bounded ([4], Theorem 4.2, and [9], p. 11, Prop. 3.4) and $\lim_{n\to+\infty} T^n = \lim_{n\to+\infty} P^n$ where $P = \lim_{n\to+\infty}\left(\frac{I+\ldots+T^{n-1}}{n}\right)$ ([9], Lemma 3.3, p. 11); therefore one has $\lim_{n\to+\infty} \|T^n\| \leqslant 1$ for any positive C-contraction matrix.

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Borel's theorem for generalized functions

by

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Abstract. Generalized complex numbers and new generalized functions were introduced in order to give a meaning to both the value of any distribution at any point and to any finite product of distributions. In this paper we prove: Given any sequence (c_n) of generalized complex numbers, there is a generalized function f on $\boldsymbol{R}$ such that $f^{(n)}(0) = c_n$ for all n. This result shows a coherence between generalized numbers and functions similar to that of the classical case.

Introduction. One of the authors introduced a generalized mathematical analysis in order to give a mathematical sense to any finite product of distributions and to classical heuristic computations done by physicists, see Colombeau [1], [2], [3], [4]. This generalized mathematical analysis deals with new generalized functions, more general than distributions, and with generalized complex numbers such that, if G is any generalized function on $\Omega \subset \boldsymbol{R}^n$ open and if $x \in \Omega$ then $G(x)$ is defined as a generalized complex number.

In this paper we prove Borel's theorem in this setting: given any family $\{c_\alpha\}_{\alpha\in N^n}$ of generalized complex numbers, there is a generalized function G on $\boldsymbol{R}^n$ such that, for any $\alpha \in N^n$,

$$\left(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} G\right)(0) = c_\alpha .$$

This shows a deep connection between our generalized functions and our generalized complex numbers, similar to the classical case. The proof is more technical than the classical one given in Narasimhan [5], since we have to do more detailed computations and estimates.

We use the concepts of generalized functions and the terminology defined in Colombeau [3]. According to Colombeau [3], we consider an algebra $\bar{\boldsymbol{C}}^*$ such that if $G \in \mathscr{G}^*(\Omega)$ and $x \in \Omega$ then the value $G(x)$ is defined as an element of $\bar{\boldsymbol{C}}^*$.

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THEOREM. *Given, for each* $\alpha=(\alpha_1,\ldots,\alpha_n)\in N^n$, *an element* c_α *in* $\bar{C}^*$, *there exists* $G\in\mathscr{G}^*(R^n)$ *such that*

$$D^\alpha G(0)=\alpha!\,c_\alpha.$$

Proof. Let, for each $\beta\in N^n$, $\tilde{c}_\beta\in\mathscr{E}_M^*$ be a representative of c_β. By definition there exists $N\in N$ such that, for each $\varphi\in\mathscr{A}_N$ with $\operatorname{diam}(\operatorname{supp}\varphi)=1$, there is $\eta(\varphi)>0$ such that $\tilde{c}_\beta(\varphi_\varepsilon)$ is defined for $0<\varepsilon<\eta(\varphi)$. Notice that for all $\psi\in\mathscr{A}_N$ there exist a unique $\varphi\in\mathscr{A}_N$ with $\operatorname{diam}(\operatorname{supp}\varphi)=1$ and a unique ε such that $\psi=\varphi_\varepsilon$.

Let $\bar{c}_\beta\colon \mathscr{A}_1\to C$ be defined by:

(i) $\bar{c}_\beta(\psi)=0$ if $\psi\in\mathscr{A}_1$ and $\psi\notin\mathscr{A}_N$,

(ii) $\bar{c}_\beta(\psi)=0$ if $\psi=\varphi_\varepsilon\in\mathscr{A}_N$ (φ and ε as above) and $\varepsilon\geqslant\eta(\varphi)$,

(iii) $\bar{c}_\beta(\psi)=\tilde{c}_\beta(\varphi_\varepsilon)$ if $\psi=\varphi_\varepsilon\in\mathscr{A}_N$ (φ and ε as above) and $0<\varepsilon<\eta(\varphi)$.

It is immediate that $\bar{c}_\beta\in\mathscr{E}_M^*$ is also a representative of c_β.

For each $\varphi_\varepsilon\in\mathscr{A}_1$, $x\in R^n$ and $m\in N$, let us define

$$T_m(\varphi_{\varepsilon,x})=\sum_{|\beta|=m}\bar{c}_\beta(\varphi_\varepsilon)x^\beta.$$

Then

(1) $$D^\alpha[x\to T_{m+1}(\varphi_{\varepsilon,x})](0)=0$$

for all α with $|\alpha|\leqslant m$.

ASSERTION 1. *For each* $m\in N$ *there is* $N_m\in N$ *such that for all* $\varphi\in\mathscr{A}_{N_m}$ *with* $\operatorname{diam}(\operatorname{supp}\varphi)=1$ *there exists* $\eta_m>0$ *such that if* $0<\varepsilon<\eta_m$ *there exists* $g_{\delta(m,\varphi_\varepsilon)}\in\mathscr{E}(R^n)$ *vanishing in a neighborhood of zero such that*

(2) $$\sum_{|\alpha|\leqslant m}\frac{1}{\alpha!}\sup_{x\in R^n}|D^\alpha[T_{m+1}(\varphi_{\varepsilon,x})-g_{\delta(m,\varphi_\varepsilon)}(x)]|\leqslant\frac{1}{2^m}.$$

Proof. In fact, since $\bar{c}_\beta\in\mathscr{E}_M^*$, there is $N_m\in N$ such that if $\varphi\in\mathscr{A}_{N_m}$ there are $D_m>0$, $\eta_m>0$ such that

$$|\bar{c}_\beta(\varphi_\varepsilon)|\leqslant\frac{D_m}{\varepsilon^{N_m}}\quad\text{if}\quad 0<\varepsilon<\eta_m,$$

for all β with $|\beta|=m+1$.

Let $f\in\mathscr{E}(R^n)$ be such that $f(x)\geqslant0$ for all x and $f(x)=0$ if $|x|\leqslant 1/2$, $f(x)=1$ if $|x|\geqslant1$. For each $\alpha\in N^n$ let

(3) $$M_\alpha=\max_{|\nu|\leqslant|\alpha|}\binom{\alpha}{\nu}\sup_{x\in R^n}|D^\nu f(x)|<\infty,$$

let

(4) $$\delta=\delta(m,\varphi_\varepsilon)=\min\left\{\frac{\varepsilon^{N_m}}{C_m 2^{m+1}},1\right\}$$

where

(5) $$C_m=\max\left\{D_m\sum_{\substack{|\alpha|\leqslant m\\|\beta|=m+1}}\binom{\beta}{\alpha},\ D_m\sum_{\substack{|\alpha|\leqslant m\\|\beta|=m+1}}\frac{M_\alpha}{\alpha!}\sum_{|\mu|\leqslant|\alpha|}\frac{\beta!}{(\beta-\mu)!}\right\}.$$

If we define g_δ by

$$g_\delta(x)=f\left(\frac{x}{\delta}\right)T_{m+1}(\varphi_{\varepsilon,x}),$$

then, clearly, $g_\delta\in\mathscr{E}(R^n)$ and vanishes near zero. Also

$$g_\delta(x)=T_{m+1}(\varphi_{\varepsilon,x})\quad\text{if}\quad |x|\geqslant\delta.$$

It is therefore sufficient to consider, in (2), the supremum over $\{x\in R^n\colon |x|\leqslant\delta\}$. We have

$$\sup_{|x|\leqslant\delta}\Big|D^\alpha\Big(\sum_{|\beta|=m+1}\bar{c}_\beta(\varphi_\varepsilon)x^\beta\Big)\Big|\leqslant\sup_{|x|\leqslant\delta}\sum_{|\beta|=m+1}|\bar{c}_\beta(\varphi_\varepsilon)|\frac{\beta!}{(\beta-\alpha)!}|x|^{|\beta|-|\alpha|}$$
$$\leqslant\frac{D_m}{(\varepsilon)^{N_m}}\sum_{|\beta|=m+1}\frac{\beta!}{(\beta-\alpha)!}\delta^{m+1-|\alpha|}$$

if $0<\varepsilon<\eta_m$ and $|\alpha|\leqslant m$. Then from (4) and (5)

(6) $$\sum_{|\alpha|\leqslant m}\frac{1}{\alpha!}\sup_{|x|\leqslant\delta}\Big|D^\alpha\Big(\sum_{|\beta|=m+1}\bar{c}_\beta(\varphi_\varepsilon)x^\beta\Big)\Big|\leqslant\frac{D_m}{(\varepsilon)^{N_m}}\sum_{\substack{|\alpha|\leqslant m\\|\beta|=m+1}}\binom{\beta}{\alpha}\delta\leqslant\frac{C_m\delta}{(\varepsilon)^{N_m}}\leqslant\frac{1}{2^{m+1}}$$

if $0<\varepsilon<\eta_m$. Now we have, from the definition of g_δ and (3)

(7) $$|D^\alpha g_\delta(x)|\leqslant\sum_{\mu+\nu=\alpha}\binom{\alpha}{\nu}\delta^{-|\nu|}\left|D^\nu f\left(\frac{x}{\delta}\right)\right|\left|D^\mu\sum_{|\beta|=m+1}\bar{c}_\beta(\varphi_\varepsilon)x^\beta\right|$$
$$\leqslant M_\alpha\sum_{\mu+\nu=\alpha}\delta^{-|\nu|}\sum_{|\beta|=m+1}|\bar{c}_\beta(\varphi_\varepsilon)|\frac{\beta!}{(\beta-\mu)!}|x|^{m+1-|\mu|}$$
$$\leqslant\frac{M_\alpha D_m}{(\varepsilon)^{N_m}}\sum_{\mu+\nu=\alpha}\delta^{-|\nu|}\sum_{|\beta|=m+1}\frac{\beta!}{(\beta-\mu)!}|x|^{m+1-|\mu|}$$

if $0<\varepsilon<\eta_m$. From (5) and (7)

(8) $$\sum_{|\alpha|\leqslant m}\frac{1}{\alpha!}\sup_{|x|\leqslant\delta}|D^\alpha g_\delta(x)|\leqslant\frac{C_m\delta}{(\varepsilon)^{N_m}}\leqslant\frac{1}{2^{m+1}}$$

if $0<\varepsilon<\eta_m$. (6) and (8) prove the assertion.

Let $U=\bigcup_{x\in R^n}\tau_x\mathscr{A}_1\in\mathscr{F}^*$ and let $F\colon U\to C$ be defined by

$$F(\varphi_{\varepsilon,x})=T_0(\varphi_{\varepsilon,x})+\sum_{m=0}^{\infty}[T_{m+1}(\varphi_{\varepsilon,x})-g_\delta(x)]$$

which makes sense from (2). Still from (2) $[x \to F(\varphi_{\varepsilon,x})] \in \mathscr{E}(\boldsymbol{R}^n)$; therefore $F \in \mathscr{E}^*(U)$.

Note that we have chosen the number δ given by (4) in order to prove that

ASSERTION 2. $F \in \mathscr{E}^*_M(\boldsymbol{R}^n_{\mathscr{D}(\boldsymbol{R}^n)})$.

Proof. Given $K \subset \boldsymbol{R}^n$ compact and $\alpha \in N^n$ with $|\alpha| = r$, we have

$$|D^\alpha F(\varphi_{\varepsilon,x})| \leqslant |D^\alpha T_0(\varphi_{\varepsilon,x})| + \sum_{m=0}^{r-1} |D^\alpha T_{m+1}(\varphi_{\varepsilon,x})| + \sum_{m=0}^{r-1} |D^\alpha g_\delta(x)| + \\ + \sum_{m=r}^{\infty} |D^\alpha [T_{m+1}(\varphi_{\varepsilon,x}) - g_\delta(x)]|.$$

As in (7), if $x \in K$,

$$\sum_{m=0}^{r-1} |D^\alpha g_\delta(x)| \leqslant M_\alpha \sum_{m=0}^{r-1} \sum_{\mu+\nu=\alpha} \delta^{-|\nu|} \sum_{|\beta|=m+1} |\bar{c}_\beta(\varphi_\varepsilon)|\, |D^\mu x^\beta| \\ \leqslant M_\alpha \sum_{m=0}^{r-1} \frac{D_m L_m}{(\varepsilon)^{N_m}} \sum_{|\nu| \leqslant |\alpha|} \max\left\{\left(\frac{C_m 2^{m+1}}{(\varepsilon)^{N_m}}\right)^{|\nu|}, 1\right\}$$

where

$$L_m = \sup_{\substack{|\mu| \leqslant |\alpha| \\ x \in K}} \Big\{ \sum_{|\beta|=m+1} |D^\mu x^\beta| \Big\}.$$

Let $N = \max_{0 \leqslant m \leqslant r-1} N_m$ and $\eta = \min_{0 \leqslant m \leqslant r-1} \{1, \eta_m\}$. Then

$$\sum_{m=0}^{r-1} |D^\alpha g_\delta(x)| \leqslant M_\alpha \sum_{m=0}^{r-1} \frac{D_m L_m}{\varepsilon^{N(1+r)}} \sum_{|\nu| \leqslant |\alpha|} \max\{(C_m 2^{m+1})^{|\nu|}, 1\} \leqslant \frac{C'}{\varepsilon^{N(1+r)}} \tag{9}$$

where $C' > 0$, and this bound is independent of $x \in K$ and $0 < \varepsilon < \eta$.

Now,

$$\sum_{m=0}^{r-1} |D^\alpha T_{m+1}(\varphi_{\varepsilon,x})| = \sum_{m=0}^{r-1} \Big|D^\alpha \Big(\sum_{|\beta|=m+1} \bar{c}_\beta(\varphi_\varepsilon) x^\beta \Big)\Big| \\ \leqslant \sum_{m=0}^{r-1} \sum_{|\beta|=m+1} \frac{D_m}{(\varepsilon)^{N_m}} |D^\alpha x^\beta| \leqslant \sum_{m=0}^{r-1} \frac{D_m L_m}{(\varepsilon)^{N_m}} \\ \leqslant \frac{\sum_{m=0}^{r-1} D_m L_m}{\varepsilon^N} = \frac{C''}{\varepsilon^N} \tag{10}$$

if $0 < \varepsilon < \eta$ and $x \in K$.

If $r = 0$, $|D^\alpha T_0(\varphi_{\varepsilon,x})| = |\bar{c}_0(\varphi_\varepsilon)|$ then there is $N' \in N$ such that if $\varphi \in \mathscr{A}_1$

there are $\bar{C}_0$, $\bar{\eta}_0 > 0$ such that

$$|\bar{c}_0(\varphi_\varepsilon)| \leqslant \frac{\bar{C}_0}{\varepsilon^N} \quad \text{if} \quad 0 < \varepsilon < \bar{\eta}_0.$$

If $r > 0$, $D^\alpha T_0(\varphi_{\varepsilon,x}) = 0$.

With the above, (9), (10) and Assertion 1 we have proved Assertion 2.

Let $G \in \mathscr{G}^*(\boldsymbol{R}^n)$ be the class of F. We will prove the theorem.

If $\varphi_\varepsilon \in \mathscr{A}_1$ we have

$$D^\alpha [x \to F(\varphi_{\varepsilon,x})](0) = D^\alpha \Big[x \to T_0(\varphi_{\varepsilon,x}) + \sum_{m=0}^{\infty} \big(T_{m+1}(\varphi_{\varepsilon,x}) - g_\delta(x)\big)\Big](0).$$

By (1) and since g_δ is zero for $|x|$ small enough, we have, if $k \in \boldsymbol{N}$,

$$D^\alpha \Big[x \to \sum_{m=k}^{\infty} \big(T_{m+1}(\varphi_{\varepsilon,x}) - g_\delta(x)\big)\Big](0) = 0$$

for all α with $|\alpha| \leqslant k$.

Then, if $|\alpha| \leqslant k$,

$$D^\alpha [x \to F(\varphi_{\varepsilon,x})](0) = D^\alpha \Big[x \to \sum_{m=0}^{k} T_m(\varphi_{\varepsilon,x})\Big](0) = D^\alpha \Big[x \to \sum_{|\beta| \leqslant k} \bar{c}_\beta(\varphi_\varepsilon) x^\beta\Big](0) \\ = \Big[x \to \sum_{|\alpha| \leqslant |\beta| \leqslant k} \bar{c}_\beta(\varphi_\varepsilon) \frac{\beta!}{(\beta-\alpha)!} x^{\beta-\alpha}\Big](0) = \alpha!\, \bar{c}_\beta(\varphi_\varepsilon).$$

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