

namely the total $\varkappa$-variation of f on $[0, x]$, need *not* be increasing with x, a situation that certainly does not prevail in any of the concepts of variation mentioned in the introduction.

We begin with a simple fact concerning functions with infinite slope at the origin (see (1) of Section 2).

LEMMA 1. *There exists an infinite sequence* $\{a_i\}$ *of positive numbers such that* $\sum a_i = 1$ *and* $\sum \varkappa(a_i) = +\infty$.

Proof. Choose any sequence of positive numbers α_m so that $\sum \alpha_m = 1$. Since $\varkappa$ is concave down, it follows from (1) that $\varkappa(x)/x$ is *increasing* to $+\infty$ as $x \to 0+$. Hence for each $m = 1, 2, \ldots$ one can choose a positive b_m so small that $\varkappa(b_m)/b_m \geqslant 1/\alpha_m$ and so that $\alpha_m/b_m = k_m$ is an integer.

Now choose the sequence $\{a_i\}$ to be the numbers b_m, each b_m repeated k_m times. Then

$$\sum a_i = \sum k_m b_m = \sum \alpha_m = 1,$$

and

$$\sum \varkappa(a_i) = \sum k_m \varkappa(b_m) \geqslant \sum k_m b_m/\alpha_m = \sum 1 = +\infty.$$

EXAMPLE 1. For the a_i of Lemma 1 set $x_0 = 0$, $x_i = a_1 + \ldots + a_i$, $i = 1, 2, \ldots$ and define for $0 \leqslant x < 1$

$$f(x) = \varkappa(x - x_i), \quad x_i \leqslant x < x_{i+1}, \quad i = 0, 1, 2, \ldots,$$

and let $f(1) = 0$. Then f is not of bounded variation for the total variation is clearly $2\sum \varkappa(a_i) = +\infty$. However, f is $\varkappa$-decreasing and therefore of bounded $\varkappa$-variation. To see this let $0 \leqslant x < y < 1$ be arbitrary. Then there exist unique integers $m \leqslant n$ such that $x_n \leqslant x < x_{n+1}$ and $x_m \leqslant y < x_{m+1}$. But $f(y) - f(x) = \varkappa(y - x_m) - \varkappa(x - x_n) \leqslant \varkappa(y - x_n) - \varkappa(x - x_n) \leqslant \varkappa(y - x)$, where the last inequality follows from (2), the subadditivity of $\varkappa$. Hence f is $\varkappa$-decreasing with constant 1, and by Theorem 1, f is of bounded $\varkappa$-variation with $\varkappa V(f) \leqslant 2$.

EXAMPLE 2. Fix $0 < a < 1$ and $0 \leqslant b \leqslant 1$. Let f be the continuous function, linear on the intervals $[0, a]$ and $[a, 1]$, and such that $f(0) = 0$, $f(a) = 1$, $f(1) = b$. To compute the total $\varkappa$-variation of f, $\varkappa V(f)$, one must maximize the competing ratios $\sum |f(I_i)| / \sum \varkappa(|I_i|)$ over all partitions $P = \{I_i\}$ of $[0, 1]$. But using the piecewise linearity of f and the concavity (and therefore subadditivity) of $\varkappa$, it is elementary to show that the only partitions that need be considered are $\{0, 1\}$ and $\{0, a, 1\}$. The conclusion is

$$\varkappa V(f) = \max\left\{b, \frac{2-b}{\varkappa(a) + \varkappa(1-a)}\right\}. \tag{16}$$

We make the following observations. First of all, if $f(1) = b \geqslant 1$, f is increasing and as remarked earlier $\varkappa V(f) = V(f) = b$. However, it follows (by equating the two quantities in (16) and solving for b), that there exists a b_0, $0 < b_0 < 1$, namely

$$b_0 = \frac{2}{1 + \varkappa(a) + \varkappa(1-a)}, \tag{17}$$

such that if $b_0 \leqslant b \leqslant 1$, the total $\varkappa$-variation of f remains equal to $b = f(1)$!

Finally we show the following. If $\frac{1}{2} \leqslant a < 1$, and if $0 < b < 1$ is chosen sufficiently close to 1, the total variation function

$$\varkappa V_f(x) = f(x).$$

In other words, the variation function need not be increasing! In order to see this, it follows from definition (15) that for each $0 \leqslant x \leqslant 1$, $\varkappa V_f(x) = \varkappa V(f_x)$, where $f_x(t) = f(xt)$, $0 \leqslant t \leqslant 1$. Hence, if $0 \leqslant x \leqslant a$, $f_x(t)$ is increasing and so $\varkappa V_f(x) = V(f_x) = f(x)$.

Next if $a \leqslant x \leqslant 1$ it follows from (16) and (17) that

$$\varkappa V_f(x) = \varkappa V(f_x) = \begin{cases} f_x(1) = f(x) & \text{if} \quad f(x) \geqslant \dfrac{2}{1 + H(a/x)}, \\ \dfrac{2 - f(x)}{H(a/x)} & \text{if} \quad f(x) \leqslant \dfrac{2}{1 + H(a/x)}, \end{cases}$$

where $H(x) = \varkappa(x) + \varkappa(1-x)$. Now $\varkappa$ is concave and therefore so is H. Furthermore H is decreasing on $\frac{1}{2} \leqslant x \leqslant 1$ and hence $H(a/x)$ is concave for $a \leqslant x \leqslant 1$ provided $\frac{1}{2} \leqslant a < 1$. Thus the function $g(x) = 2/(1 + H(a/x))$ is convex for $a \leqslant x \leqslant 1$. Since $g(a) = 1 = f(a)$ and $g(1) < 1$, it follows that if $b = f(1)$ is chosen so that $g(1) \leqslant f(1) \leqslant 1$, then the straight line $f(x)$ dominates $g(x)$ for $a \leqslant x \leqslant 1$ and therefore $\varkappa V_f(x) = f(x)$ for $a \leqslant x \leqslant 1$ also.

4. The decomposition theorem.

THEOREM 3. *Every function f of bounded $\varkappa$-variation, $\varkappa V(f) = C$, is the difference of two $\varkappa$-decreasing functions*: $f = g - h$,

$$g(y) - g(x) \leqslant \bar{C}\varkappa(y - x), \quad h(y) - h(x) \leqslant \bar{C}\varkappa(y - x), \quad 0 \leqslant x < y \leqslant 1. \tag{18}$$

If $f(0) = f(1)$, then one can choose g and h to agree with f at 0 *and* 1, *and in this case $\bar{C} = \frac{1}{2}C$. In general, $\bar{C} \leqslant \frac{5}{4}C$.*

The proof of Theorem 3 is in three stages. Without loss of generality we can assume $f(0) = 0$. Suppose first that the theorem has been proved for functions that vanish also at $x = 1$, and that f is now given with $f(1) \neq 0$. Define $\bar{f}(x)$ to be $f(x)$ for $0 \leqslant x < 1$ and $\bar{f}(1) = 0$.

If $P = \{x_i\}$ is a partition of $[0, 1]$, then

$$\sum_{i=1}^{n} |\bar{f}(x_i) - \bar{f}(x_{i-1})| \leqslant \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + |f(x_{n-1})|.$$

But f is bounded by $\frac{3}{2}C$ (see (6)) and so the right-hand side above is bounded by

$$C \sum_{i=1}^{n} \varkappa(x_i - x_{i-1}) + \tfrac{3}{2}C \leqslant \tfrac{5}{2}C \sum_{i=1}^{n} \varkappa(x_i - x_{i-1}).$$

Hence $\bar{f}$ is of bounded $\varkappa$-variation and $\varkappa V(\bar{f}) \leqslant 5C/2$.

It follows from our assumption that $\bar{f} = \bar{g} - \bar{h}$ where $\bar{g}$ and $\bar{h}$ both vanish at 0 and 1, and are $\varkappa$-decreasing with constant $5C/4$. Now if $f(1) < 0$, define g by $g(x) = \bar{g}(x)$, $0 \leqslant x < 1$, $g(1) = f(1)$ and set $h = \bar{h}$. If $f(1) > 0$, set $g = \bar{g}$ and define h by $h(x) = \bar{h}(x)$, $0 \leqslant x < 1$, $h(1) = -f(1)$. In either case $f = g - h$. To see that g and h are $\varkappa$-decreasing, consider, for example, g in the former case. One need only show that (7) is valid for intervals $I = [x, y]$ where $0 \leqslant x < y = 1$. But then

$$g(1) - g(x) = g(1) + \bar{g}(1) - \bar{g}(x) \leqslant f(1) + \tfrac{5}{4} C\varkappa(1-x) \leqslant \tfrac{5}{4} C\varkappa(1-x),$$

since $f(1) < 0$. Similarly in the latter case h is $\varkappa$-decreasing with constant $5C/4$.

The second stage of the proof proceeds as follows. Suppose Theorem 3 is true for continuous piecewise linear functions which vanish at $x = 0$ and 1. Let f be an arbitrary function of bounded $\varkappa$-variation, $\varkappa V(f) = C$, $f(0) = 0 = f(1)$. Enumerate the (countably many) points of discontinuity of f, and choose a sequence P_n of partitions of $[0, 1]$ such that the *mesh*, $|P_n| \to 0$ as $n \to \infty$, and being sure to include in P_n the first n points of discontinuity of f from the above list of discontinuities. Then if f_n is the continuous piecewise linear function which coincides with f at the points of P_n, one easily shows $f_n \to f$, $0 \leqslant x \leqslant 1$. By Theorem 3 of [4], p. 204, each f_n is of bounded $\varkappa$-variation and $\varkappa V(f_n) \leqslant \varkappa V(f) = C$. By our assumption each $f_n = g_n - h_n$, where g_n and h_n are all $\varkappa$-decreasing with constant $C/2$ and uniformly bounded by $3C/2$ (see Theorem 1 and (6)). Hence by the Helly-type selection Theorem 2, there exist subsequences of g_n and h_n converging respectively to functions g and h which are $\varkappa$-decreasing with constant $C/2$. But $f_n \to f$ and so $f = g - h$, which completes the proof.

The third and final stage of the proof is to prove Theorem 3 for continuous piecewise linear functions which vanish at 0 and 1. Suppose then that P: $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$ is a *fixed* partition of $[0, 1]$ and let f be a continuous function which is linear in each interval $[x_{i-1}, x_i]$, with $f(0) = f(1) = 0$. Suppose f is of bounded $\varkappa$-variation, $\varkappa V(f) = C$. Then inequality (3) is valid for each partition of $[0, 1]$. In particular, (3) is valid for each subpartition of P. There are 2^n such subpartitions (there being $\binom{n}{i}$ subpartitions containing $i+2$ points, 0 and 1 and i interior points). Thus we have 2^n inequalities of type (3) which we take as hypotheses.

The task at hand is to construct two functions g and h such that $f = g - h$. We want g and h to be continuous, to vanish at 0 and 1, to be linear in each interval $[x_{i-1}, x_i]$, $i = 1, \ldots, n+1$, and to satisfy inequality (7) with constant $C/2$ for every interval $I \subset [0, 1]$. But using the piecewise linearity of g and h and the concavity of $\varkappa$, it is easy to show that g and h need satisfy inequality (7) only for intervals $I = [x_i, x_j]$, where $x_i < x_j$ are arbitrary points of the fixed partition P.

Thus the problem becomes a finite-dimensional one. Find $2n$ numbers, $g(x_i)$, $h(x_i)$, $i = 1, \ldots, n$, such that $f(x_i) = g(x_i) - h(x_i)$ (f, g and h all vanish at 0 and 1), and subject to the inequalities

$$g(x_j) - g(x_i) \leqslant \tfrac{1}{2} C\varkappa(x_j - x_i), \qquad h(x_j) - h(x_i) \leqslant \tfrac{1}{2} C\varkappa(x_j - x_i),$$
$$0 \leqslant i < j \leqslant n+1.$$

By putting $g(x_i) = f(x_i) + h(x_i)$ into these two sets of inequalities, we can summarize the problem as follows.

Let P: $0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1$ be a fixed partition of $[0, 1]$. Let $f(0) = f(1) = 0$ and let $f(x_i)$, $i = 1, \ldots, n$, be n arbitrary real numbers subject to 2^n inequalities of type (3), *one to each subpartition of P. Find n numbers $h(x_i)$, $i = 1, \ldots, n$, $h(0) = h(1) = 0$, subject to the inequalities*

$$h(x_j) - h(x_i) \leqslant \tfrac{1}{2} C\varkappa(x_j - x_i)$$
$$h(x_j) - h(x_i) \leqslant \tfrac{1}{2} C\varkappa(x_j - x_i) - [f(x_j) - f(x_i)],$$

$0 \leqslant i < j \leqslant n+1$. Using the notation

$$a^+ = \max(a, 0), \qquad a^- = \max(-a, 0),$$

we can combine the above two sets of inequalities to the one set

$$h(x_j) - h(x_i) \leqslant \tfrac{1}{2} C\varkappa(x_j - x_i) - [f(x_j) - f(x_i)]^+, \qquad 0 \leqslant i < j \leqslant n+1. \tag{19}$$

First of all, an admissible range for each $h(x_i)$ can be found by first setting $0 = x_i < x_j < 1$ and then setting $0 < x_i < x_j = 1$ into (19). The result is

$$f(x_i)^- - \tfrac{1}{2} C\varkappa(1 - x_i) \leqslant h(x_i) \leqslant \tfrac{1}{2} C\varkappa(x_i) - f(x_i)^+. \tag{20}$$

The intervals

$$I_i = [f(x_i)^- - \tfrac{1}{2} C\varkappa(1 - x_i), \tfrac{1}{2} C\varkappa(x_i) - f(x_i)^+],$$

$i = 1, \ldots, n$, are not empty. For if one writes down inequality (3) using the partition $\{0, x_i, 1\}$, the result is $|f(x_i) - f(0)| + |f(1) - f(x_i)| = 2|f(x_i)| = 2f(x_i)^+ + 2f(x_i)^- \leqslant C\varkappa(x_i) + C\varkappa(1 - x_i)$, which implies the left endpoint of I_i is indeed to the left of the right endpoint.

We now define each $h(x_j)$ using the following recursive algorithm. Set

$$h(x_1) = \tfrac{1}{2} C\varkappa(x_1) - f(x_1)^+, \tag{21}$$

and for each $j = 2, \ldots, n$ define

$$h(x_j) = \min_{1 \leqslant i < j} \{\tfrac{1}{2} C\varkappa(x_j) - f(x_j)^+,\ h(x_i) + \tfrac{1}{2} C\varkappa(x_j - x_i) - [f(x_j) - f(x_i)]^+\}. \tag{22}$$

It follows from the definition that each $h(x_j)$ satisfies (19). The construction will be complete if we can show that each $h(x_j)$ satisfies (20) as well, i.e. $h(x_j) \in I_j$. Since the first quantity in the bracket of (22) is the right endpoint of I_j, $h(x_j)$ automatically satisfies the right inequality of (20). The number $h(x_j)$ will also satisfy the left inequality of (20) provided that for *each* $i = 1, \ldots, j-1$, the second quantity in the bracket of (22) is $\geqslant$ the left endpoint of I_j:

(23) $$f(x_j)^- - \tfrac{1}{2} C\varkappa(1-x_j) \leqslant h(x_i) + \tfrac{1}{2} C\varkappa(x_j - x_i) - [f(x_j) - f(x_i)]^+.$$

However, inequality (23) reduces exactly to inequality (3) for a certain subpartition of P, that subpartition depending upon which choice of the *minima* was made in (22) for each $h(x_i)$, $i < j$, that appears in (23). The details can be easily verified by the reader. One needs to use the simple fact that for any sequence of real numbers, $0 = a_0, a_1, \ldots, a_n, a_{n+1} = 0$,

$$\sum_{i=1}^{n} (a_i - a_{i-1})^+ + a_n^- = \tfrac{1}{2} \sum_{i=1}^{n+1} |a_i - a_{i-1}|.$$

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