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On the pointwise ergodic theorems in L_p $(1 < p < \infty)$

by

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Abstract. Using M. A. Akcoglu's estimate [1] we show that

$$\Big\|\sup_{\substack{n\geqslant 1\\ n\in N}}\Big|\frac{I+\ldots+T^{n-1}}{n}f\Big|\Big\|_p \leqslant e\frac{p}{p-1}\|f\|_p$$

for any $f\in L_p$ $(1<p<\infty)$ and any positive operator T on L_p which verifies

$$\sup_{\substack{n\geqslant 1\\ n\in N}}\Big\|\frac{I+\ldots+T^{n-1}}{n}\Big\|_p \leqslant 1$$

or more generally $\sup_{0<k<1}\|(1-k)\sum_{i=0}^{\infty}k^i T^i\|_p \leqslant 1$. For such operators (which are not necessarily contractions) we also obtain the pointwise ergodic theorem in L_p.

Introduction. Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and T a positive operator on $L_p(X, \mathscr{F}, \mu) = L_p$, $1 < p < \infty$.

M. A. Akcoglu's powerful estimate [1] is

$$\Big\|\sup_{\substack{n\in N\\ n\geqslant 1}}\Big|\frac{I+\ldots+T^{n-1}}{n}f\Big|\Big\|_p \leqslant \frac{p}{p-1}\|f\|_p$$

for any $f\in L_p$ and any positive contraction T on L_p.

A trivial example shows that a positive operator T on L_p $(1<p<\infty)$ which verifies

$$(*)\qquad \sup_{\substack{n\geqslant 1\\ n\in N}}\frac{\|I+\ldots+T^{n-1}\|}{n}\leqslant 1$$

is not necessarily a positive contraction on L_p $(1<p<\infty)$: take $X=\{1, 2\}$, $\mu\{1\}=\mu\{2\}=1$ and $T=\begin{bmatrix}0 & 1+\varepsilon\\ 0 & 0\end{bmatrix}$ with $\varepsilon>0$ small enough.

Of course, the converse is true.

In this paper we show that M. A. Akcoglu's estimate [1] yields an

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estimate and the pointwise ergodic theorem for positive operators on L_p $(1 < p < \infty)$ which verify $(*)$ or more generally $\sup_{0<k<1} (1-k)\left\|\sum_{i=0}^{\infty} k^i T^i\right\| \leqslant 1$.

Acknowledgement. The problem of mean and pointwise ergodic convergence for mean-bounded positive operators on L_p $(1<p<\infty)$ has been introduced by Professor A. Brunel. The mean ergodic theorem is proved in [4]. I would like to express my gratitude to Professor A. Brunel for his interest in the present work.

1. We recall the

DEFINITION. A *resolvent* on a vector space B is a family $(\lambda V_\lambda)_{\lambda>0}$ of linear operators on B such that $V_\lambda - V_\mu = -(\lambda-\mu) V_\lambda V_\mu$ for all $\lambda, \mu > 0$.

Examples of resolvent are

$$V_\lambda = \sum_{i=0}^{\infty} \frac{T^i}{(\lambda+1)^{i+1}} \quad \text{where } T \text{ is a linear operator on } B;$$

$$V_\lambda = \int_0^\infty e^{-\lambda s} T_s\, ds \quad \text{where } (T_s)_{s\geqslant 0} \text{ is a semi-group of operators.}$$

Ergodic theorem for resolvents on L_1 were obtained by D. Feyel [2] and R. Sato [7], [8].

The following appears as a consequence of M. A. Akcoglu's estimate [1] and the Hille–Yosida theorem ([6], p. 261).

(1.1) THEOREM. *Let $V = (\lambda V_\lambda)_{\lambda>0}$ be a resolvent on $L_p(X, \mathscr{F}, \mu)$ $(1<p<\infty)$ such that λV_λ is a positive contraction for any $\lambda > 0$.*

Then, for any $f \in L_p$ one has

$$\left\|\sup_{\lambda>0} |\lambda V_\lambda f|\right\| \leqslant \frac{p}{p-1}\|f\|, \tag{1.2}$$

$$\lim_{\lambda\to 0^+} \lambda V_\lambda f \ \text{ exists and is finite a.e. on } X, \tag{1.3}$$

$$\lim_{\lambda\to +\infty} \lambda V_\lambda f \ \text{ exists and is finite a.e. on } X. \tag{1.4}$$

Proof. Since L_p is reflexive and $\|\lambda V_\lambda\| \leqslant 1$, one sees that $T_0 =$ strong-$\lim_{\lambda\to+\infty} \lambda V_\lambda$ exists (see e.g. [4]). T_0 is a positive contraction on L_p and the resolvent equation $V_\lambda - V_\mu = -\lambda V_\lambda V_\mu + \mu V_\mu V_\lambda$ (as $\mu\to+\infty$) shows that $V_\lambda = T_0 V_\lambda (= V_\lambda T_0)$. This implies that $T_0 = T_0^2$ and thus $H = T_0(L_p) = \overline{T_0(L_p)}$ is a Banach space. See also R. Sato [7].

Therefore, $(\lambda V_\lambda)_{\lambda>0}$ can be considered as a resolvent on H which verifies s-$\lim_{\lambda\to+\infty} \lambda V_\lambda h = h$ for any $h \in H$ and consequently the Hille–Yosida theorem ([6], p. 261) shows that $\lambda V_\lambda h = \int_0^\infty e^{-\lambda t} A_t h\, dt$ for any $h \in H$, where $A_t h = \lim_{n\to+\infty} e^{-nt} \exp(nt\cdot nV_n)(h)$. Since nV_n is a positive contraction, one sees that $T_t = A_t \circ T_0$ is a strongly continuous semigroup of positive contractions on L_p. Note that $(T_t)_{t=0} = T_0$.

Now, put $S_t f = \int_0^t T_s f\, ds$ for any $f \in L_p$ and $t \geqslant 0$.

If $D_n = \{k2^{-n}, k = 1, 2, \ldots\}$ and $D = \bigcup_{n=0}^{\infty} D_n$, one has

$$\sup_{t>0} \frac{S_t f}{t} = \sup_{t\in D} \frac{S_t f}{t} = \lim_{n\to+\infty} \sup_{t\in D_n} \frac{S_t f}{t} \quad \text{for any } f \in L_p^+.$$

Also note that

$$\frac{S_t f}{t} = \frac{1}{k} \sum_{j=0}^{k-1} T_{2^{-n}}^j \left(\frac{1}{2^{-n}} \int_0^{2^{-n}} T_u f\, du\right) \quad \text{for any } t = k2^{-n} \in D_n.$$

Hence M. A. Akcoglu's estimate [1] applied to $T_{2^{-n}}$ gives us

$$\left\|\sup_{t\in D_n} \frac{S_t f}{t}\right\|_p = \left\|\sup_{\substack{k\in N\\ k\geqslant 1}} \frac{1}{k} \sum_{j=0}^{k-1} T_{2^{-n}}^j \left(\frac{1}{2^{-n}} \int_0^{2^{-n}} T_u f\, du\right)\right\|$$

$$\leqslant \frac{p}{p-1} \left\|\frac{1}{2^{-n}} S_{2^{-n}} f\right\| \leqslant \frac{p}{p-1}\|f\|.$$

This implies

$$\left\|\sup_{t>0} \frac{S_t f}{t}\right\| \leqslant \frac{p}{p-1}\|f\| \quad \text{for any } f \in L_p^+. \tag{1.5}$$

Now, for any $\lambda > 0$ and $h = T_0 f$, $f \in L_p^+$, an integration by parts gives

$$V_\lambda h = \int_0^\infty \lambda e^{-\lambda t} S_t f\, dt \leqslant \int_0^\infty \left(\sup_{t>0} \frac{S_t f}{t}\right) \lambda e^{-\lambda t}\, t\, dt.$$

Therefore,

$$\lambda V_\lambda h \leqslant \sup_{t>0} \frac{S_t f}{t} \int_0^\infty \lambda^2 e^{-\lambda t}\, t\, dt = \sup_{t>0} \frac{S_t f}{t}$$

and

$$\left\|\sup_{\lambda>0} \lambda V_\lambda h\right\| \leqslant \frac{p}{p-1}\|f\|.$$

Thus, if $f\in L_p^+$ and $h = T_0 f$, one obtains

$$\|\sup_{\lambda>0}\lambda V_\lambda f\| = \|\sup_{\lambda>0}\lambda V_\lambda(T_0 f)\| \leqslant \frac{p}{p-1}\|f\|.$$

Hence, for any $f\in L_p$, $\|\sup_{\lambda>0}|\lambda V_\lambda f|\,\| \leqslant \frac{p}{p-1}\|f\|$. This proves (1.2).

We now prove (1.3). Inequality (1.2) shows that the set $C = \{f \mid \lim_{\lambda\to0^+}\lambda V_\lambda f$ exists a.e. on $X\}$ is closed in the strong topology of L_p.

Indeed, if s-lim $f_n = f$ with $f_n\in C$, then, for any $\varepsilon>0$ and $f_n\in C$ such that $\|f_n-f\|\leqslant\varepsilon$, we have

$$\|\overline{\lim_{\substack{\lambda\to0^+\\ \lambda'\to0^+}}}\,|\lambda V_\lambda f-\lambda' V_{\lambda'} f|\,\| \leqslant \|\overline{\lim_{\lambda\to0^+}}\,|\lambda V_\lambda(f-f_n)| + \overline{\lim_{\lambda'\to0^+}}\,|\lambda' V_{\lambda'}(f-f_n)|\,\| \leqslant \frac{2p}{p-1}\varepsilon.$$

Hence $f\in C$.

Now, $L_p = \operatorname{Inv} V \oplus \overline{(I-\alpha V_\alpha)(L_p)}$, where $\operatorname{Inv} V = \{f\in L_p \mid \lambda V_\lambda f = f$ for any $\lambda>0\}$ (see e.g. [4]). It is clear that $\operatorname{Inv} V\subset C$.

On the other hand, if $f=(I-\alpha V_\alpha)(g)$ for some $\alpha>0$ and $g\in L_p$ then

$$\lambda V_\lambda f = \lambda V_\lambda(I-\alpha V_\alpha)(g) = \lambda V_\alpha(I-\lambda V_\lambda)g = \lambda V_\alpha g - \lambda^2 V_\lambda(V_\alpha g).$$

Since $V_\alpha g\in L_p$, $\lim_{\lambda\to0^+}\lambda V_\alpha g = 0$ a.e. on X and since $\sup_{\lambda>0}\lambda V_\lambda(V_\alpha g) < +\infty$ a.e. on X (1.2), one has $\lim_{\lambda\to0^+}\lambda^2 V_\lambda(V_\alpha g)=0$ a.e. on X. Therefore, $\lim_{\lambda\to0^+}\lambda V_\lambda f = 0$ a.e. on X whenever $f=(I-\alpha V_\alpha)(g)$.

Hence C contains $\operatorname{Inv} V + (I-\alpha V_\alpha)(L_p)$.

Since C is closed, $C = L_p$. This proves (1.3).

To prove (1.4) consider similarly the set $C' = \{f\in L_p \mid \lim_{\lambda\to+\infty}\lambda V_\lambda f$ exists a.e. on $X\}$. C' is closed in the norm-topology of L_p (1.2). Let $f\in L_p$. $f^* = T_0 f = \lim_{\mu\to+\infty}\mu V_\mu f$ verifies $\lambda V_\lambda f = \lambda V_\lambda f^*$ for each $\lambda>0$ (see the proof of (1.2)). Therefore, to prove that $f\in C'$ it suffices to show that $f^*\in C'$ and since C' is closed, it also suffices to prove that $\mu V_\mu f\in C'$ for each $\mu>0$. This is easy: one has

$$\lambda V_\lambda \mu V_\mu f = \frac{\lambda\mu}{\lambda-\mu}(V_\mu f - V_\lambda f) = \frac{\lambda}{\lambda-\mu}\mu V_\mu f - \frac{\mu}{\lambda-\mu}\lambda V_\lambda f.$$

Since $\mu V_\mu f\in L_p$, one has $\mu V_\mu f<+\infty$ a.e. on X and $\lim_{\lambda\to+\infty}\frac{\lambda}{\lambda-\mu}\mu V_\mu f = \mu V_\mu f$ a.e. Since $\sup_{\lambda>0}|\lambda V_\lambda f| < +\infty$ a.e. (1.2), one has $\lim_{\lambda\to+\infty}\frac{\mu}{\lambda-\mu}\lambda V_\lambda f = 0$ a.e. Therefore $\lim_{\lambda\to+\infty}\lambda V_\lambda\mu V_\mu f = \mu V_\mu f$ a.e. on X. Thus, $\mu V_\mu f\in C'$ and $C'=L_p$.

Note that $\lim_{\lambda\to+\infty}\lambda V_\lambda f = T_0 f$ a.e. on X.

Remark. In the proof of (1.1) one has seen that $\lambda V_\lambda f = \lambda V_\lambda T_0 f = \int_0^\infty e^{-\lambda s}A_s(T_0 f)\,ds$ where $(A_s)_{s\geqslant0}$ is a strongly continuous semi-group of contractions on $T_0(L_p)$. Note that $(A_s\circ T_0)_{s\geqslant0}$ is a semi-group on L_p.

Now, if $(U_s)_{s\geqslant0}$ is an arbitrary semi-group of L_p-positive contractions, (1.5) shows that

$$\left\|\sup_{t>0}\frac{\left|\int_0^t U_s f\,ds\right|}{t}\right\| \leqslant \frac{p}{p-1}\|f\| \quad\text{for any } f\in L_p.$$

This implies (see [3])

$$\lim_{t\to0^+}\frac{\int_0^t U_s f\,ds}{t} = U_0 f \quad \text{a.e. on } X. \tag{1.6}$$

On the other hand, for any $f = L_p^+$, any $t\geqslant1$, one has

$$\frac{n}{n+1}\cdot\frac1n\sum_{j=0}^{n-1}U_1^j f \leqslant \frac{\int_0^t U_s f\,ds}{t} \leqslant \frac{n+1}{n}\,\frac{1}{n+1}\sum_{j=0}^{n}U_1^j f,\quad\text{where}\quad n=[t].$$

Since $\lim_{n\to+\infty}\frac1n\sum_{j=0}^{n-1}U_1^j f$ exists and is finite a.e. on X [1], one has

$$\lim_{t\to+\infty}\frac{\int_0^t U_s f\,ds}{t} \quad\text{exists and is finite a.e. on } X. \tag{1.7}$$

Note that (1.6) (resp. (1.7)) implies (1.4) (resp. (1.3)) (Abelian theorem [10], p. 197), and conversely (1.4) (resp. (1.3)) implies (1.6) (resp. (1.7)) (Tauberian theorem [10], p. 209).

2. We recall [4] that in any Banach space B a sequence $(a_n)_{n\in N}$ which verifies $M = \sup_{\substack{n\in N\\ n\geqslant1}}\left\|\frac{a_0+\ldots+a_n}{n}\right\| < +\infty$ necessarily verifies

$$(1-k)\sum_{i=0}^{\infty} k^i a_i \quad\text{is defined for any } k\colon\ 0<k<1 \tag{2.0}$$

and

$$\sup_{0<k<1}\left\|(1-k)\sum_{i=0}^{\infty} k^i a_i\right\| \leqslant M. \tag{2.1}$$

(2.2) Definition. An operator T on B will be called a *C-contraction* (resp. *A-contraction*) if

$$\sup_{\substack{n\in N\\ n\geq 1}} \frac{\|I+\ldots+T^{n-1}\|}{n} \leq 1$$

$$\text{(resp. } \sup_{0<k<1} \Big\|(1-k)\sum_{i=0}^{\infty} k^i T^i\Big\| \leq 1)$$

As we said, a C-contraction is necessarily an A-contraction (2.1), and a C-contraction on L_p $(1<p<\infty)$ is not necessarily a contraction (see the Introduction).

We can now state the dominated ergodic

(2.3) Theorem. *Let T be a positive A-(or C-)contraction on L_p $(1<p<\infty)$; then, for any $f\in L_p$, one has*

$$\Big\|\sup_{0<k<1}\Big|(1-k)\sum_{i=0}^{\infty} k^i T^i f\Big|\Big\| \leq \frac{p}{p-1}\|f\|, \tag{2.4}$$

$$\Big\|\sup_{\substack{n\geq 1\\ n\in N}}\Big|\frac{I+\ldots+T^{n-1}}{n}f\Big|\Big\| \leq e\frac{p}{p-1}\|f\|, \tag{2.5}$$

$$f^*(x) = \lim_{n\to+\infty}\Big(\frac{I+\ldots+T^{n-1}}{n}f\Big)(x) \text{ exists and is finite a.e. on } X, \tag{2.6}$$

$$\lim_{n\to+\infty}\frac{T^n f}{n} = 0 \quad \text{a.e. on } X, \tag{2.7}$$

$$\operatorname*{strong-lim}_{n\to+\infty}\frac{I+\ldots+T^{n-1}}{n} \text{ exists and } L_p = \operatorname{Inv} T + \overline{(I-T)(L_p)} \ [4]. \tag{2.8}$$

Remarks.

- (2.4) generalizes a result of S. A. Mc. Grath [5].
- (2.5) and (2.6) generalize [1].
- $\lim(1-k)\sum_{i=0}^{\infty} k^i T^i f$ exists a.e. on X as $k\to 1^-$ (1.3) or as $k\to 0^+$ (1.4).
- $Tf^* = f^*$.

Proof. (2.4) is a consequence of (1.2), (2.1) and (2.2). Indeed,

$$(1-k)\sum_{i=0}^{\infty} k^i T^i f = \lambda\sum_{i=0}^{\infty}\frac{T^i}{(\lambda+1)^{i+1}} \quad (\text{with } k = 1/(\lambda+1))$$

is a particular case of positive contractions resolvent ((2.2) or (2.1)).

(2.5) is an immediate consequence of (2.4). Indeed, for any $n\in N$, $n\geq 1$,

any k: $0<k<1$ and any $f\in L_p$ one has

$$\Big|\frac{I+\ldots+T^{n-1}}{n}f\Big| \leq \frac{I+\ldots+T^{n-1}}{n}|f| \leq \frac{1}{nk^{n-1}}\sum_{i=0}^{n-1} k^i T^i|f|$$
$$\leq \frac{1}{nk^{n-1}(1-k)}(1-k)\sum_{i=0}^{\infty} k^i T^i|f|$$
$$\leq \frac{1}{nk^{n-1}(1-k)}\sup_{0<k<1}(1-k)\sum_{i=0}^{\infty} k^i T^i|f|$$
$$\leq e\sup_{0<k<1}(1-k)\sum_{i=0}^{\infty} k^i T^i|f| \quad (\text{take } k = 1-1/n).$$

This inequality also appears in [2], p. 154, [8] and [3] (in the continuous case).

(2.6) is a consequence of (1.3). Indeed, for any $f\in L_p^+$

$$\lim_{\lambda\to 0^+}\lambda\sum_{i=0}^{\infty}\frac{T^i f}{(\lambda+1)^{i+1}} = \lim_{\lambda\to 0^+}\lambda\sum_{i=0}^{\infty} e^{-\lambda i}T^i f$$
$$= \lim_{k\to 1^-}(1-k)\sum_{i=0}^{\infty} k^i T^i f \text{ exists and is finite a.e. on } X \ (1.3).$$

Since $T^i f$ is positive, the tauberian theorem [10] in the form given in [4] shows that

$$\lim_{\nearrow n\to+\infty}\frac{I+\ldots+T^{n-1}}{n}f \text{ exists a.e. on } X \text{ for any } f\in L_p^+$$

and thus for any $f\in L_p$.

(2.7) is an immediate consequence of (2.6) as $\frac{T^n}{n} = \frac{n+1}{n}\frac{S_{n+1}}{n+1} - \frac{S_n}{n}$ (with $S_n = I+\ldots+T^{n-1}$).

3. Remarks. Any positive C-contraction on L_1 (resp. on L_∞) is necessarily a contraction [4].

As we said in [4], any strongly continuous semi-group $(T_t)_{t\geq 0}$ on a Banach space, such that $\sup_{t>0} t^{-1}\big\|\int_0^t T_s\,ds\big\| \leq 1$ is necessarily a contraction semi-group; (1.5) is an estimate for such semi-groups (if T_t is positive on L_p).

If $p=2$ then $T = \begin{bmatrix} 0 & 1+\varepsilon \\ 0 & 0\end{bmatrix}$ is a C-contraction on L_2 if and only if $4\varepsilon^2+8\varepsilon-5\leq 0$; the best value possible is $T = \begin{bmatrix} 0 & 3/2 \\ 0 & 0\end{bmatrix}$ and $\|T\| = 3/2 > 1$.

If T is a mean-bounded positive matrix then T is necessarily powerbounded ([4], Theorem 4.2, and [9], p. 11, Prop. 3.4) and $\lim_{n\to+\infty} T^n = \lim_{n\to+\infty} P^n$ where $P = \lim_{n\to+\infty}\left(\frac{I+\ldots+T^{n-1}}{n}\right)$ ([9], Lemma 3.3, p. 11); therefore one has $\lim_{n\to+\infty} \|T^n\| \leqslant 1$ for any positive C-contraction matrix.

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Borel's theorem for generalized functions

by

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Abstract. Generalized complex numbers and new generalized functions were introduced in order to give a meaning to both the value of any distribution at any point and to any finite product of distributions. In this paper we prove: Given any sequence (c_n) of generalized complex numbers, there is a generalized function f on $\boldsymbol{R}$ such that $f^{(n)}(0) = c_n$ for all n. This result shows a coherence between generalized numbers and functions similar to that of the classical case.

Introduction. One of the authors introduced a generalized mathematical analysis in order to give a mathematical sense to any finite product of distributions and to classical heuristic computations done by physicists, see Colombeau [1], [2], [3], [4]. This generalized mathematical analysis deals with new generalized functions, more general than distributions, and with generalized complex numbers such that, if G is any generalized function on $\Omega \subset \boldsymbol{R}^n$ open and if $x \in \Omega$ then $G(x)$ is defined as a generalized complex number.

In this paper we prove Borel's theorem in this setting: given any family $\{c_\alpha\}_{\alpha\in\boldsymbol{N}^n}$ of generalized complex numbers, there is a generalized function G on $\boldsymbol{R}^n$ such that, for any $\alpha \in \boldsymbol{N}^n$,

$$\left(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} G\right)(0) = c_\alpha.$$

This shows a deep connection between our generalized functions and our generalized complex numbers, similar to the classical case. The proof is more technical than the classical one given in Narasimhan [5], since we have to do more detailed computations and estimates.

We use the concepts of generalized functions and the terminology defined in Colombeau [3]. According to Colombeau [3], we consider an algebra $\bar{\boldsymbol{C}}^*$ such that if $G \in \mathscr{G}^*(\Omega)$ and $x \in \Omega$ then the value $G(x)$ is defined as an element of $\bar{\boldsymbol{C}}^*$.

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