On the pointwise ergodic theorems in $L_p$ ($1 < p < \infty$)

by

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Abstract. Using M. A. Akcoglu's estimate [1] we show that

$$\sup_{\alpha > 1} \left\| \frac{1}{n} \alpha + \cdots + T^{n-1} f \right\|_p \leq \frac{p}{p-1} \left\| f \right\|_p$$

for any $f \in L_p$ ($1 < p < \infty$) and any positive operator $T$ on $L_p$ which verifies

$$\sup_{\alpha > 1} \left\| \frac{1}{n} \alpha + \cdots + T^{n-1} \right\|_p \leq 1$$

or more generally $\sup_{\alpha > 1} \left\| \frac{1}{n} \alpha + \cdots + T^{n-1} \right\|_p \leq 1$. For such operators (which are not necessarily contractions) we also obtain the pointwise ergodic theorem in $L_p$.

Introduction. Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and $T$ a positive operator on $L^1(X, \mathcal{F}, \mu) = L_p$, $1 < p < \infty$.

M. A. Akcoglu's powerful estimate [1] is

$$\sup_{\alpha > 1} \left\| \frac{1}{n} \alpha + \cdots + T^{n-1} f \right\|_p \leq \frac{p}{p-1} \left\| f \right\|_p$$

for any $f \in L_p$ and any positive contraction $T$ on $L_p$.

A trivial example shows that a positive operator $T$ on $L_p$ ($1 < p < \infty$) which verifies

$$(*) \quad \sup_{\alpha > 1} \left\| \frac{1}{n} \alpha + \cdots + T^{n-1} \right\|_p \leq 1$$

is not necessarily a positive contraction on $L_p$ ($1 < p < \infty$); take $X = \{1, 2\}$, $\mu[1] = \mu[2] = 1$ and $T = \begin{bmatrix} 0 & 1 + \varepsilon \\ 0 & 0 \end{bmatrix}$ with $\varepsilon > 0$ small enough.

Of course, the converse is true.

In this paper we show that M. A. Akcoglu's estimate [1] yields an

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estimate and the pointwise ergodic theorem for positive operators on $L_p (1 < p < +\infty)$ which verify (+) or more generally, \( \sup_{0 < k < 1} |\sum_{i=0}^{\infty} k^i T_i| \leq 1 \).

**Acknowledgement.** The problem of mean and pointwise ergodic convergence for mean-bounded positive operators on $L_p (1 < p < +\infty)$ has been introduced by Professor A. Brunel. The mean ergodic theorem is proved in [4]. I would like to express my gratitude to Professor A. Brunel for his interest in the present work.

1. We recall the

**Definition.** A resolvent on a vector space $B$ is a family \((\lambda V_\lambda)_{\lambda > 0}\) of linear operators on $B$ such that $V_\lambda - V_{\mu} = -(\lambda - \mu) V_\lambda V_{\mu}$ for all $\lambda, \mu > 0$.

Examples of resolvent are

\[ V_\lambda = \sum_{k=0}^{\infty} \frac{T^k}{(\lambda + 1)^{k+1}} \text{ where } T \text{ is a linear operator on } B; \]

\[ V_\lambda = \int_0^\infty e^{-\lambda s} T_\lambda ds \text{ where } (T_\lambda)_{\lambda > 0} \text{ is a semi-group of operators}. \]

Ergodic theorem for resolvents on $L_1$ were obtained by D. Feyel [2] and R. Sato [7], [8].

The following appears as a consequence of M. A. Akcoglu's estimate [1] and the Hille-Yosida theorem ([6], p. 261).

(1.1) **Theorem.** Let \( V = (\lambda V_\lambda)_{\lambda > 0} \) be a resolvent on $L_p (X, \mathcal{F}, \mu) (1 < p < +\infty)$ such that $\lambda V_\lambda$ is a positive contraction for any $\lambda > 0$.

Then, for any $f \in L_p$, one has

\[ \frac{1}{k} \sum_{j=0}^{k-1} T_\lambda^{j+1} - \frac{1}{k} \sum_{j=0}^{k-1} T_\lambda^{j+1} f \leq \left( \frac{k}{2^{k+1}} \right) \int_0^\infty T_\lambda f du \]

for any $t = k2^{-n} \in D_\lambda$.

Hence M. A. Akcoglu's estimate [1] applied to $T_\lambda^{k2^{-n}}$ gives us

\[ \left| \sup_{k \neq 0} \left( \frac{1}{k} \sum_{j=0}^{k-1} T_\lambda^{j+1} f \right) \right| \leq \frac{p}{p-1} \left| \frac{1}{2^{k+1}} \sum_{j=0}^{k-1} T_\lambda^{j+1} f \right| \]

This implies

\[ \left| \sup_{k \neq 0} \left( \frac{1}{k} \sum_{j=0}^{k-1} T_\lambda^{j+1} f \right) \right| \leq \frac{p}{p-1} \left| \frac{1}{2^{k+1}} \sum_{j=0}^{k-1} T_\lambda^{j+1} f \right| \]

for any $f \in L_p$.
Thus, if \( f \in L_p^* \) and \( h = T_0 f \), one obtains
\[
\|\sup_{\lambda > 0} \lambda V_\lambda f\| = \|\sup_{\lambda > 0} \lambda V_\lambda (T_0 f)\| \leq \frac{p}{p-1} \|f\|.
\]
Hence, for any \( f \in L_p \), \( \|\sup_{\lambda > 0} \lambda V_\lambda f\| \leq \frac{p}{p-1} \|f\| \). This proves (1.2).

We now prove (1.3). Inequality (1.2) shows that the set \( C = \{f \in L_p : \lambda V_\lambda f \text{ exists a.e. on } X_\lambda \} \) is closed in the strong topology of \( L_p \).

Indeed, if \( x_n \) in \( C \), then, for any \( \varepsilon > 0 \) and \( f_n \in C \) such that \( \|f_n - f\| < \varepsilon \), we have
\[
\|\lim_{n \to \infty} \lambda V_\lambda f_n - \lambda V_\lambda f\| \leq \lim_{n \to \infty} \|\lambda V_\lambda (f_n - f)\| \leq \lim_{n \to \infty} \frac{2p}{p-1} \varepsilon.
\]
Hence \( f \in C \).

Now, \( L_p = \text{Inv} \oplus (I - \lambda V_\lambda^2)(L_p) \), where \( \text{Inv} V = \{f \in L_p : \lambda V_\lambda f = f \text{ for any } \lambda > 0 \} \) (see e.g. [4]). It is clear that \( \text{Inv} V \subseteq C \).

On the other hand, if \( f = (I - \lambda V_\lambda^2)(g) \) for some \( x > 0 \) and \( y \in L_p \) then
\[
\lambda V_\lambda f = \lambda V_\lambda (I - \lambda V_\lambda^2)(g) = \lambda V_\lambda (I - \lambda V_\lambda g) = \lambda V_\lambda g - \lambda^2 V_\lambda^2 g.
\]

Since \( \lambda V_\lambda g \in L_p \), \( \lambda V_\lambda f = 0 \) a.e. on \( X \) and since \( \sup_{\lambda > 0} \lambda V_\lambda (V_\lambda g) < +\infty \) a.e. on \( X \), one has \( \lambda V_\lambda f = 0 \) a.e. on \( X \). Therefore, \( \lambda V_\lambda f = 0 \) a.e. on \( X \) whenever \( f = (I - \lambda V_\lambda^2)(g) \).

Since \( C \) contains \( \text{Inv} V + (I - \lambda V_\lambda^2)(L_p) \), \( C \) is closed in the norm-topology of \( L_p \).

To prove (1.4) consider similarly the set \( C' = \{f \in L_p : \lambda V_\lambda f \text{ exists a.e. on } X_\lambda \} \) closed in the norm-topology of \( L_p \). Let \( f \in L_p \), \( f^* = T_0 f \), \( f^* \in L_p \). Then \( \lambda V_\lambda f \), \( \lambda V_\lambda f^* \) exists for each \( \lambda > 0 \) (the proof of (1.2)).

Therefore, to prove that \( f \in C' \), it suffices to show that \( f^* \in C' \) and since \( C' \) is closed, it also suffices to prove that \( \lambda V_\lambda f^* \in C' \) for each \( \mu > 0 \).

One has
\[
\lambda V_\lambda f = \lambda \lambda V_\lambda f^* = \frac{\lambda - \mu}{\lambda - \mu} \mu V_\lambda f^* - \frac{\mu}{\lambda - \mu} \mu V_\lambda f.
\]

Since \( \mu V_\lambda f \in L_p \), one has \( \mu V_\lambda f < +\infty \) a.e. on \( X \) and
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda - \mu} \mu V_\lambda f = \mu V_\lambda f
\]
a.e. Since \( \sup_{\lambda > 0} |\lambda V_\lambda f| < +\infty \) a.e. (1.2), one has \( \lim_{\lambda \to \infty} \frac{1}{\lambda - \mu} \lambda V_\lambda f = 0 \) a.e.

Therefore \( \lim_{\lambda \to \infty} \lambda V_\lambda \mu V_\lambda f = \mu V_\lambda f \) a.e. on \( X \). Thus, \( \mu V_\lambda f \in C' \) and \( C' = L_p \).

Note that \( \lim_{\lambda \to \infty} \lambda V_\lambda f = T_0 f \) a.e. on \( X \).

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**Remark.** In the proof of (1.1) one has seen that \( \lambda V_\lambda f = \lambda V_\lambda T_0 f \) if \( \lambda \in \mathbb{N} \), \( \lambda V_\lambda T_0 f \) where \( \lambda V_\lambda T_0 f \) is a strongly continuous semi-group of contractions on \( T_0(L_p) \). Note that \( \lambda V_\lambda T_0 f \) is a semi-group on \( L_p \).

Now, if \( (U_{3k})_{k=0}^{\infty} \) is an arbitrary semi-group of \( L_p \)-positive contractions, (1.5) shows that
\[
\|\sup_{\lambda > 0} \lambda V_\lambda f\| \leq \frac{p}{p-1} \|f\| \quad \text{for any } f \in L_p.
\]

This implies (see [3])
\[
\lim_{t \to 0} \frac{1}{t} U_{3t} f = U_{0} f \quad \text{a.e. on } X.
\]

On the other hand, for any \( f \in L_p \), any \( t \geq 1 \), one has
\[
\frac{1}{n+1} \sum_{j=0}^{n} U_{0} f \leq \frac{1}{n} \sum_{j=0}^{n} U_{0} f \leq \frac{1}{n} \sum_{j=0}^{n} U_{0} f \quad \text{where } n = \lfloor t \rfloor.
\]

Since \( \lim_{n \to \infty} \sum_{j=0}^{n} U_{0} f \) exists and is finite a.e. on \( X \), one has
\[
\lim_{t \to \infty} \frac{1}{t} U_{0} f \quad \text{exists and is finite a.e. on } X.
\]

Note that (1.6) (resp. (1.7)) implies (1.4) (resp. (1.3)) (Abelian theorem [10], p. 197), and conversely (1.4) (resp. (1.3)) implies (1.6) (resp. (1.7)) (Tauberian theorem [10], p. 209).

2. We recall [4] that in any Banach space \( B \) a sequence \( (a_n)_{n=1}^{\infty} \) which verifies \( M = \sup_{n=1}^{\infty} \sqrt{a_0 + \cdots + a_n} < +\infty \) necessarily verifies
\[
1 - k \leq \sum_{i=0}^{\infty} k^i a_i \quad \text{for any } k: 0 < k < 1
\]
and
\[
\sup_{0 < k < 1} \left( \prod_{i=0}^{\infty} k^i a_i \right) \leq M.
\]
(2.2) **Definition.** An operator \( T \) on \( B \) will be called a \( C \)-contraction (resp. \( A \)-contraction) if

\[
\sup_{n \in \mathbb{N}} \left( 1 - k \right) \sum_{i=0}^{n} k^i T f_i \leq 1
\]

(resp. \( \sup_{0 < k < 1} \left( 1 - k \right) \sum_{i=0}^{n} k^i T f_i \leq 1 \))

As we said, a \( C \)-contraction is necessarily an \( A \)-contraction (2.1), and a \( C \)-contraction on \( L_p \) (\( 1 < p < \infty \)) is not necessarily a contraction (see the Introduction).

We can now state the dominated ergodic

(2.3) **Theorem.** Let \( T \) be a positive \( A \)-(or \( C \))-contraction on \( L_p \) (\( 1 < p < \infty \)); then, for any \( f \in L_p \), one has

\[
\sup_{0 < k < 1} \left( 1 - k \right) \sum_{i=0}^{n} k^i T f_i \leq \frac{p}{p-1} \| f \|
\]

(2.4)

(2.5)

\[
\sup_{n \in \mathbb{N}} \left( 1 + \ldots + T^{n-1} \right) f \leq \varepsilon \frac{p}{p-1} \| f \|
\]

(2.6) \( f^*(x) = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} f \right) (x) \) exists and is finite a.e. on \( X \),

(2.7) \( \lim_{n \to \infty} T^n f = 0 \) a.e. on \( X \),

(2.8) strong-lim \( \frac{1}{n} \sum_{i=0}^{n-1} f \) exists and \( L_p = \text{Inv} \left( T \right) (L_p) \) [4].

**Remarks.**

- (2.4) generalizes a result of S. A. Mc. Grath [5].
- (2.5) and (2.6) generalize [1].
- \( \lim_{n \to \infty} \sum_{i=0}^{n-1} \| T_i f \| \) exists a.e. on \( X \) as \( k \to 1^- \) (1.3) or as \( k \to 0^+ \) (1.4).
- \( T f^* = f^* \).

**Proof.** (2.4) is a consequence of (1.3), (2.1) and (2.2). Indeed,

\[
\left( 1 - k \right) \sum_{i=0}^{n} k^i T f_i = \lambda \sum_{i=0}^{n} \frac{T^i f}{(\lambda + 1)^{i+1}} \quad (\text{with } k = 1/(\lambda + 1))
\]

is a particular case of positive contractions resolvent (2.2) or (2.1).

(2.5) is an immediate consequence of (2.4). Indeed, for any \( n \in \mathbb{N}, \ n \geq 1 \),

any \( k: 0 < k < 1 \) and any \( f \in L_p \) one has

\[
\frac{1}{n} \sum_{i=0}^{n-1} k^i T f_i \leq \frac{1}{n k^{n-1}} \sum_{i=0}^{n-1} f_i
\]

\[
\leq \frac{1}{n k^{n-1} (1 - k)} \sum_{i=0}^{n-1} k^i T f_i
\]

\[
\leq \frac{1}{n k^{n-1} (1 - k)} \sum_{0 < k < 1} \sup_{i=0}^{n-1} k^i T f_i
\]

\[
\leq \varepsilon \sup_{0 < k < 1} (1 - k) \sum_{i=0}^{n-1} k^i T f_i \quad \text{(take } k = 1/n)\]

This inequality also appears in [2], p. 154, [5] and [3] (in the continuous case).

(2.6) is a consequence of (1.3). Indeed, for any \( f \in L_p \),

\[
\lim_{n \to \infty} \lambda \sum_{i=0}^{n} \frac{T f}{(\lambda + 1)^{i+1}} = \lim_{n \to \infty} \sum_{i=0}^{n} e^{-\mu T f}
\]

\[
\mu = \lim_{k \to 1^-} \left( 1 - k \right) \sum_{i=0}^{n-1} k^i T f \quad \text{exists and is finite a.e. on } X \quad (1.3).
\]

Since \( T f \) is positive, the tauberian theorem [10] in the form given in [4] shows that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f \quad \text{exists a.e. on } X \quad \text{for any } f \in L_p
\]

and thus for any \( f \in L_\infty \).

(2.7) is an immediate consequence of (2.6) as \( T f = \frac{1}{n} \sum_{i=0}^{n-1} f \) and thus for any \( f \in L_p \).

3. **Remarks.** Any positive \( C \)-contraction on \( L_1 \) (resp. \( L_\infty \)) is necessarily a contraction [4].

As we said in [4], any strongly continuous semi-group \( (T_{h+0})_0 \) on a Banach space, such that \( \sup_{0}^{\infty} \| T_{h+0} \| \leq 1 \) is necessarily a contraction semi-group; (1.5) is an estimate for such semi-groups (if \( T \) is positive on \( L_p \)).

If \( p = 2 \) then \( T = \begin{bmatrix} 1 + e & 0 \\ 0 & 0 \end{bmatrix} \) is a \( C \)-contraction on \( L_2 \) if and only if

\( 4e^2 + 8e - 5 \leq 0 \); the best value possible is \( T = \begin{bmatrix} 0 & 3/2 \\ 0 & 0 \end{bmatrix} \) and \( \| T \| = 3/2 > 1 \).

If \( T \) is a mean-bounded positive matrix then \( T \) is necessarily power-
bounded ([4], Theorem 4.2, and [9], p. 11, Prop. 3.4) and \( \lim_{n \to +\infty} T^n = \lim_{n \to +\infty} P^n \) where \( P = \lim_{n \to +\infty} \left( I + \cdots + T^{-1} \right)^n \) ([9], Lemma 3.3, p. 11); therefore one has \( \lim_{n \to +\infty} ||T^n|| \leq 1 \) for any positive C-contraction matrix.

References


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Borel’s theorem for generalized functions

by

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Abstract. Generalized complex numbers and new generalized functions were introduced in order to give a meaning to both the value of any distribution at any point and to any finite product of distributions. In this paper we prove: Given any sequence \( \{c_n\} \) of generalized complex numbers, there is a generalized function \( f \) on \( \mathbb{R} \) such that \( f^{(n)}(0) = c_n \) for all \( n \). This result shows a coherence between generalized numbers and functions similar to that of the classical case.

Introduction. One of the authors introduced a generalized mathematical analysis in order to give a mathematical sense to any finite product of distributions and to classical heuristic computations done by physicists, see Colombeau [1], [2], [3], [4]. This generalized mathematical analysis deals with new generalized functions, more general than distributions, and with generalized complex numbers such that, if \( G \) is any generalized function on \( \Omega \subset \mathbb{R}^n \) open and if \( x \in \Omega \) then \( G(x) \) is defined as a generalized complex number.

In this paper we prove Borel’s theorem in this setting: given any family \( \{c_n\}_{n=0} \) of generalized complex numbers, there is a generalized function \( G \) on \( \mathbb{R}^n \) such that, for any \( x \in \mathbb{R}^n \),

\[
\frac{\partial^{d} G}{\partial x_1^{d_1} \cdots \partial x_n^{d_n}} (0) = c_x.
\]

This shows a deep connection between our generalized functions and our generalized complex numbers, similar to the classical case. The proof is more technical than the classical one given in Narasimhan [5], since we have to do more detailed computations and estimates.

We use the concepts of generalized functions and the terminology defined in Colombeau [3]. According to Colombeau [3], we consider an algebra \( C^{\infty} \) such that if \( G \in C^{\infty}(\Omega) \) and \( x \in \Omega \) then the value \( G(x) \) is defined as an element of \( C^{\infty} \).

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