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Banach spaces which are proper M-ideals

by

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Abstract. In the theory of Banach spaces certain subspaces J of Banach spaces X, the *M-ideals*, have been investigated in great detail. *M-summands*, i.e. subspaces J for which there exists a subspace $J^\perp$ such that $X = J \oplus J^\perp$ and $\|j+j^\perp\| = \max\{\|j\|, \|j^\perp\|\}$ for $j \in J$, $j^\perp \in J^\perp$, are special examples of M-ideals, but there is an abundance of M-ideals which are not of this simple form. They will be called *proper M-ideals*.

The more interesting examples of M-ideals are proper, and in the development of M-structure theory it turned out that all these examples share some geometric properties. This motivated the present investigations to give conditions concerning the geometry of a Banach space J such that J can be a proper M-ideal in a suitable space X. The main results are the following:

- if J can be a proper M-ideal, then J contains a copy of c_0;
- if J satisfies a certain intersection property then J is never a proper M-ideal;
- J can be a proper M-ideal iff J contains a pseudoball which is not a ball (a pseudoball is a closed convex subset B of diameter two such that for every finite collection $x_1, \ldots, x_n$ of elements with $\|x_i\| < 1$ there is an $x \in B$ such that $x + x_i \in B$ for every i).

1. Introduction. At first we recall some basic definitions from M-structure theory.

1.1. Definition. Let X be a (real or complex) Banach space, J a closed subspace of X.

(i) J is called an *L-summand* (resp. *M-summand*) if there exists a subspace $J^\perp$ such that $X = J \oplus J^\perp$ and $\|j+j^\perp\| = \|j\| + \|j^\perp\|$ (resp. $\|j+j^\perp\| = \max\{\|j\|, \|j^\perp\|\}$) for $j \in J$, $j^\perp \in J^\perp$.

(ii) J is called an *M-ideal* if the annihilator J^π of J in X' is an L-summand.

Note. It is easy to see that every M-summand is an M-ideal. M-ideals which are not of this simple form will be called *proper M-ideals* in the sequel.

These notions play an important rôle in the applications of M-structure to approximation theory and the theory of L^1-preduals (for references see [1] or [9]).

If X is a given space it is often important to determine the collection of M-ideals and M-summands of X. Here we are interested in the *converse problem*: Given a Banach space J, can J be a proper M-ideal in a suitable space X? (Trivially every J can be an M-summand in a suitably chosen X.) How can this be characterized by means of the geometry of J?

The *second section* deals with examples as well as some useful simple characterizations. In *section three* we introduce *pseudoballs* (a definition which is due to R. Evans): a *pseudoball* B in a Banach space J is a closed convex subset of J with diameter two such that for $x_1, \ldots, x_n \in J$ with $\|x_i\| < 1$ $(i = 1, \ldots, n)$ there is an $x \in B$ with $x + x_i \in B$ for every i. Every ball in J with radius one is a pseudoball, and we show that J can be a proper M-ideal iff there are pseudoballs which are not balls.

In *section four* we consider Banach spaces with the following *intersection property*: for every $\varepsilon > 0$ there are $x_1, \ldots, x_n$ in J with $\|x_i\| < 1$ $(i = 1, \ldots, n)$ such that $\|x\| \leqslant \varepsilon$ if $\|x - x_i\| \leqslant 1$ for every i (i.e. the intersection of the balls $B(x_i, 1)$ is contained in $B(0, \varepsilon)$). We show that there are many classes of Banach spaces which have this intersection property. One of our main results is a theorem which states that Banach spaces with the intersection property are never proper M-ideals. Using this fact, we are able to prove that every space which can be a proper M-ideal contains an isomorphic copy of c_0.

In the last section, in *section five*, we point out that the concepts we dealt with all have a quantitative aspect. This gives rise to the definition of a number associated with each Banach space J which measures how J fails to have the intersection property or how complicated pseudoballs can be in J.

2. Examples and some preliminary results. The most well-known examples of proper M-ideals are c_0 (= the null sequences) in c (= the convergent sequences) and $K(H)$ (= the compact operators on a Hilbert space H) in $L(H)$ (= the bounded operators). On the other hand, certain spaces never occur as proper M-ideals, e.g. reflexive spaces (Prop. 2.2 in [1]). What are the geometric properties which make it possible that c_0 and $K(H)$ are proper M-ideals, why are reflexive spaces never proper M-ideals?

Clearly, the property "J can be a proper M-ideal" is invariant with respect to isometric isomorphism, and simple counter-examples show that invariance with respect to isomorphisms and hereditary results are not to be expected. Given an M-ideal J in X several sufficient conditions are known such that J is an M-summand. One of these results is applied in the next proposition:

2.1. PROPOSITION. *Let J be a Banach space such that there is a norm one projection from J'' onto (the canonical image of) J. Then J is never a proper M-ideal.*

Proof. Suppose that J is an M-ideal in a Banach space X; we will show that J is in fact an M-summand.

By assumption, X' splits into two L-summands $X' = J^\pi \oplus (J^\pi)^\perp$ so that X'' splits into the M-summands $X'' = (J^\pi)^\pi \oplus ((J^\pi)^\perp)^\pi$ ([1], Prop. 1.5). In particular, there is a norm one projection P on X'' with range $(J^\pi)^\pi$. If Q denotes any norm one projection from J'' onto J then QPi_X

$$X \xrightarrow{i_X} X'' \xrightarrow{P} (J^\pi)^\pi \cong J'' \xrightarrow{Q} J$$

is a norm one projection from X onto J (here we identified $(J^\pi)^\pi$ with J'', and $i_X\colon X \to X''$ means the canonical embedding). Since M-ideals which admit such projections are M-summands (cf. Prop. 2.1 in [7]), the proposition is proved.

This proposition provides us at once with many classes of Banach spaces which are never proper M-ideals, e.g. dual spaces or L^1-spaces.

The following result is of technical nature. It states that the property of never being a proper M-ideal can be checked by considering the case of codimension one.

2.2. LEMMA. *Let J be a Banach space. Then the following are equivalent:*

(i) *J can be a proper M-ideal in a Banach space X,*

(ii) *J can be a proper M-ideal in a Banach space X such that* $\dim X/J = 1$.

Proof. Independently A. Lima [8] and the first-named author ([2]) have observed that an M-ideal J in a Banach space X is an M-summand iff the sets of best approximation $P_J(x) := \{y \mid y \in J,\ \|x-y\| = \inf_{z \in J} \|x-z\|\}$ are balls in J for every $x \in X$. The lemma is an immediate consequence of this fact.

3. Pseudoballs. Here we consider certain subsets of Banach spaces which behave in some respects like balls.

3.1. DEFINITION. Let B be a closed convex subset of a Banach space J of diameter two. B is called a *pseudoball* if, for $x_1, \ldots, x_n \in J$ with $\|x_i\| < 1$ $(i = 1, \ldots, n)$ there is an $x_0 \in B$ such that $x_0 + x_i \in B$ for every i.

EXAMPLES. 1. Balls with radius one are obviously pseudoballs.

2. $\{(x_n) \mid (x_n) \in c_0,\ 0 \leqslant x_n \leqslant 2 \text{ for every } n\}$ is a pseudoball in c_0; more generally, $\{(x_n) \mid (x_n) \in c_0,\ a_n \leqslant x_n \leqslant a_n + 2\}$ is a pseudoball in c_0 whenever (a_n) is a sequence such that $\overline{\lim}\, a_n \leqslant 0$, $\underline{\lim}\, a_n \geqslant -2$.

3.2. PROPOSITION. *Let B be a closed convex subset of a Banach space J of diameter two. Then the following are equivalent:*

(i) *B is a pseudoball.*

(ii) *The weak*-closure of B in J'' is a ball with radius one.*

(iii) *For every subspace J_0 of J with* $\dim J/J_0 < \infty$ *there is an* $[x_0] \in J/J_0$ *such that* $\operatorname{int} B([x_0], 1) \subset \omega(B) \subset B([x_0], 1)$ (*where* $\omega\colon J \to J/J_0$ *denotes the natural map*; $B([x_0], 1)$ *means the closed ball with center* $[x_0]$ *and radius one*).

Proof. (i) $\Rightarrow$ (iii): It is obvious that $\tilde{B} := \omega(B) \subset J/J_0$ has the following property:

$\tilde{B}$ is a convex subset of J/J_0 of diameter two, and for $[x_i] \in J/J_0$ *with*

$\|[x_i]\| < 1$ $(i = 1, \ldots, n)$ there is an $[x] \in J/J_0$ such that $[x]+[x_i] \in \tilde{B}$ for every i.

From this (iii) follows by a simple compactness argument (using the fact that $\dim(J/J_0) < \infty$).

(iii)$\Rightarrow$(ii): Let $f \in J'$ be given. We consider the canonical map ω_f: $J \to J/\ker f$, choose an $x_f \in J$ such that $\omega_f(x_f)$ is the "midpoint" of $\omega_f(B)$ (i.e. $\operatorname{int} B(\omega_f(x_f), 1) \subset \omega_f(B) \subset B(\omega_f(x_f), 1)$) and define $\Phi(f) := f(x_f)$; note that this implies

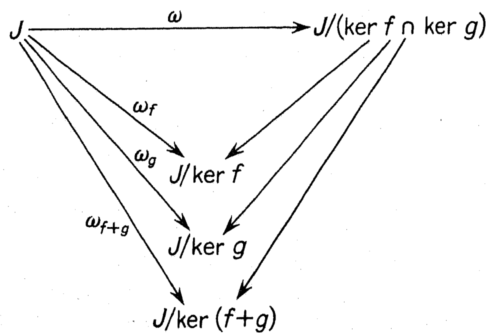
$$(*) \qquad \operatorname{int} B(\Phi(f), \|f\|) \subset f(B) \subset B(\Phi(f), \|f\|).$$

Φ: $J' \to \boldsymbol{K}$ is clearly well-defined and homogeneous.

For the proof of the additivity of Φ let $f, g \in J'$ be given. There is an $x \in J$ such that, in $J/(\ker f \cap \ker g)$,

$$\operatorname{int} B([x], 1) \subset \omega(B) \subset B([x], 1).$$

Since the following diagram (where the maps are the natural ones)



is commutative, we necessarily have

$$\omega_f(x) = \omega_f(x_f), \quad \omega_g(x) = \omega_g(x_g), \quad \omega_{f+g}(x) = \omega_{f+g}(x_{f+g}).$$

This proves that

$$\Phi(f+g) = (f+g)(x_{f+g}) = (f+g)(x) = f(x)+g(x) = \Phi(f)+\Phi(g).$$

Since $|\Phi(f)| \leqslant (\sup_{x \in B}\|x\|)\|f\|$, we have shown that $\Phi \in J''$.

Using $(*)$, it is easy to see that $B \subset B(\Phi, 1)$ and consequently $B^{-w^*} \subset B(\Phi, 1)$, and the reverse inclusion is proved by the usual separation techniques.

(ii)$\Rightarrow$(i): (This part of the proof is due to A. Lima). A moment's reflection shows that we have to prove that

$$\operatorname{int}(B_J \times \ldots \times B_J) \subset (B \times \ldots \times B) - \Delta_B,$$

where $B_J := B(0, 1)$ and $\Delta_B := \{(b, \ldots, b) | \ b \in B\} \subset J^n$. By the Tuckey–Klee–Ellis theorem (see [5], p. 734) this is true if we verify that

$$(**) \qquad B_J \times \ldots \times B_J \subset [(B \times \ldots \times B) - \Delta_B]^{-\| \ \|}.$$

To prove this inclusion we first note that (as a consequence of the weak*-continuity of $(x_1, \ldots, x_n, x) \mapsto (x_1, \ldots, x_n) - (x, \ldots, x)$)

$$B^{-w^*} \times \ldots \times B^{-w^*} - \Delta_{B^{-w^*}} \subset (B \times \ldots \times B - \Delta_B)^{-w^*}.$$

Since B^{-w^*} is a ball, the left-hand side contains $B_{J''} \times \ldots \times B_{J''}$, the n-fold product of the unit ball of J'', and thus $B_J \times \ldots \times B_J$. Consequently $\sup \operatorname{Re} f|_{B_J \times \ldots \times B_J} \leqslant 1$ for every $f \in (J^n)'$ such that $\sup \operatorname{Re} f|B \times \ldots \times B - \Delta_B \leqslant 1$. Now $(**)$ follows from the Hahn–Banach theorem.

The reason why we are interested in pseudoballs is the following theorem which provides us with an internal characterization of Banach spaces which can be proper M-ideals. The proof of this theorem is prepared by

3.3. Lemma. *For $m \in \boldsymbol{N}$, $\lambda_k \geqslant 0$, $\sum_{k=1}^{m} \lambda_k = 1$, and $\theta_k \in \boldsymbol{C}$ with $|\theta_k| = 1$ we have*

$$\sum_{k=1}^{m} \lambda_k |1-\theta_k| \leqslant \sqrt{2\sqrt{\varepsilon}} + 2\sqrt{\varepsilon}$$

provided that $|\sum \lambda_k(1-\theta_k)| \leqslant \varepsilon$.

Proof. We define

$$N_1 := \{k \in \{1, \ldots, m\} \,|\, \operatorname{Re}\theta_k > 1-\sqrt{\varepsilon}\},$$

$$N_2 := \{k \in \{1, \ldots, m\} \,|\, \operatorname{Re}\theta_k \leqslant 1-\sqrt{\varepsilon}\},$$

$\lambda := \sum_{k \in N_1} \lambda_k$, and we claim that

(1) $1-\lambda \leqslant \sqrt{\varepsilon}$,

(2) $|1-\theta_k| \leqslant \sqrt{2\sqrt{\varepsilon}}$ for every $k \in N_1$.

ad (1): By assumption we have $|1-z| \leqslant \varepsilon$ and therefore $\operatorname{Re} z \geqslant 1-\varepsilon$, where $z := \sum \lambda_k \theta_k$. Let $z_1 := \sum_{k \in N_1} (\lambda_k/\lambda)\theta_k$ and $z_2 := \sum_{k \in N_2} (\lambda_k/(1-\lambda))\theta$. The convexity of $\{w | \ w \in \boldsymbol{C}, \ |w| \leqslant 1, \ \operatorname{Re} w \leqslant 1-\sqrt{\varepsilon}\}$ implies that $\operatorname{Re} z_2 \leqslant 1-\sqrt{\varepsilon}$. Hence

$$1-\varepsilon \leqslant \operatorname{Re} z = \operatorname{Re}(\lambda z_1 + (1-\lambda) z_2) \leqslant \lambda + (1-\lambda)(1-\sqrt{\varepsilon}),$$

and this gives $1-\lambda \leqslant \sqrt{\varepsilon}$.

ad (2): For $k \in N_1$ we have $1-\operatorname{Re}\theta_k < \sqrt{\varepsilon}$ and thus

$$|1-\theta_k|^2 = 2(1-\operatorname{Re}\theta_k) < 2\sqrt{\varepsilon}.$$

The lemma follows by combining (1) and (2):

$$\sum \lambda_k |1-\theta_k| = \sum_{k\in N_1} \lambda_k |1-\theta_k| + \sum_{k\in N_2} \lambda_k |1-\theta_k|$$

$$\leqslant \lambda\sqrt{2\sqrt{\varepsilon}} + (1-\lambda)2 \leqslant \sqrt{2\sqrt{\varepsilon}} + 2\sqrt{\varepsilon}.$$

3.4. THEOREM. *For a Banach space J the following are equivalent:*

(i) *J can be a proper M-ideal,*

(ii) *J contains a pseudoball which is not a ball.*

Proof. (i)$\Rightarrow$(ii): Using Lemma 2.2 we may suppose that J is a proper M-ideal in a Banach space X and that $\dim(X/J) = 1$. Choose $x\in X$ such that $d(x, J)$ (= the distance from x to J) $= 1$ and define $B := P_J(x) := \{y \mid y\in J,\ \|y-x\| = d(x, J)\}$. Theorem 1.2 and Theorem 1.1(2) in [8] then show that B is a pseudoball which is not a ball.

(ii)$\Rightarrow$(i): Let B be a pseudoball in J which is not a ball; we can and will assume that $0\in B$. We identify J with $J\times\{0\}$ and embed $J \cong J\times\{0\}$ into $X := J\times \boldsymbol{K}$. X is provided with a norm by taking as the unit ball the set

$$K := \overline{\operatorname{co}}\,\{(\theta b, \theta) \mid \theta\in \boldsymbol{K},\ |\theta| = 1,\ b\in B\}$$

(closure with respect to the product topology).

To show that J is a proper M-ideal in X we proceed as follows:

(a) X is a Banach space, and the norm of X on $J\times\{0\}$ coincides with the norm of J;

(b) $B\times\{0\} = P_{J\times\{0\}}(0, -1)$ (= the set of best approximations in $J\times\{0\}$ to $(0, -1)$);

(c) $J \cong J\times\{0\}$ is an M-ideal in X which is not an M-summand.

ad (a) It is easy to see that K is absolutely convex and absorbing so that ϱ, the Minkowski functional associated with K, is a semi-norm. ϱ is in fact a norm since B is a bounded set. Now suppose that we have shown that

$$(*) \qquad K\cap(J\times\{0\}) \subset B_J\times\{0\}.$$

Then (a) is proved since $(\operatorname{int} B_J)\times\{0\} \subset K\cap(J\times\{0\})$ is obviously valid so that ϱ induces $\|\ \|$ on J, and the completeness of (X, ϱ) is a consequence of the completeness of $(J, \|\ \|)$. It thus remains to prove $(*)$, i.e. $(x, 0)\in K$ implies $\|x\| \leqslant 1$. Let $(x, 0)\in K$ be given and $((x_n, r_n))_{n\in N}$ a sequence in

$$\operatorname{co}\{(\theta b, \theta) \mid b\in B,\ |\theta| = 1\} \quad \text{with} \quad (x_n, r_n)\to(x, 0).$$

We write (x_n, r_n) as

$$(x_n, r_n) = \left(\sum\lambda_{k,n}\theta_{k,n}b_{k,n},\ \sum\lambda_{k,n}\theta_{k,n}\right).$$

Choose $\lambda_n\in[0, 1]$ and θ_n with $|\theta_n| = 1$ such that $0 = \lambda_n r_n + (1-\lambda_n)\theta_n$. Since $r_n\to 0$ we have $\lambda_n\to 1$. Thus, for an arbitrary $b\in B$, the elements $\tilde{x}_n := \lambda_n\left(\sum\lambda_{k,n}\theta_{k,n}b_{k,n}\right) + (1-\lambda_n)\theta_n b$ converge to x.

We now observe: If $B(y_0, 1)$ is a ball with radius one in any space Y, $y_i\in B(y_0, 1)$, μ_i scalars with $\sum\mu_i = 0$ and $\sum|\mu_i| \leqslant 1$, then $\left\|\sum\mu_i y_i\right\| \leqslant 1$. Since the $b_{k,n}$, b lie in such a ball (in J'') it follows that $\|\tilde{x}_n\| \leqslant 1$ and consequently $\|x\| \leqslant 1$.

ad (b) We first note that $d((x, r), J\times\{0\}) = |r|$ for every $(x, r)\in X$ so that in particular $P_{J\times\{0\}}(0, -1) = \{(x, 0) \mid (x, 1)\in K\}$. It is therefore obvious that $B\times\{0\} \subset P_{J\times\{0\}}(0, -1)$. Conversely, let $(x, 1)\in K$ be given; we have to show that $x\in B$. For a suitable sequence

$$(x_n, r_n) = \left(\sum\lambda_{k,n}\theta_{k,n}b_{k,n},\ \sum\lambda_{k,n}\theta_{k,n}\right)$$

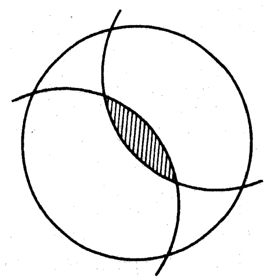
in $\operatorname{co}\{(\theta b, \theta) \mid b\in B,\ |\theta| = 1\}$ we have $(x_n, r_n)\to(x, 1)$. Since $r_n\to 1$ the preceding lemma implies that $\sum\lambda_{k,n}|1-\theta_{k,n}|\to 0$. Thus, with

$$b_n := \sum\lambda_{k,n}b_{k,n} = \sum\lambda_{k,n}\theta_{k,n}b_{k,n} + \sum\lambda_{k,n}(1-\theta_{k,n})b_{k,n}$$

we have $b_n\to x$ and $b_n\in B$ so that $x\in B$.

ad (c) For arbitrary $(x, r)\in X$ we get from (b) that $P_{J\times\{0\}}(x, r) = (-r(B-x))\times\{0\}$. But $B-x$ is a pseudoball so that $-r(B-x)$ satisfies the intersection property of Theorem 1.1 in [8]. Therefore $J-\{0\}$ is an M-ideal. It is not an M-summand since B is not a ball and $P_{J\times\{0\}}(0, -1) = B\times\{0\}$.

4. An intersection property. Here we will investigate an intersection property which implies that the Banach space under consideration can never occur as a proper M-ideal (Theorem 4.3). A space J has this intersection property if, roughly speaking, the center of a ball can be arbitrarily well approximated by the intersection of finitely many translates of this ball:



More precisely:

4.1. Definition. A Banach space J is said to have the *intersection property* if for every $\varepsilon > 0$ there are $x_1, \ldots, x_n$ in J with $\|x_i\| < 1$ ($i = 1, \ldots, n$) such that $\|x - x_i\| \leqslant 1$ (for every i) always implies that $\|x\| \leqslant \varepsilon$. To illustrate this definition by some examples one should observe that c_0 fails to have the intersection property but every CK-space (K a compact Hausdorff space) has it. Further examples are considered in the following proposition:

4.2. Proposition. *Each of the following conditions implies that J has the intersection property*:

(i) *the unit ball B_J of J is dentable,*

(ii) *B_J has a strongly exposed point,*

(iii) *J has the Radon–Nikodým property,*

(iv) *J is reflexive,*

(v) *J is a separable dual space,*

(vi) *J contains a nontrivial L^p-summand for some p with $1 \leqslant p < \infty$,*

(vii) *there is an $x \in J$ such that $|p(x)| = 1$ for every extreme functional p.*

Proof. (i) Suppose that J fails to have the intersection property. We will show that there is an $\varepsilon > 0$ such that

$$x \in \overline{\mathrm{co}}\,(B_J \backslash B(x, \varepsilon))$$

for every $x \in B_J$, i.e. the unit ball is not dentable (see [4], p. 133).

Since J does not have the intersection property, there is an $\alpha > 0$ such that for every y in the open unit ball of J there is a $z \in J$ with $\|z\| > \alpha$ and $\|y \pm z\| \leqslant 1$. Define $\varepsilon := \alpha/2$ and choose a sequence (r_n) of real numbers such that $1 - \varepsilon < r_n < 1$, $r_n \to 1$. Now, for given $x \in B_J$, choose $z_n \in J$ such that $\|z_n\| > \alpha$ and $\|r_n x \pm z_n\| \leqslant 1$. Since $r_n x = (1/2)\big((r_n x + z_n) + (r_n x - z_n)\big)$ tends to x and $\|x - (r_n x \pm z_n)\| > \varepsilon$ the claim is proved.

(ii)–(v) These assertions are implied by (i); cf. Th.V.3.10, Th. VII.3.3, Cor. III.2.13, Th. III.3.1 in [4].

(vi) Suppose that $J = X \oplus Y$ and that $\|x + y\|^p = \|x\|^p + \|y\|^p$ for $x \in X$ and $y \in Y$. We fix vectors $x \in X$, $y \in Y$ with $\|x\| = \|y\| = 1$ and choose, for given $\varepsilon > 0$, numbers $\delta, \delta' > 0$ such that

$$2\delta' \leqslant \varepsilon^p, \qquad 1 - (1-\delta)^p \leqslant \delta'.$$

We claim that $\|z\| \leqslant \varepsilon$ whenever $\|z \pm (1-\delta)x\| \leqslant 1$, $\|z \pm (1-\delta)y\| \leqslant 1$. Let such a z be given. We write $z = z_1 + z_2$, where $z_1 \in X$, $z_2 \in Y$. It follows that

$$\|z \pm (1-\delta)x\|^p = \|(z_1 \pm (1-\delta)x) + z_2\|^p = \|z_1 \pm (1-\delta)x\|^p + \|z_2\|^p \leqslant 1$$

so that $2(1-\delta) = \|((1-\delta)x + z_1) + ((1-\delta)x - z_1)\| \leqslant 2(1 - \|z_2\|^p)^{1/p}$ and consequently $\|z_2\|^p \leqslant 1 - (1-\delta)^p \leqslant \delta'$. Similarly one obtains $\|z_1\|^p \leqslant \delta'$ so that $\|z\|^p = \|z_1\|^p + \|z_2\|^p \leqslant 2\delta' \leqslant \varepsilon^p$.

(vii) For $\varepsilon > 0$ choose $\delta > 0$ such that $|\lambda| \leqslant \varepsilon$ for every $\lambda \in \boldsymbol{K}$ for which $|\lambda \pm (1-\delta)\sigma| \leqslant 1$ for some σ with $|\sigma| = 1$. With x as in the assumption it follows that $\|z\| \leqslant \varepsilon$ for every z such that $\|z \pm (1-\delta)x\| \leqslant 1$ (note that $\|x\| = 1$).

Note. The reader may have observed that the number n of Definition 4.1 can be chosen to be not greater than four in all examples of Proposition 4.2. We do not know whether this is accidental.

4.3. Theorem. *Let J be a Banach space with the intersection property. Then J can never be a proper M-ideal.*

Proof. Suppose that J can be a proper M-ideal. By Theorem 3.4 J contains a pseudoball B which is not a ball. The weak*-closure of B in J'' is a ball $B_{J''}(\psi, 1)$ in J'' by Proposition 3.4, and $d(\psi, J) =: \alpha > 0$ since B is not a ball and $J \cap B_{J''}(\psi, 1) = B$. We will show that J fails to satisfy the property in Definition 4.1 for every $\varepsilon < \alpha$.

Let $0 < \varepsilon < \alpha$ and a finite family $x_1, \ldots, x_n$ in the open unit ball of J be given; we will show that there is a $z \in J$ with $\|z\| \geqslant \varepsilon$ and $\|z - x_i\| \leqslant 1$ for every i. We choose an $\eta > 0$ such that

$$(1+\eta)\max\|x_i\| < 1 \quad \text{and} \quad \varepsilon < \alpha/(1+\eta)^2.$$

The defining property of pseudoballs provides us with an $x \in J$ such that $x + (1+\eta)x_i \in B$ and thus $\|\psi - x - (1+\eta)x_i\| \leqslant 1$. By the principle of local reflexivity we get an operator T from $\operatorname{span}\{\psi, x, x_1, \ldots, x_n\} =: E$ to J with $T|_{E \cap J} = \mathrm{Id}$ and $\|T\|, \|T^{-1}\| \leqslant 1 + \eta$ so that $\|T(\psi - x) - (1+\eta)x_i\| \leqslant 1 + \eta$ and $\|T(\psi - x)\| \geqslant \alpha/(1+\eta)$ (note that $\|\psi - x\| \geqslant \alpha$). Hence $z := T(\psi - x)/(1+\eta)$ has the claimed properties.

It is an *open problem* whether the converse of the preceding theorem is also true. In this connection it would be interesting to know whether dual spaces have the intersection property (since dual spaces are never proper M-ideals) or whether C_σ-spaces which are not C_Σ-spaces can have pseudoballs which are not balls (since these spaces obviously don't have the intersection property).

4.4. Theorem. *A Banach space J which does not have the intersection property contains a subspace isomorphic to c_0.*

Proof. From the assumption it easily follows that

(*) *There is an $\alpha > 0$ such that for every finite family $x_1, \ldots, x_n$ in the open unit ball of J there exists z with $\|z\| > \alpha$ and $\|z - x_i\| < 1$ for every i (we may assume that $\|z\| < 1$).*

We define

$$k = 2,\ t_1 = 1,\ t_2 = -1 \text{ if } \boldsymbol{K} = \boldsymbol{R} \quad \text{and}$$

$$k = 8,\ t_j = \exp(2\pi i(j-1)/8) \text{ for } j = 1, \ldots, 8 \text{ if } \boldsymbol{K} = \boldsymbol{C};$$

note that then $\sum |a_n| < \infty$ for every sequence (a_n) in K such that $(\sum_1^m r_j a_j)_m$ is bounded whenever $r_1, r_2, \ldots$ are in $\{t_1, \ldots, t_k\}$. Now a sequence (x_n) in J with

$$(**) \qquad \alpha < \|x_n\| < 1, \ \|r_1 x_1 + \ldots + r_n x_n\| < 1 \text{ for arbitrary } r_1, \ldots \ldots, r_n \in \{t_1, \ldots, t_k\}$$

can easily be constructed:

x_1 is chosen arbitrarily with $\alpha < \|x_1\| < 1$, and if $x_1, \ldots, x_n$ have already been obtained, then an application of $(*)$ to the finite family $\{r_1 x_1 + \ldots \ldots + r_n x_n \mid r_i \in \{t_1, \ldots, t_k\}\}$ provides us with an element x_{n+1} such that $x_1, \ldots, x_{n+1}$ behave as claimed.

By $(**)$ the sequence (x_n) has the following two properties:

– $\sum_{k=1}^{\infty} |f(x_n)| < \infty$ for every $f \in J'$,

– $\sum x_k$ does not converge.

Now the theorem follows from a result of Bessaga and Pełczyński ([3]) which asserts that a space contains c_0 iff such a sequence exists.

4.5. Corollary. *Every Banach space which can be a proper M-ideal contains a subspace isomorphic to c_0.*

Note. Since there are spaces J which are strictly convex and which are proper M-ideals in their bidual ($C(T)/A(T)$ is such a space) the corollary cannot be strengthened by replacing "isomorphic" by "isometric".

5. Grades of M-ideals and pseudoballs. When analyzing the difference between M-ideals and M-summands one is naturally led to introduce quantitative versions of the above-used notions. In the following we only quote the definitions and some of the results and refer to [6] for proofs and a more detailed discussion.

Definition. (i) Let J be a (not necessarily proper) M-ideal in X, where $J \subsetneqq X$. By the *grade of J in X*, $g(J, X)$, we denote the characteristic of $(J^\pi)^\perp$ in X', i.e.

$$g(J, X) := \max\{\alpha \mid \alpha \geqslant 0,\ \alpha B_{X'} \subset (\text{unit ball of } (J^\pi)^\perp)^{-w^*}\}.$$

(ii) Let B be a pseudoball in J. We define $g(B)$, the *grade of B*, by $g(B) := 1 - \sup\{\varepsilon \mid \varepsilon > 0,\ B \text{ contains a ball with radius } \varepsilon\}$. It is easy to see that $g(J, X) = 0$ iff J is an M-summand (cf. Prop. 2.2 in [1]) and that $g(B) = 0$ iff B is a ball so that the opposite limiting case, i.e. $g(J, X) = 1$ resp. $g(B) = 1$, describes in a certain sense the "extreme" proper M-ideals resp. pseudoballs. It can be shown that

(i) *there exists a pseudoball B in J with $g(B) = 1$ iff there exists a $y \in J'' \backslash J$ such that J is an M-ideal in* $\operatorname{lin}\{y, J\}$;

(ii) *if there is a pseudoball B in J with $g(B) = 1$ then 0 is in the weak*-closure of the extreme functionals;*

(iii) $g(B) = d(\Phi, J)$, *where* $B^{-w^*} = B_{J''}(\Phi, 1)$ (cf. Proposition 3.2);

(iv) $g(B) = g(J, X)$ (B, J, X as in Theorem 3.4).

It is not difficult to prove that for a proper M-ideal the grade can be decreased: for $\alpha < g(J, X)$ there is a Banach space X_α such that J is a proper M-ideal in X_α and $g(J, X_\alpha) = \alpha$.

One might suspect that, conversely, the grade can be increased if $g(J, X) < 1$. This is not true in general: The space $J := \{(x_n) \mid (x_n) \in c,\ x_1 + 2\lim x_n = 0\}$ can be a proper M-ideal (e.g. in c, c provided with the norm $|x| := \max\{\|x\|_\infty, |x_1 + 2\lim x_n|\}$). But $g(J, X) \leqslant 1/2$ whenever J is an M-ideal in X. It follows from (iv) that $g(B) \leqslant 1/2$ for every pseudoball B in J.

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