STUDIA MATHEMATICA, T. LXXXI. (1985)

Characterization of $H^p(R^n)$ in terms of generalized Littlewood–Paley g-functions

by

AKIHITO UCHIYAMA (Sendai)

Abstract. We generalize Littlewood-Paley g-functions and show that these generalized g-functions characterize $H^p(R^n)$ under certain conditions.

1. Introduction. In this note functions considered are complex-valued and measurable. For $f \in \bigcup_{1 \leqslant p \leqslant +\infty} L^p(R^n)$ let

$$u(x, t) = \int_{R^n} P_t(y) f(x-y)\, dy, \tag{1.1}$$

where $x \in R^n$, $t \in (0, +\infty)$ and

$$P_t(x) = C_n t/(|x|^2 + t^2)^{(n+1)/2},$$

that is, the Poisson kernel. With this notation, we define

$$g(f)(x) = \Big(\int_0^{+\infty} |D_t u(x, t)|^2 t\, dt\Big)^{1/2},$$

where D_t denotes $\partial/\partial t$. It is known that if $p \in (1, +\infty)$ and if $f \in L^p(R^n)$, then

$$c_p \|f\|_{L^p} \leqslant \|g(f)\|_{L^p} \leqslant C_p \|f\|_{L^p}, \tag{1.2}$$

where $0 < c_p$ and $C_p < \infty$. (See E. Stein [12], p. 82.)

Following C. Fefferman and E. Stein [7] we define $H^p(R^n)$. Let $\eta \in \mathscr{S}(R^n)$ be such that

$$\operatorname{supp} \eta \subset \{x \in R^n: |x| < 1\} \quad \text{and} \quad \int \eta(x)\, dx = 1. \tag{1.3}$$

For $f \in \mathscr{S}'(R^n)$ let

$$f^*(x) = \sup_{t>0} |f * (\eta)_t(x)| \quad \text{and} \quad \|f\|_{H^p} = \Big(\int_{R^n} f^*(x)^p\, dx\Big)^{1/p},$$

Supported in part by NSF MCS-8203319.

where

$$(\eta)_t(x) = t^{-n}\eta(x/t).$$

It is known that $\|\cdot\|_{H^p}$ is essentially independent of the choice of η. Set

$$H^p(R^n) = \{f \in \mathscr{S}'(R^n)\colon \|f\|_{H^p} < +\infty\}.$$

If $p > 1$, then H^p coincides with L^p by the Hardy–Littlewood maximal theorem.

If $f \in H^p(R^n)$, then we define

$$u(x, t) = \lim_{\varepsilon \downarrow 0} \int \big(f * (\eta)_\varepsilon\big)(x-y)\, P_t(y)\, dy. \tag{1.4}$$

(It is known that $f*(\eta)_\varepsilon \in L^\infty(R^n)$ and that this limit exists. If $f \in L^q(R^n)$ and $q \geqslant 1$, then the above two definitions (1.1) and (1.4) coincide. See Lemmas 2.B and 2.C in Section 2.) It was shown by C. Fefferman and E. Stein [7] that if $p \in (0, +\infty)$ and if $f \in H^p(R^n)$, then

$$c_p \|f\|_{H^p} \leqslant \|g(f)\|_{L^p} \leqslant C_p \|f\|_{H^p}, \tag{1.5}$$

where $0 < c_p$ and $C_p < +\infty$.

In this note we replace the Poisson kernel $P_t(x)$ by more general kernels and show that g-functions defined from general kernels satisfy inequality (1.5) under certain conditions. As far as the author knows, C. Fefferman–Stein's proof of the first inequality of (1.5) for the case $p \leqslant 1$ crucially uses the harmonicity and the semigroup property of the Poisson kernel. So, we have to develop a method that does not appeal to harmonicity. Our idea is to extend the method in Stein [12], Chapter 4, which uses vector-valued singular integral operators.

Let $E \subset (0, +\infty)$ be a measurable set. Let μ be a positive measure on E such that

$$1 \geqslant \mu\big((t, 2t) \cap E\big) \tag{1.6}$$

for any $t > 0$. Let α_0 be a positive integer. Let $\alpha_1 > 0$. Let $\{\varphi_i(x, t)\}_{i=1}^N$ be measurable functions defined on $R^n \times E$ such that

$$|D_x^\gamma \varphi_i(x, t)| \leqslant t^{-n-l(\gamma)}(1+|x|/t)^{-n-1-l(\gamma)} \tag{1.7}$$

for any multi-index $\gamma = (\gamma_1, \ldots, \gamma_n)$ with

$$l(\gamma) = \sum_{j=1}^n \gamma_j \leqslant \alpha_0,$$

$$|D_\xi^\gamma \mathscr{F}\varphi_i(\xi, t)| \leqslant t|\xi|^{1-l(\gamma)}, \quad \xi \neq 0, \tag{1.8}$$

for any multi-index γ with $l(\gamma) \leqslant n+\alpha_0+1$ and such that

$$\mu\Big\{t \in E\colon \sum_{i=1}^N |\mathscr{F}\varphi_i(\xi, t)| > \alpha_1\Big\} > \alpha_1 \tag{1.9}$$

for any $\xi \in R^n \setminus \{0\}$, where D_x^γ denotes $\partial^{\gamma_1+\ldots+\gamma_n}/\partial x_1^{\gamma_1}\ldots\partial x_n^{\gamma_n}$ and $\mathscr{F}\varphi(\xi, t)$ denotes the Fourier transform of $\varphi(x, t)$ with respect to the variable x.

If $p \in [n/(n+\alpha_0), +\infty)$ and if $f \in H^p(R^n)$, then we define

$$f * \varphi_i(x, t) = \lim_{\varepsilon \downarrow 0} \int \big(f*(\eta)_\varepsilon\big)(x-y)\, \varphi_i(y, t)\, dy.$$

(It is known that this limit exists. See Lemmas 2.B and 2.C.)

THEOREM. *If $p \in \big(n/(n+\alpha_0), +\infty\big)$ and if $f \in H^p(R^n)$, then*

$$c\|f\|_{H^p} \leqslant \sum_{i=1}^N \Big\{\int_{R^n}\Big(\int_E |f*\varphi_i(x, t)|^2\, d\mu(t)\Big)^{p/2} dx\Big\}^{1/p} \leqslant C\|f\|_{H^p},$$

where c and C are positive constants depending only on α_0, α_1, N, p and n.

Remark 1.1. Except the first inequality for the case $p \in \big(n/(n+\alpha_0), 1\big]$, our Theorem is essentially known.

EXAMPLE 1. The case $N = 1$, $E = (0, +\infty)$, $d\mu = dt/t$ and $\varphi(x, t) = tD_t P_t(x)$ is the usual Littlewood–Paley g-function. Since (1.7) and (1.8) hold for any α_0, we get (1.5) as a result of our Theorem.

EXAMPLE 2. $N = 1$, $E = \{2^k\colon k = 0, \pm 1, \pm 2, \ldots\}$, $\mu(A) =$ the number of elements in A and $\varphi(x, 2^k) = [tD_t P_t(x)]_{t=2^k}$. As a result of our theorem we get

$$c_p\|f\|_{H^p} \leqslant \Big\|\Big\{\sum_{k=-\infty}^{+\infty} 2^{2k}|D_t u(x, 2^k)|^2\Big\}^{1/2}\Big\|_{L^p} \leqslant C_p\|f\|_{H^p}$$

for any $p \in (0, +\infty)$ and for any $f \in H^p$, where u is defined by (1.4).

EXAMPLE 3. $N = n$, $E = (0, +\infty)$, $d\mu = dt/t$, $\varphi_i(x, t) = tD_{x_i}P_t(x)$ for $i \in \{1, \ldots, n\}$. As a result of our theorem we get

$$c_p\|f\|_{H^p} \leqslant \sum_{i=1}^n \Big\|\Big\{\int_0^{+\infty} |D_{x_i}u(x, t)|^2\, t\, dt\Big\}^{1/2}\Big\|_{L^p} \leqslant C_p\|f\|_{H^p} \tag{1.10}$$

for any $p \in (0, +\infty)$ and any $f \in H^p$, where u is defined by (1.4). This is also called the Littlewood–Paley g-function and inequality (1.10) is known.

EXAMPLE 4. Let $\varphi_1, \ldots, \varphi_N \in \mathscr{S}(R^n)$ be such that $\int_{R^n} \varphi_i(x)\, dx = 0$ and such that $\sup_{t \in (0,+\infty)} \sum_{i=1}^N |\mathscr{F}\varphi_i(t\xi)| > 0$ for any $\xi \in R^n \setminus \{0\}$. (The author learned this condition from Calderón and Torchinsky [1].) Put $\varphi_i(x, t) = (\varphi_i)_t(x)$ for $i = 1, \ldots, N$. Then $\{\varphi_i(x, t)\}_{i=1}^N$ satisfy (1.6)–(1.9) with $E = (0, +\infty)$, $d\mu = dt/t$, any α_0 and with appropriate α_1. Thus as a result of our theorem we get

$$c_p\|f\|_{H^p} \leqslant \sum_{i=1}^N \Big\|\Big\{\int_0^{+\infty} |(\varphi_i)_t * f(x)|^2\, dt/t\Big\}^{1/2}\Big\|_{L^p} \leqslant C_p\|f\|_{H^p}$$

for any $p \in (0, +\infty)$ and any $f \in H^p$.

In the following part of this paper, we give a proof of the Theorem. We show this only for the case $N=1$. The general case follows from a very easy modification. We write $\varphi(x,t)$ instead of $\varphi_1(x,t)$.

In Section 2, we prepare several basic lemmas. In Section 3 we explain vector-valued singular integral operators. In Section 4 we give the proof of our theorem, where Lemma 4.1 is crucial. We prove this lemma in Section 5.

Notation. $[\alpha]$ denotes the integral part of a real number α. $\chi_K(x)$ denotes the characteristic function of a set K. For a function $f(x)$, $\check{f}(x)$ denotes $f(-x)$. For a function $\psi(x,t)$, $\check{\psi}(x,t)$ denotes $\psi(-x,t)$. For $(x,t)\in R_+^{n+1} = \{(x,t):\ x\in R^n,\ t>0\}$, $B(x,t)$ denotes the ball $\{y\in R^n:\ |x-y|<t\}$. $2B(x,t)$ denotes $B(x,2t)$. $Q(B(x,t))$ denotes $\{(y,s)\in R_+^{n+1}:\ y\in B(x,t) \text{ and } s\in(0,t]\}$. The letter C denotes various constants >1. The letter c denotes various positive constants <1.

2. Basic lemmas.

DEFINITION 2.1. Let $p\in(0,1]$. A function $a(x)$ is called a *p-atom* if there exists $B=B(x_0,t_0)$ such that

(2.1) $$\operatorname{supp} a\subset B,$$

(2.2) $$\|a\|_{L^\infty}\leqslant |B|^{-1/p},$$

(2.3) $$\int a(x)x^\gamma\,dx=0 \text{ for any multi-index } \gamma \text{ with } l(\gamma)\leqslant n(1/p-1).$$

LEMMA 2.A. *Let $p\in(0,1]$ and let $f\in\mathscr{S}'(R^n)$. Then*

(2.4) $$c\|f\|_{H^p}\leqslant \inf\Big\{\Big(\sum_{i=1}^{+\infty}|\lambda_i|^p\Big)^{1/p}: \text{there exists a sequence of } p\text{-atoms } \{a_i(x)\}_{i=1}^{\infty} \text{ such that } f=\lim_{n\to\infty \text{ in } \mathscr{S}'}\sum_{i=1}^{n}\lambda_i a_i\Big\}\leqslant C\|f\|_{H^p},$$

where C and c are positive constants depending only on p and n and where we define $\inf(\emptyset)=+\infty$.

This was shown by R. Coifman [3] and R. Latter [10]. (See also R. Latter and A. Uchiyama [11].)

Remark 2.1. If $f\in H^p(R^n)\cap L^2(R^n)$, then in Lemma 2.A we can replace the limit in (2.4) by

$$f=\lim_{n\to\infty \text{ in } L^2}\sum_{i=1}^{n}\lambda_i a_i.$$

DEFINITION 2.2. Let $\alpha>0$. For $g\in L^1_{\text{loc}}(R^n)$ let

$$\|g\|_{\text{Lip}\alpha}=\sup_B\ \inf_{P:\deg P\leqslant\alpha}|B|^{-1-\alpha/n}\int_B|g(x)-P(x)|\,dx,$$

where the supremum is taken over all balls B in R^n and the infimum is taken over all polynomials $P(x)$ of degree $\leqslant\alpha$. Let

$$\text{Lip}\,\alpha=\{g\in L^1_{\text{loc}}(R^n):\ \|g\|_{\text{Lip}\alpha}<+\infty\}.$$

Remark 2.2. Let $\alpha'\geqslant\alpha$. Then it is known that

$$\sup_B\ \inf_{P:\deg P\leqslant\alpha'}|B|^{-1-\alpha/n}\int_B|g(x)-P(x)|\,dx$$

gives an equivalent norm with $\|g\|_{\text{Lip}\alpha}$ for any g with compact support.

LEMMA 2.B. *Let $0<p<1$, $\alpha=n(1/p-1)$, $f\in H^p(R^n)\cap L^\infty(R^n)$ and $g\in\text{Lip}\,\alpha\cap L^1(R^n)$. Then*

$$\Big|\int f(x)g(x)\,dx\Big|\leqslant C\|f\|_{H^p}\|g\|_{\text{Lip}\alpha},$$

where C is a constant depending only on p and n.

See R. Coifman and G. Weiss [4], C. Fefferman and E. Stein [7] or P. Duren, B. Romberg and A. Schields [6]. The following is also well known.

LEMMA 2.C. *Let $\eta\in\mathscr{S}(R^n)$ satisfy (1.3). Then*

$$\lim_{\varepsilon\downarrow 0}\|f-f*(\eta)_\varepsilon\|_{H^p}=0$$

for any $p\in(0,+\infty)$ and any $f\in H^p(R^n)$.

DEFINITION 2.3. For $f\in L^1_{\text{loc}}(R^n)$, $x\in R^n$ and $q>0$, let

$$M_q(f)(x)=\sup_{t>0}\Big(|B(x,t)|^{-1}\int_{B(x,t)}|f(y)|^q\,dy\Big)^{1/q}.$$

LEMMA 2.D. *Let $p>q$ and $f\in L^1_{\text{loc}}(R^n)$. Then*

$$\|M_q(f)\|_{L^p}\leqslant C\|f\|_{L^p},$$

where C is a constant depending only on p, q and n.

This is an easy consequence of the Hardy–Littlewood maximal theorem.

In the following part of this paper, we assume that $\varphi(x,t)$, μ, E, α_0, α_1 and $\eta(x)$ satisfy all the assumptions in Section 1.

LEMMA 2.1.

$$|\mathscr{F}\varphi(\xi,t)|\leqslant C\min(|t\xi|,\ |t\xi|^{-1}),$$

where C is a constant depending only on n.

This is clear from (1.7)–(1.8).

LEMMA 2.2. *For each $t\in E$ there exist $\{g_{t,i}(x)\}_{i=0}^{\infty}\subset\text{Lip}\,\alpha_0$ such that*

$$\varphi(x,t)=\sum_{i=0}^{\infty}2^{-i}(g_{t,i})_{2^i t}(x),$$

(2.5) $$\operatorname{supp} g_{t,i}\subset B(0,1),$$

(2.6) $$\|D_x^\gamma g_{t,i}\|_{L^\infty}\leqslant C \quad \text{if} \quad l(\gamma)\leqslant\alpha_0,$$

and

(2.7) $$\int g_{t,i}(x)\,dx = 0,$$

where C is a constant depending only on α_0 and n.

Proof. We show this only for the case $t = 1$. Put $\varphi(x) = \varphi(x, 1)$. Let $h \in \mathscr{S}(R^n)$ be a nonnegative function such that

(2.8) $$\operatorname{supp} h \subset B(0, 1)\backslash B(0, 1/4)$$

and that

(2.9) $$\sum_{i=1}^{\infty} h(2^{-i}x) = 1 \quad \text{on } B(0, 1)^c.$$

Set

$$\varphi(x) = \Big(1 - \sum_{i=1}^{\infty} h(2^{-i}x)\Big)\varphi(x) + \sum_{i=1}^{\infty} h(2^{-i}x)\varphi(x)$$

$$= \theta_0(x) + \sum_{i=1}^{\infty} \theta_i(x)$$

$$= \{\theta_0(x) + \int \sum_{k=1}^{\infty} \theta_k(y)\,dy\, h(x)/\int h(y)\,dy\} +$$

$$+ \sum_{i=1}^{\infty} \{\theta_i(x) - \int \sum_{k=i}^{\infty} \theta_k(y)\,dy\, h(2^{-i+1}x)/\int h(2^{-i+1}y)\,dy +$$

$$+ \int \sum_{k=i+1}^{\infty} \theta_k(y)\,dy\, h(2^{-i}x)/\int h(2^{-i}y)\,dy\}$$

$$= g_0(x) + \sum_{i=1}^{\infty} 2^{-i}(g_i)_{2^i}.$$

(2.5) follows from (2.8)–(2.9). (2.6) follows from (1.7). (2.7) for $i = 0$ follows from $\int \varphi(x)\,dx = 0$. ■

LEMMA 2.3. *There exists a measurable function $\psi(x, t)$ defined on $R^n \times E$ such that*

(2.10) $$\int_E \mathscr{F}\varphi(\xi, t)\,\mathscr{F}\psi(\xi, t)\,d\mu(t) = 1$$

for any $\xi \in R^n\backslash\{0\}$,

(2.11) $$\operatorname{supp} \mathscr{F}\psi(\cdot, t) \subset B(0, Ct^{-1})\backslash B(0, ct^{-1}),$$

(2.12) $$|D_\xi^\gamma \mathscr{F}\psi(\xi, t)| \leqslant Ct^{l(\gamma)} \text{ for any } \gamma \text{ with } l(\gamma) \leqslant n + \alpha_0 + 1,$$

and such that

(2.13) $$|D_x^\gamma \psi(x, t)| \leqslant Ct^{-n-l(\gamma)}(1 + |x|/t)^{-n-\alpha_0-1} \text{ for any } \gamma \text{ with } l(\gamma) \leqslant \alpha_0,$$

where C and c are positive constants depending only on α_0, α_1 and n.

Proof. By (1.8), (1.9) and Lemma 2.1 for each $\xi \in R^n\backslash\{0\}$ there exists $E(\xi) \subset E$ such that

(2.14) $$|\mathscr{F}\varphi(\xi, t)| \geqslant \alpha_1 \quad \text{if} \quad t \in E(\xi),$$

(2.15) $$E(\xi) \subset (c_0|\xi|^{-1}, C_0|\xi|^{-1}),$$

(2.16) $$\mu(E(\xi)) > \alpha_1,$$

(2.17) $$|\mathscr{F}\varphi(\xi, t) - \mathscr{F}\varphi(\xi', t)| \leqslant \alpha_1/2 \text{ if } t \in E(\xi) \text{ and if } |\xi' - \xi| < c_0|\xi|,$$

where $c_0 \in (0, 1/2)$ and $C_0 > 1$ are constants depending only on α_1 and n. Take a sequence $\{\xi_k\}_{k=-\infty}^{+\infty} \subset R^n$ such that

(2.18) $$\#\{k:\ t < |\xi_k| < 2t\} < C$$

for any $t > 0$ and such that

$$\bigcup_{k=-\infty}^{+\infty} B(\xi_k, 2^{-1}c_0|\xi_k|) = R^n\backslash\{0\}.$$

Let $\{u_k(\xi)\}_{k=-\infty}^{+\infty} \subset \mathscr{S}(R^n)$ be a partition of unity of $R^n\backslash\{0\}$ such that $\sum_{k=-\infty}^{+\infty} u_k(\xi) \equiv 1$ on $R^n\backslash\{0\}$,

(2.19) $$\operatorname{supp} u_k \subset B(\xi_k, c_0|\xi_k|),$$

(2.20) $$\|D^\gamma u_k\|_{L^\infty} \leqslant C_\gamma |\xi_k|^{-l(\gamma)} \quad \text{for any } \gamma.$$

Set

$$v_k(\xi) = \{\int_{E(\xi_k)} \mathscr{F}\varphi(\xi, t)/\mathscr{F}\varphi(\xi_k, t)\,d\mu(t)\}^{-1} u_k(\xi).$$

Note that by (2.14), (2.17) and (2.19)

(2.21) $$\operatorname{Re}\big(\mathscr{F}\varphi(\xi, t)/\mathscr{F}\varphi(\xi_k, t)\big) > 1/2 \quad \text{if } u_k(\xi) \neq 0 \text{ and if } t \in E(\xi_k).$$

Thus $|v_k(\xi)| \leqslant 2/\alpha_1$ by (2.16). Set $\psi(x, t)$ so that

$$\mathscr{F}\psi(\xi, t) = \sum_{k: E(\xi_k) \ni t} v_k(\xi)/\mathscr{F}\varphi(\xi_k, t).$$

Then

$$\int_E \mathscr{F}\varphi(\xi, t)\,\mathscr{F}\psi(\xi, t)\,d\mu(t)$$

$$= \sum_{k=-\infty}^{+\infty} u_k(\xi) \int_{E(\xi_k)} \mathscr{F}\varphi(\xi, t)/\mathscr{F}\varphi(\xi_k, t)\,d\mu(t) \cdot$$

$$\cdot \{\int_{E(\xi_k)} \mathscr{F}\varphi(\xi, t)/\mathscr{F}\varphi(\xi_k, t)\,d\mu(t)\}^{-1}$$

$$= 1 \quad \text{if} \quad \xi \in R^n\backslash\{0\}.$$

Condition (2.11) follows from (2.19) and (2.15). Condition (2.12) follows from (1.8), (2.18), (2.20) and (2.21). Condition (2.13) follows easily from (2.11)–(2.12). ■

3. Vector-valued functions. The author learned main ideas in this section from Stein [12], Chapter 4.

Let $\mathscr{H}$ be the Hilbert space of measurable functions $\theta(t)$ defined on E satisfying

$$|||\theta||| = \left(\int_E |\theta(t)|^2 \, d\mu(t)\right)^{1/2} < +\infty.$$

For θ and $\zeta \in \mathscr{H}$ let

$$\langle \theta, \zeta \rangle = \int_E \theta(t) \overline{\zeta(t)} \, d\mu(t).$$

For $p \in (1, +\infty)$, $L^p(R^n, \mathscr{H})$ denotes the set of strongly measurable $\mathscr{H}$-valued functions $F(x)$ satisfying

$$\|F\|_{L^p(R^n, \mathscr{H})} = \left(\int |||F(x)|||^p \, dx\right)^{1/p} < +\infty.$$

For $F, G \in L^2(R^n, \mathscr{H})$ and $f \in L^2(R^n)$ let

$$F * f(x) = \int F(x-y) f(y) \, dy$$

and

$$F * G(x) = \int \langle F(x-y), \bar{G}(y) \rangle \, dy,$$

where $\bar{G}$ denotes the element in $L^2(R^n, \mathscr{H})$ such that $\bar{G}(y, t) = \overline{G(y, t)}$.

Let $\varepsilon > 0$. Let

$$\varphi_\varepsilon(x, t) = \varphi(x, t) \chi_{[\varepsilon, +\infty)}(t).$$

By (1.7) for any $x \in R^n$ $\varphi_\varepsilon(x, t)$ belongs to $\mathscr{H}$ as a function of t. We define

$$\Phi_\varepsilon(x) = \varphi_\varepsilon(x, \cdot)$$

as an $\mathscr{H}$-valued function defined on R^n. Similarly, let

$$\psi_\varepsilon(x, t) = \psi(x, t) \chi_{[\varepsilon, +\infty)}(t)$$

and

$$\Psi_\varepsilon(x) = \psi_\varepsilon(x, \cdot).$$

LEMMA 3.1. *If $f \in L^2(R^n)$, then*

$$\Psi_\varepsilon * (\Phi_\varepsilon * f) \to f \quad \text{in} \quad L^2(R^n) \quad \text{as} \quad \varepsilon \to +0.$$

This is clear from (2.10).

LEMMA 3.2.

$$|||\Phi_\varepsilon(x)||| \leqslant C \min(|x|^{-n}, \varepsilon^{-n}),$$

where C is a constant depending only on n.

This follows easily from (1.7).

Since $\Phi_\varepsilon \in L^2(R^n, \mathscr{H})$ by Lemma 3.2, we can define its Fourier transform.

LEMMA 3.3.

$$|||\mathscr{F}\Phi_\varepsilon(\xi)||| \leqslant C,$$

where C is a constant depending only on n.

This follows easily from Lemma 2.1.

LEMMA 3.4. *If $f \in L^2(R^n)$ and if $F \in L^2(R^n, \mathscr{H})$, then*

(3.1) $$\|\Phi_\varepsilon * f\|_{L^2(R^n, \mathscr{H})} \leqslant C \|f\|_{L^2},$$

and

(3.2) $$\|\check{\Phi}_\varepsilon * F\|_{L^2} \leqslant C \|F\|_{L^2(R^2, \mathscr{H})},$$

where $\check{\Phi}_\varepsilon(x) = \Phi_\varepsilon(-x)$ and where C is a constant depending only on n.

This is clear from Lemma 3.3.

LEMMA 3.5. *Let $s > 0$, $l(\gamma) \leqslant \alpha_0$ and $x \in R^n \setminus \{0\}$. Let $\eta \in \mathscr{S}(R^n)$ and $\operatorname{supp} \eta \subset B(0, 1)$. Then*

$$|||D_x^\gamma(\Phi_\varepsilon * (\eta)_s)(x)||| \leqslant C|x|^{-n-l(\gamma)},$$

where C is a constant depending only on α_0, n and η.

Proof. It is clear that $\Phi_\varepsilon * (\eta)_s(x)$ is an $\mathscr{H}$-valued C^∞-function and that $D_x^\gamma(\Phi_\varepsilon * (\eta)_s)(x)$ assigns $D_x^\gamma((\eta)_s * \varphi_\varepsilon(x, t))$ to each $t \in E$.

If $s < \max(t, |x|/2)$, then by (1.7)

$$\begin{aligned} |D_x^\gamma((\eta)_s * \varphi(x, t))| &= |(\eta)_s * D_x^\gamma \varphi(x, t)| \\ &\leqslant \int |(\eta)_s(x-y)| \, t^{-n-l(\gamma)} (1 + |y|/t)^{-n-1-l(\gamma)} \, dy \\ &\leqslant C t^{-n-l(\gamma)} (1 + |x|/t)^{-n-1-l(\gamma)}. \end{aligned}$$

If $s \geqslant \max(t, |x|/2)$, then

$$\begin{aligned} |D_x^\gamma((\eta)_s * \varphi(x, t))| &= s^{-l(\gamma)} |(D^\gamma \eta)_s * \varphi(x, t)| \\ &= s^{-l(\gamma)} \int \{(D^\gamma \eta)_s(x-y) - (D^\gamma \eta)_s(x)\} \varphi(y, t) \, dy \\ &\leqslant C s^{-n-l(\gamma)} \int \min(|y|/s, 1) \, t^{-n} (1 + |y|/t)^{-n-1} \, dy \\ &\leqslant C s^{-n-1-l(\gamma)} t \int_{|y| < s} (|y|/t) \, t^{-n} (1 + |y|/t)^{-n-1} \, dy + \end{aligned}$$

$$+Cs^{-n-l(\gamma)}\int_{|y|\geqslant s} t^{-n}(1+|y|/t)^{-n-1}\,dy$$
$$\leqslant Cs^{-n-l(\gamma)}(t/s)\log(s/t+1)+Cs^{-n-l(\gamma)}(t/s)$$
$$\leqslant C(t+|x|)^{-n-l(\gamma)}(1+|x|/t)^{-1}\log(2+|x|/t)$$
$$\leqslant Ct^{-n-l(\gamma)}(1+|x|/t)^{-n-1-l(\gamma)}\log(2+|x|/t).$$

By the above two estimates we have

$$|D_x^\gamma((\eta)_s*\varphi(x,t))|\leqslant Ct^{-n-l(\gamma)}(1+|x|/t)^{-n-1-l(\gamma)}\log(2+|x|/t).$$

Thus

$$|||D_x^\gamma((\eta)_s*\Phi_\varepsilon)(x)|||=\{\int_E|D_x^\gamma((\eta)_s*\varphi_\varepsilon(x,t))|^2\,d\mu(t)\}^{1/2}$$
$$\leqslant C|x|^{-n-l(\gamma)}. \qquad ■$$

Next we define the H^p-norm of $\mathscr{H}$-valued functions. Let $0<p<\infty$. Let $\eta\in\mathscr{S}(R^n)$ satisfy (1.3). For $F\in L^2(R^n,\mathscr{H})$ let

$$F^*(x)=\sup_{s>0}|||F*(\eta)_s(x)|||. \tag{3.3}$$

Set

$$\|F\|_{H^p(R^n,\mathscr{H})}=\|F^*\|_{L^p}.$$

We can show that $\|\cdot\|_{H^p(R^n,\mathscr{H})}$ is essentially independent of the choice of η in the same way as in the scalar-valued case. We define $H^p(R^n,\mathscr{H})$ to be the completion of

$$\{F\in L^2(R^n,\mathscr{H}):\ \|F\|_{H^p(R^n,\mathscr{H})}<+\infty\}$$

with respect to the quasi-metric $\|\cdot\|_{H^p(R^n,\mathscr{H})}$. If $1<p<\infty$, this space coincides with $L^p(R^n,\mathscr{H})$.

LEMMA 3.6. *If $p\in(n/(n+\alpha_0),1]$ and if $f\in H^p(R^n)\cap L^2(R^n)$, then*

$$\|\Phi_\varepsilon*f\|_{H^p(R^n,\mathscr{H})}\leqslant C\|f\|_{H^p},$$

where C is a constant depending only on p, α_0 and n.

Proof. By Lemma 2.A and Remark 2.1, it is enough to show this only for the case where f equals a p-atom $a(x)$ that satisfies (2.1)–(2.3). Let $B=B(x_0,t_0)$ be as in (2.1)–(2.3). Let $\eta\in\mathscr{S}(R^n)$ satisfy (1.3).

Let $|x-x_0|>2t_0$. Put $\Phi_{\varepsilon,s}(x)=(\eta)_s*\Phi_\varepsilon(x)$ and $\alpha'=[n(1/p-1)]+1$. Note that $\alpha'\leqslant\alpha_0$. Then

$$|||(\eta)_s*(\Phi_\varepsilon*a)(x)|||=|||\Phi_{\varepsilon,s}*a(x)|||$$
$$\leqslant t_0^{-n/p}\int_B\Big|\Big|\Big|\Phi_{\varepsilon,s}(x-y)-\sum_{|\gamma|<\alpha'}(\gamma!)^{-1}D^\gamma\Phi_{\varepsilon,s}(x-x_0)(x_0-y)^\gamma\Big|\Big|\Big|\,dy$$
$$\leqslant Ct_0^{-n/p+n+\alpha'}|x-x_0|^{-n-\alpha'}\quad\text{by Lemma 3.5.}$$

Thus

$$(\Phi_\varepsilon*a)^*(x)\leqslant Ct_0^{-n/p+n+\alpha'}|x-x_0|^{-n-\alpha'}. \tag{3.4}$$

On the other hand, by (3.1)

$$\int_{B(x_0,2t_0)}(\Phi_\varepsilon*a)^*(x)^p\,dx\leqslant|B|^{1-p/2}\Big(\int_{B(x_0,2t_0)}(\Phi_\varepsilon*a)^*(x)^2\,dx\Big)^{p/2} \tag{3.5}$$
$$\leqslant C|B|^{1-p/2}\|a\|_{L^2}^p\leqslant C.$$

Combining (3.4) and (3.5), we get

$$\int_{R^n}(\Phi_\varepsilon*a)^*(x)^p\,dx\leqslant C. \qquad ■$$

Since the atomic decomposition of $\mathscr{H}$-valued H^p functions holds, by an argument similar to that of Lemma 3.6 we get

LEMMA 3.7. *If $p\in(n/(n+\alpha_0),1]$ and if $F\in H^p(R^n,\mathscr{H})\cap L^2(R^n,\mathscr{H})$, then*

$$\|\check{\Phi}_\varepsilon*F\|_{H^p}\leqslant C\|F\|_{H^p(R^n,\mathscr{H})},$$

where C is a constant depending only on p, α_0 and n.

Interpolating the lemmas above, we get

LEMMA 3.8. *If $p\in(n/(n+\alpha_0),+\infty)$ and if $f\in H^p(R^n)\cap L^2(R^n)$, then*

$$\|\Phi_\varepsilon*f\|_{H^p(R^n,\mathscr{H})}\leqslant C\|f\|_{H^p},$$

where C is a constant depending only on p, α_0 and n.

Proof. *Case* 1: $n/(n+\alpha_0)<p\leqslant 2$. This case follows from (3.1), Lemma 3.6 and interpolation theorems.

Case 2: $2<p<+\infty$. Let $1/p+1/p'=1$. Interpolating the estimates in Lemma 3.7 and (3.2), we get

$$\|\check{\Phi}_\varepsilon*F\|_{L^{p'}}\leqslant C\|F\|_{L^{p'}(R^n,\mathscr{H})} \tag{3.6}$$

for any $F\in L^{p'}(R^n,\mathscr{H})\cap L^2(R^n,\mathscr{H})$. Therefore

$$\|\Phi_\varepsilon*f\|_{L^p(R^n,\mathscr{H})}=\sup\Big\{\Big|\int\langle\Phi_\varepsilon*f(x),F(x)\rangle\,dx\Big|:$$
$$F\in L^{p'}(R^n,\mathscr{H})\cap L^2(R^n,\mathscr{H}),\|F\|_{L^{p'}}<1\Big\}$$
$$=\sup\Big\{\Big|\int f(x)\check{\Phi}_\varepsilon*\bar{F}(x)\,dx\Big|:\ldots\Big\}$$
$$\leqslant C\|f\|_{L^p}\quad\text{by (3.6).}\qquad ■$$

By almost the same argument, we can show that if $p\in(n/(n+\alpha_0),+\infty)$ and if $F\in H^p(R^n,\mathscr{H})\cap L^2(R^n,\mathscr{H})$, then

$$\|\check{\Phi}_\varepsilon*F\|_{H^p}\leqslant C\|F\|_{H^p(R^n,\mathscr{H})},$$

where C is a constant depending only on p, α_0 and n.

Applying the same argument to Ψ_ε instead of $\check{\Phi}_\varepsilon$, we get

LEMMA 3.9. *If* $p\in(n/(n+\alpha_0), +\infty)$ *and if* $F\in H^p(R^n, \mathscr{H})\cap L^2(R^n, \mathscr{H})$, *then*

$$\|\Psi_\varepsilon * F\|_{H^p} \leqslant C\|F\|_{H^p(R^n, \mathscr{H})},$$

where C *is a constant depending only on* p, α_0, α_1 *and* n.

4. Proof of the Theorem. For $f\in \bigcup_{p\in(n/(n+\alpha_0), +\infty)} H^p(R^n)$ let

$$\mathfrak{g}(f)(x) = \{\int_E |f*\varphi(x, t)|^2 d\mu(t)\}^{1/2}.$$

By Lemma 3.8 we get that if $p\in(n/(n+\alpha_0), +\infty)$ and if $f\in H^p(R^n)\cap L^2(R^n)$, then

$$\|\mathfrak{g}(f)\|_{L^p} \leqslant C\|f\|_{H^p},$$

where C is a constant depending only on p, α_0 and n. Thus for the proof of the Theorem it is enough to show that if $p\in(n/(n+\alpha_0), +\infty)$ and if $f\in H^p(R^n)\cap L^2(R^n)$, then

$$\|f\|_{H^p} \leqslant C\|\mathfrak{g}(f)\|_{L^p}. \tag{4.1}$$

(The nonessential restriction $f\in L^2$ will be removed at the end of this section.) To show (4.1) we need the following Lemma 4.1, which we prove in Section 5.

LEMMA 4.1. *Let* $\beta\in(0, \alpha_0)$, $q\geqslant n/(n+\beta)$ *and* $s>0$. *Let* $f\in L^2(R^n)$. *Let* $\varkappa\in \mathrm{Lip}\,\beta$ *be such that*

$$\operatorname{supp}\varkappa\subset B(0, 1), \tag{4.2}$$

$$\|\varkappa\|_{\mathrm{Lip}\beta}\leqslant 1 \tag{4.3}$$

and

$$\int\varkappa(x)\,dx = 0. \tag{4.4}$$

Then

$$\left|\int f(x)(\varkappa)_s(x)\,dx\right| \leqslant CM_q(\mathfrak{g}(f))(0),$$

where C *is a constant depending only on* β, α_0, α_1 *and* n.

Now we begin the proof of (4.1).

LEMMA 4.2. *If* $p\in(n/(n+\alpha_0), +\infty)$ *and if* $f\in H^p(R^n)\cap L^2(R^n)$, *then*

$$\|f\|_{H^p} \leqslant C\liminf_{\varepsilon\downarrow 0}\|\Phi_\varepsilon * f\|_{H^p(R^n, \mathscr{H})},$$

where C *is a constant depending only on* p, α_0, α_1 *and* n.

Proof.

$$\|f\|_{H^p} \leqslant \liminf_{\varepsilon\downarrow 0}\|\Psi_\varepsilon * \Phi_\varepsilon * f\|_{H^p} \quad \text{by Lemma 3.1}$$

$$\leqslant C\liminf_{\varepsilon\downarrow 0}\|\Phi_\varepsilon * f\|_{H^p(R^n, \mathscr{H})} \quad \text{by Lemma 3.9. } \blacksquare$$

LEMMA 4.3. *Let* $\beta\in(\alpha_0-1, \alpha_0)$, $q\geqslant n/(n+\beta)$ *and* $f\in L^2(R^n)$. *Let* $\eta\in\mathscr{S}(R^n)$ *satisfy* (1.3).

(i) *If* $0<t\leqslant 1$, *then*

$$|\eta * f * \varphi(0, t)| \leqslant Ct(1-\log_2 t)M_q(\mathfrak{g}(f))(0).$$

(ii) *If* $t>1$, *then*

$$|\eta * f * \varphi(0, t) - f * \varphi(0, t)| \leqslant Ct^{\beta-\alpha_0}M_q(\mathfrak{g}(f))(0),$$

where C *is a constant depending only on* β, α_0, α_1, η *and* n.

Proof of (i). Using Lemma 2.2, put

$$\eta * f * \varphi(0, t) = \sum_{i=0}^{\infty} 2^{-i} f * \eta * (g_{t,i})_{2^i t}(0) = \sum 2^{-i} f * v_i(0). \tag{4.5}$$

By (2.7)

$$\int v_i(x)\,dx = 0.$$

If $2^i t\leqslant 1$ (i.e. $i\leqslant[-\log_2 t]$), then by (1.3) and (2.5)–(2.7) we get that $\operatorname{supp} v_i\subset B(0, 2)$ and that

$$\|D_x^\gamma v_i\|_{L^\infty} = \|D_x^\gamma \eta * (g_{t,i})_{2^i t}\|_{L^\infty} \leqslant C_\gamma 2^i t$$

for any γ. Thus by Lemma 4.1

$$|f * v_i(0)| \leqslant C2^i t M_q(\mathfrak{g}(f))(0). \tag{4.6}$$

If $2^i t>1$ (i.e. $i>[-\log_2 t]$) and if $l(\gamma)\leqslant\alpha_0$, then by (1.3) and (2.5)–(2.6) we get that $\operatorname{supp} v_i\subset B(0, 2\cdot 2^i t)$ and that

$$\|D_x^\gamma v_i\|_{L^\infty} = \|\eta * D_x^\gamma (g_{t,i})_{2^i t}\|_{L^\infty} \leqslant C(2^i t)^{-n-l(\gamma)}.$$

Thus by Lemma 4.1

$$|f * v_i(0)| \leqslant CM_q(\mathfrak{g}(f))(0). \tag{4.7}$$

By (4.6) and (4.7) we have

$$|(4.5)| \leqslant C\sum_{i=0}^{[-\log_2 t]} 2^{-i}2^i t M_q(\mathfrak{g}(f))(0) + C\sum_{i=[-\log_2 t]+1}^{\infty} 2^{-i}M_q(\mathfrak{g}(f))(0)$$
$$\leqslant Ct(1-\log_2 t)M_q(g(f))(0).$$

Proof of (ii). Put

$$\eta * f * \varphi(0, t) - f * \varphi(0, t) = \int f(-x)\{\eta * \varphi(x, t) - \varphi(x, t)\}\,dx = \int f(-x)\xi(x)\,dx. \tag{4.8}$$

Using Lemma 2.2 and $\int\eta(y)\,dy = 1$, put

$$\xi(x) = \sum_{i=0}^{\infty} 2^{-i}\int\eta(y)\,\{(g_{t,i})_{2^i t}(x-y)-(g_{t,i})_{2^i t}(x)\}\,dy$$
$$= \sum_{i=0}^{\infty} 2^{-i}\int\eta(y)\,\zeta_i(x, y)\,dy$$
$$= \sum 2^{-i}\theta_i(x).$$

Then

(4.9) $$\operatorname{supp}\theta_i \subset B(0, 2\cdot 2^i t) \quad\text{and}\quad \int\theta_i(x)\,dx = 0.$$

If $l(\gamma) = \alpha_0-1$, then by (2.6) and the mean value theorem we have

$$|D_x^\gamma\zeta_i(x, y)| \leqslant C(2^i t)^{-n-\alpha_0}|y|.$$

If $l(\gamma) = \alpha_0$, then by (2.6) we have

$$|D_x^\gamma\zeta_i(x, y)| \leqslant C(2^i t)^{-n-\alpha_0}.$$

Hence for any x and $x_1\in R^n$ we have

(4.10) $$\Big|\zeta_i(x, y)-\sum_{|\gamma|<\alpha_0-1}(\gamma!)^{-1}D_x^\gamma\zeta_i(x_1, y)(x-x_1)^\gamma\Big| \leqslant C(2^i t)^{-n-\alpha_0}|y|\,|x-x_1|^{\alpha_0-1}$$

and

(4.11) $$\Big|\zeta_i(x, y)-\sum_{|\gamma|<\alpha_0}(\gamma!)^{-1}D_x^\gamma\zeta_i(x_1, y)(x-x_1)^\gamma\Big| \leqslant C(2^i t)^{-n-\alpha_0}|x-x_1|^{\alpha_0}.$$

Take any ball $B = B(x_1, s)$. Since

$$\min(|y|s^{\alpha_0-1},\ s^{\alpha_0}) \leqslant |y|^{\alpha_0-\beta}s^\beta \quad\text{by}\quad \alpha_0-1<\beta<\alpha_0,$$

using (4.10) and (4.11) we get

$$\inf_{P:\deg P<\alpha_0}\ \sup_{x\in B}|\zeta_i(x, y)-P(x)| \leqslant C(2^i t)^{-n-\alpha_0}|y|^{\alpha_0-\beta}s^\beta,$$

which means

$$\|\zeta_i(\cdot, y)\|_{\mathrm{Lip}\beta} \leqslant C(2^i t)^{-n-\alpha_0}|y|^{\alpha_0-\beta}.$$

Hence

(4.12) $$\|\theta_i\|_{\mathrm{Lip}\beta} \leqslant \int|\eta(y)|\,\|\zeta_i(\cdot, y)\|_{\mathrm{Lip}\beta}\,dy \leqslant \int|\eta(y)|\,C(2^i t)^{-n-\alpha_0}|y|^{\alpha_0-\beta}\,dy \leqslant C(2^i t)^{-n-\alpha_0} = C(2^i t)^{\beta-\alpha_0}(2^i t)^{-n-\beta}.$$

Thus by Lemma 4.1, (4.9) and (4.12),

$$|(4.8)| = \Big|\sum_{i=0}^{\infty}2^{-i}\int f(-x)\,\theta_i(x)\,dx\Big| \leqslant \sum 2^{-i}C(2^i t)^{\beta-\alpha_0}M_q(\mathfrak{g}(f))(0) \leqslant Ct^{\beta-\alpha_0}M_q(\mathfrak{g}(f))(0). \ \blacksquare$$

LEMMA 4.4. *Let $x_0\in R^n$, $q>n/(n+\alpha_0)$ and $f\in L^2(R^n)$. Let $\eta\in\mathscr{S}(R^n)$ satisfy (1.3). Then*

$$(\Phi_\varepsilon * f)^*(x_0) \leqslant CM_q(\mathfrak{g}(f))(x_0),$$

where C is a constant depending only on q, α_0, α_1, η and n.

Proof. (For the definition of $(\Phi_\varepsilon * f)^*$ recall (3.3).) We have to show

$$\{\int_E|(\eta)_s * f * \varphi(x_0, t)|^2\,d\mu(t)\}^{1/2} \leqslant CM_q(\mathfrak{g}(f))(x_0)$$

for any $s>0$ and $x_0\in R^n$. We may assume $q<n/(n+\alpha_0-1)$. Let $\beta = n(1/q-1)$. By translation and dilation we may assume $x_0 = 0$ and $s = 1$. By Lemma 4.3

$$\{\int_E|\eta * f * \varphi(0, t)-f * \varphi(0, t)\,\chi_{(1,+\infty)}(t)|^2\,d\mu(t)\}^{1/2} \leqslant C\{\int_{E\cap(0,1]}(t(1-\log_2 t))^2\,d\mu(t)\}^{1/2}M_q(\mathfrak{g}(f))(0)+ +C\{\int_{E\cap(1,+\infty)}t^{(\beta-\alpha_0)2}\,d\mu(t)\}^{1/2}M_q(\mathfrak{g}(f))(0) \leqslant CM_q(\mathfrak{g}(f))(0).$$

Thus

$$\{\int_E|\eta * f * \varphi(0, t)|^2\,d\mu(t)\}^{1/2} \leqslant CM_q(\mathfrak{g}(f))(0)+\mathfrak{g}(f)(0) \leqslant CM_q(\mathfrak{g}(f))(0). \ \blacksquare$$

Proof of (4.1). Let $p>n/(n+\alpha_0)$. Take q so that $n/(n+\alpha_0)<q<p$. Then by Lemmas 4.4 and 2.D,

$$\|\Phi_\varepsilon * f\|_{H^p(R^n,\mathscr{H})} = \|(\Phi_\varepsilon * f)^*\|_{L^p} \leqslant C\|M_q(\mathfrak{g}(f))\|_{L^p} \leqslant C\|\mathfrak{g}(f)\|_{L^p}.$$

Thus we get (4.1) from Lemma 4.2. ■

Finally we remove the restriction $f\in L^2$. Let $p\in(n/(n+\alpha_0), +\infty)$ and let $f\in H^p$. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}\subset H^p\cap L^2$ such that $\|f_k-f\|_{H^p}\to 0$. From the result obtained so far, we get

(4.13) $$c\|f_k\|_{H^p} \leqslant \|\mathfrak{g}(f_k)\|_{L^p} \leqslant C\|f_k\|_{H^p}.$$

Since for every $(x, t)\in R^n\times E$ we have

$$f * \varphi(x, t) = \lim_{m\to\infty} f_m * \varphi(x, t),$$

we have

$$\int_{R^n} \Big(\int_E |f*\varphi(x,t)-f_k*\varphi(x,t)|^2\,d\mu(t)\Big)^{p/2}dx$$

$$\leqslant \int \big(\liminf_{m\to\infty} \int |f_m*\varphi(x,t)-f_k*\varphi(x,t)|^2\,d\mu(t)\big)^{p/2}dx$$

$$\leqslant \liminf_{m\to\infty} \int \big(\int |f_m*\varphi(x,t)-f_k*\varphi(x,t)|^2\,d\mu(t)\big)^{p/2}dx$$

$$\leqslant C\liminf_{m\to\infty} \|f_m-f_k\|_{H^p}^p \to 0 \quad \text{as} \quad k\to\infty.$$

Therefore

$$\|\mathfrak{g}(f)\|_{L^p} = \lim_{k\to\infty} \|\mathfrak{g}(f_k)\|_{L^p}.$$

Letting $k\to\infty$ in (4.13), we get the desired result.

5. Proof of Lemma 4.1.

LEMMA 5.1. *For each $t\in E$ there exist $\{v_{t,i}(x)\}_{i=0}^{\infty}$ such that*

$$\psi(x,t) = \sum_{i=0}^{\infty} 2^{-i(\alpha_0+1)}(v_{t,i})_{2^i t}(x),$$

(5.1) $$\operatorname{supp} v_{t,i} \subset B(0,1),$$

(5.2) $$\|v_{t,i}\|_{L^\infty} \leqslant C,$$

and

(5.3) $$\int v_{t,i}(x)x^\gamma\,dx = 0 \quad \textit{for any } \gamma \textit{ with } l(\gamma)<\alpha_0,$$

where C is a constant depending only on α_0, α_1, and n.

Proof. The following argument is very similar to the proof of Lemma 2.2. We show this for the case $t=1$ only. Put $\psi(x)=\psi(x,1)$. Let $h\in\mathscr{S}(R^n)$ be as in the proof of Lemma 2.2. Let $\{\pi_j(x)\}_{j=1}^{L}$ be an orthonormal basis for the Hilbert space of polynomials of degree $<\alpha_0$ with norm

$$\|P\| = \{\int |P(x)|^2 h(x)\,dx\}^{1/2}.$$

Put

$$\psi(x) = \big(1-\sum_{i=1}^{\infty} h(2^{-i}x)\big)\psi(x) + \sum_{i=1}^{\infty} h(2^{-i}x)\psi(x) = \theta_0(x)+\sum_{i=1}^{\infty}\theta_i(x)$$

and

$$\zeta_i(x) = \sum_{j=1}^{L}\int \sum_{k=i+1}^{\infty} \theta_k(y)\pi_j(2^{-i}y)\,dy\,\pi_j(2^{-i}x)\,2^{-in}h(2^{-i}x).$$

Note that by (2.13) and by $\deg\pi_j<\alpha_0$ the above integrand is integrable and

that

(5.4) $$\|\zeta_i\|_{L^\infty} \leqslant C2^{-i(n+\alpha_0+1)}.$$

Put

$$\psi = (\theta_0+\zeta_0)+\sum_{i=1}^{\infty}(\theta_i-\zeta_{i-1}+\zeta_i) = v_0+\sum_{i=1}^{\infty} 2^{-i(\alpha_0+1)}(v_i)_{2^i}.$$

Then, condition (5.1) is clear. Condition (5.2) follows from (2.13) and (5.4). Condition (5.3) for $i\geqslant 1$ is easy. Since

$$\int \psi(x)x^\gamma\,dx = 0$$

for any γ with $l(\gamma)<\alpha_0$ by (2.11) and (2.13), condition (5.3) holds for the case $i=0$, too. ∎

DEFINITION 5.1. Let ν be a complex measure defined on R_+^{n+1}. Let $|\nu|$ be its total variation. For $\alpha\geqslant 0$ let

$$\|\nu\|_\alpha = \sup_B |\nu|(Q(B))/|B|^{1+\alpha/n},$$

where the supremum is taken over all balls B in R^n.

DEFINITION 5.2. For $f\in L^1_{\mathrm{loc}}(R^n)$ let

$$G(f,x,t) = |B(x,t)|^{-1}\int_{B(x,t)} |f(y)|\,dy.$$

LEMMA 5.A. *Let $\alpha\geqslant 0$ and $\|\nu\|_\alpha\leqslant 1$. Let $f\in L^1_{\mathrm{loc}}(R^n)$ and $p\in(1,+\infty)$. Then*

$$\Big\{\iint_{R_+^{n+1}} G(f,x,t)^{p(1+\alpha/n)}\,d|\nu|\Big\}^{1/(p(1+\alpha/n))} \leqslant C\|f\|_{L^p},$$

where C is a constant depending only on p, α and n.

For the case $\alpha=0$, this was shown by L. Carleson. (See Stein [12], p. 236, Carleson [2] and Hörmander [9].) For the case $\alpha>0$ this was shown by P. Duren [5].

LEMMA 5.2. *Let $0<\beta<\alpha$. Let $g(x,t)$ be a measurable function defined on R_+^{n+1} such that*

(5.5) $$g(x,t)=0 \quad \textit{if} \quad |x|>t,$$

and that

(5.6) $$\|t^n g(t\cdot,t)\|_{\mathrm{Lip}\alpha}\leqslant 1.$$

Let ν be a complex measure on R_+^{n+1} such that

(5.7) $$\|\nu\|_\beta\leqslant 1$$

and that $\operatorname{supp} v$ *is a bounded set. Let*

$$f(x) = \iint_{R_+^{n+1}} g(x-y,\, t)\, dv(y,\, t).$$

Then

$$\|f\|_{\mathrm{Lip}\,\beta} \leqslant C, \tag{5.8}$$

where C *is a constant depending only on* α, β *and* n.

Proof. Take an arbitrary ball $B = B(z,\, s)$. Put

$$D_1 = \{(y,\, t)\colon\ t \in (0,\, s),\ |y-z| < t+s\},$$

$$D_2 = \{(y,\, t)\colon\ t \geqslant s,\ |y-z| < t+s\}.$$

Then, for $x \in B$ we have

$$f(x) = \iint_{D_1} g(x-y,\, t)\, dv(y,\, t) + \iint_{D_2} g(x-y,\, t)\, dv(y,t) = \zeta(x) + \theta(x).$$

Since $\int |g(x,\, t)|\, dx < C$ by (5.5)–(5.6), we get

$$\int_B |\zeta(x)|\, dx \leqslant C \iint_{D_1} d|v|(y,\, t) \leqslant C|B|^{1+\beta/n} \tag{5.9}$$

by (5.7). Since

$$\Big\|\iint_{D_2} g(\cdot - y,\, t)\, dv(y,\, t)\Big\|_{\mathrm{Lip}\,\alpha} \leqslant \iint_{D_2} \|g(\cdot - y,\, t)\|_{\mathrm{Lip}\,\alpha}\, d|v|(y,\, t)$$

$$\leqslant \iint_{D_2} t^{-n-\alpha}\, d|v|(y,\, t) \qquad \text{by (5.6)}$$

$$\leqslant \sum_{i=0}^{\infty} (2^i s)^{-n-\alpha} \iint_{D_2 \cap (Q(2^{i+1}B) \setminus Q(2^i B))} d|v|(y,\, t)$$

$$\leqslant C \sum (2^i s)^{-n-\alpha} (2^i s)^{n+\beta} \qquad \text{by (5.7)}$$

$$\leqslant C s^{\beta-\alpha} \qquad \text{by } \alpha > \beta,$$

we get

$$\inf_{\deg P \leqslant \alpha} \int_B |\theta(x) - P(x)|\, dx \leqslant C s^{n+\alpha} s^{\beta-\alpha} = C|B|^{1+\beta/n}. \tag{5.10}$$

Combining (5.9) and (5.10), we get

$$\inf_{\deg P \leqslant \alpha} \int_B |f(x) - P(x)|\, dx \leqslant C|B|^{1+\beta/n}.$$

Then (5.8) follows from Remark 2.2. ■

Now we begin the proof of Lemma 4.1. Let $\varkappa$ satisfy (4.2)–(4.4). Let $\beta \in (0,\, \alpha_0)$. We may assume $q = n/(n+\beta)$.

DEFINITION 5.3. Let

$$Q_0 = Q(B(0,\, 1)),$$

$$Q_i = Q(B(0,2^i)) \setminus Q(B(0,2^{i-1})), \qquad i = 1,\, 2,\, 3,\, \ldots,$$

$$\mathcal{Q}(x,\, t) = \{(y,\, s) \in R_+^{n+1}\colon\ y \in B(x,\, t/2),\ t < s < 2t\},$$

$$k(x,t) = \varkappa * \check{\psi}(x,\, t), \qquad L > 1,$$

$$\mathcal{E} = \{(y,\, s) \in R_+^{n+1}\colon\ |f * \varphi(y,\, s)| > LG(\mathfrak{g}(f)^{q/2},\, y,\, s)^{2/q}\},$$

where $\check{\psi}(x,\, t)$ denotes $\psi(-x,\, t)$.

LEMMA 5.3.

$$|k(x,\, t)| \leqslant Ct^{\beta} \quad \textit{for any } (x,\, t) \in R_+^{n+1}, \tag{5.11}$$

$$|k(x,\, t)| \leqslant Ct^{-n-1}(1+|x|/t)^{-n-\alpha_0-1} \quad \textit{if } |x| > 2 \textit{ or if } t > 1, \tag{5.12}$$

where C *is a constant depending only on* α_0, α_1, β *and* n.

Proof. By Lemma 5.1

$$|\varkappa * \check{\psi}(x,\, t)| = \Big|\sum_{i=0}^{\infty} 2^{-i(\alpha_0+1)} \int \varkappa(x-y)(\tilde{v}_{t,i})_{2^i t}(y)\, dy\Big|$$

$$\leqslant \sum 2^{-i(\alpha_0+1)} \inf_{\deg P \leqslant \beta} \int |\varkappa(x-y) - P(y)|\, |(\tilde{v}_{t,i})_{2^i t}(y)|\, dy$$

$$\leqslant \sum 2^{-i(\alpha_0+1)} (2^i t)^{\beta} \leqslant Ct^{\beta}.$$

Thus we get (5.11). If $|x| > 2$ or if $t > 1$, then by (4.2)–(4.4) and (2.13) we get

$$|\varkappa * \check{\psi}(x,\, t)| = \Big|\int (\check{\psi}(x-y,\, t) - \check{\psi}(x,\, t))\varkappa(y)\, dy\Big|$$

$$\leqslant Ct^{-n-1}(1+|x|/t)^{-n-\alpha_0-1},$$

which means (5.12). ■

LEMMA 5.4.

$$\varkappa(x) = \iint \check{\varphi}(x-y,\, t)\, k(y,\, t)\, dy d\mu(t).$$

By Lemma 5.3 the above integrand is integrable. Then this equality follows from (2.10).

LEMMA 5.5.

$$\|k(y,\, t)\, \chi_{Q_i}(y,\, t)\, dy d\mu(t)\|_{\beta} \leqslant C2^{-i(n+\beta+1)}$$

for $i = 0,\, 1,\, 2,\, \ldots$, *where* C *is a constant depending only on* α_0, α_1, β *and* n.

Proof. The case $i = 0$ is clear from (5.11). Let $i \geqslant 1$ and $(y,\, t) \in Q_i$. By (5.12)

$$|k(y,\, t)| \leqslant C2^{-i(n+\alpha_0+1)} t^{\alpha_0} \leqslant C2^{-i(n+\beta+1)} t^{\beta}.$$

Thus we get the desired result. ■

LEMMA 5.6.

$$\left|\iint f*\varphi(y,t)\,k(y,t)\,\chi_{\mathscr{E}^c}(y,t)\,dy\,d\mu(t)\right| \leqslant CLM_q(\mathfrak{g}(f))(0),$$

where C is a constant depending only on α_0, α_1, β and n.

Proof. The left-hand side

$$\leqslant \iint LG\big(\mathfrak{g}(f)^{q/2}, y, t\big)^{2/q}|k(y,t)|\,dy\,d\mu(t) \text{ by the definition of } \mathscr{E}$$

$$= L\sum_{i=0}^{\infty}\iint G\big(\mathfrak{g}(f)^{q/2}, y, t\big)^{2/q}|k(y,t)|\,\chi_{Q_i}(y,t)\,dy\,d\mu(t)$$

$$= L\sum\iint G\big(\mathfrak{g}(f)^{q/2}\chi_{B(0,2^{i+1})}, y, t\big)^{2/q}|k(y,t)|\,\chi_{Q_i}(y,t)\,dy\,d\mu(t)$$

$$\leqslant L\sum C2^{-i(n+\beta+1)}\Big(\int \mathfrak{g}(f)(y)^{(q/2)2}\chi_{B(0,2^{i+1})}(y)\,dy\Big)^{1/q}$$

by Lemmas 5.5 and 5.A

$$= LC\sum 2^{-i}\Big(2^{-in}\int_{B(0,2^{i+1})}\mathfrak{g}(f)(y)^q\,dy\Big)^{1/q}$$

$$\leqslant LCM_q(\mathfrak{g}(f))(0). \quad ■$$

LEMMA 5.7. *Let $(x, s)\in R_+^{n+1}$. Then*

$$\iint_{\mathscr{E}\cap\mathscr{Q}(x,t)} dy\,d\mu(t) \leqslant CL^{-q/2}s^n, \tag{5.13}$$

where C is a constant depending only on n.

Proof. Note that if $(y,t)\in\mathscr{Q}(x,s)$, then

$$4^n G\big(\mathfrak{g}(f)^{q/2}, y, t\big) \geqslant G\big(\mathfrak{g}(f)^{q/2}, x, s/2\big).$$

Thus if $(y,t)\in\mathscr{E}\cap\mathscr{Q}(x,s)$, then

$$|f*\varphi(y,t)|^{q/2}/L^{q/2}G\big(\mathfrak{g}(f)^{q/2}, x, s/2\big) \geqslant 4^{-n}.$$

Therefore,

$$\iint_{\mathscr{E}\cap\mathscr{Q}(x,t)} dy\,d\mu(t) \leqslant CL^{-q/2}G\big(\mathfrak{g}(f)^{q/2}, x, s/2\big)^{-1}\iint_{\mathscr{Q}(x,t)}|f*\varphi(y,t)|^{q/2}\,dy\,d\mu(t)$$

$$\leqslant CL^{-q/2}G(\ldots)^{-1}\int_{B(x,s/2)}\mathfrak{g}(f)(y)^{q/2}\,dy$$

$$\leqslant CL^{-q/2}s^n. \quad ■$$

LEMMA 5.8.

$$\|k(y,t)\,\chi_{Q_i\cap\mathscr{E}}(y,t)\,dy\,d\mu(t)\|_\beta \leqslant CL^{-q/2}2^{-i(n+\beta+1)},$$

where C is a constant depending only on α_0, α_1, β and n.

This follows from Lemmas 5.3, 5.7 and the same argument as in the proof of Lemma 5.5.

LEMMA 5.9. *There exist $\{\varkappa_m\}_{m=1}^{\infty}\subset \mathrm{Lip}\,\beta$ such that*

$$\varkappa(x) = \iint\check{\varphi}(x-y,t)\,k(y,t)\,\chi_{\mathscr{E}^c}(y,t)\,dy\,d\mu(t) + {} \tag{5.14}$$

$$+CL^{-q/2}\sum_{m=1}^{\infty}m2^{-m}(\varkappa_m)_{2^m}(x),$$

$$\|\varkappa_m\|_{\mathrm{Lip}\beta}\leqslant 1, \tag{5.15}$$

$$\operatorname{supp}\varkappa_m\subset B(0,1), \tag{5.16}$$

$$\int\varkappa_m(x)\,dx = 0, \tag{5.17}$$

where C is a constant depending only on α_0, α_1, β and n.

Proof. Put

$$\tilde{\varkappa}(x) = \iint\check{\varphi}(x-y,t)\,k(y,t)\,\chi_{\mathscr{E}}(y,t)\,dy\,d\mu(t).$$

By Lemma 5.4 it is enough to show that $\tilde{\varkappa}$ can be written in the form of the second term on the right-hand side of (5.14). By Lemma 2.2 $\check{\varphi}(x,t)$ can be decomposed into the sum $\sum_{j=0}^{\infty}2^{-j}(\check{g}_{t,j})_{t2^j}(x)$ with (2.5)–(2.7). Put

$$\iint(\check{g}_{t,j})_{t2^j}(x-y)\,k(y,t)\,\chi_{Q_i\cap\mathscr{E}}(y,t)\,dy\,d\mu(t)$$

$$= \iint_{\{(y,t):t\leqslant 2^{i-j}\}}\ldots + \sum_{h=0}^{j-1}\iint_{\{(y,t):2^{i-h-1}<t\leqslant 2^{i-h}\}}\ldots$$

$$= \tilde{\varkappa}_{i,j,j}(x) + \sum_{h=0}^{j-1}\tilde{\varkappa}_{i,j,h}(x).$$

Then

$$\int\tilde{\varkappa}_{i,j,h}(x)\,dx = 0, \qquad \operatorname{supp}\tilde{\varkappa}_{i,j,h}\subset B(0, 2\cdot 2^{i+j-h})$$

and

$$\tilde{\varkappa}(x) = \sum_{j=0}^{\infty}2^{-j}\sum_{i=0}^{\infty}\sum_{h=0}^{j}\tilde{\varkappa}_{i,j,h}(x).$$

Put

$$\tilde{\varkappa}_{m+1}(x) = \sum_{i=0}^{m}\sum_{j=m-i}^{\infty}2^{-j}\tilde{\varkappa}_{i,j,i+j-m} \quad \text{for} \quad m = 0, 1, 2, \ldots$$

Then

$$\tilde{\varkappa}(x) = \sum_{m=1}^{\infty}\tilde{\varkappa}_m(x),$$

$$\int\tilde{\varkappa}_m(x)\,dx = 0 \quad\text{and}\quad \operatorname{supp}\tilde{\varkappa}_m\subset B(0,2^m). \tag{5.18}$$

Applying Lemma 5.2 with $g(x, t) = (\check{g}_{t/2^j,j})_t(x)$ and $\alpha = \alpha_0$, we get

$$(5.19)\quad \|\tilde{\varkappa}_{i,j,j}\|_{\mathrm{Lip}\beta} = \Big\| \iint_{\{(y,t):\, t<2^i\}} (\check{g}_{t/2^j,j})_t(\cdot - y)\, k(y, t/2^j) \cdot \chi_{Q_i \cap \mathscr{E}}(y, t/2^j)\, dy\, d\mu(t/2^j)\Big\|_{\mathrm{Lip}\beta}$$
$$\leqslant C \|k(y, t/2^j)\chi_{Q_i\cap\mathscr{E}}(y, t/2^j)\, dy\, d\mu(t/2^j)\|_\beta \quad \text{by Lemma 5.2}$$
$$\leqslant CL^{-q/2} 2^{-i(n+\beta+1)-j\beta} \quad \text{by Lemma 5.8.}$$

If $h < j$, then

$$(5.20)\quad \|\tilde{\varkappa}_{i,j,h}\|_{\mathrm{Lip}\beta}$$
$$\leqslant \iint_{\{(y,t):\, 2^{i-h-1} < t \leqslant 2^{i-h}\}} \|(\check{g}_{t,j})_{t2^j}\|_{\mathrm{Lip}\beta} |k(y, t)|\, \chi_{Q_i\cap\mathscr{E}}(y, t)\, dy\, d\mu(t)$$
$$\leqslant \iint_{\{\ldots\}} (t2^j)^{-n-\beta} |k|\, \chi\, dy\, d\mu \quad \text{by (2.6)}$$
$$\leqslant (2^{i-h+j})^{-n-\beta} CL^{-q/2} 2^{-i(n+\beta+1)} 2^{in+(i-h)\beta} \quad \text{by Lemma 5.8}$$
$$= CL^{-q/2} 2^{-i(n+\beta+1)+hn-j(n+\beta)}.$$

Hence, by (5.19)–(5.20), we get

$$(5.21)\quad \|\tilde{\varkappa}_{m+1}\|_{\mathrm{Lip}\beta} \leqslant \sum_{i=0}^{m} \sum_{j=m-i}^{\infty} 2^{-j} CL^{-q/2} 2^{-i(n+\beta+1)+(i+j-m)n-j(n+\beta)}$$
$$\leqslant CL^{-q/2} 2^{-m(n+\beta+1)} \sum_{i=0}^{m} 1$$
$$\leqslant CL^{-q/2} 2^{-m(n+\beta+1)} m.$$

Thus $\tilde{\varkappa}_m$ can be writter in the form

$$CL^{-q/2} m 2^{-m} (\varkappa_m)_{2^m}$$

with (5.15)–(5.17). Conditions (5.16)–(5.17) follow from (5.18). Condition (5.15) follows from (5.21). ■

Now, we conclude the proof of Lemma 4.1. By Lemmas 5.9 and 5.6 we get

$$(5.22)\quad \Big|\int f(x)\varkappa(x)\,dx\Big|$$
$$\leqslant CLM_q\big(\mathfrak{g}(f)\big)(0) + CL^{-q/2} \sum_{m=1}^{\infty} m2^{-m} \Big|\int f(x)(\varkappa_m)_{2^m}(x)\,dx\Big|.$$

For $s > 0$ put

$$A_s = \sup\Big\{\Big|\int f(x)(\varkappa)_s(x)\,dx\Big|:\ \varkappa \in \mathrm{Lip}\,\beta \text{ with (4.2)–(4.4)}\Big\}.$$

For $\varepsilon \geqslant 0$ put

$$B_\varepsilon = \sup_{s>\varepsilon} A_s.$$

By (5.22)

$$A_1 \leqslant CLM_q\big(\mathfrak{g}(f)\big)(0) + CL^{-q/2} B_1.$$

By the argument of dilation we get

$$A_\varepsilon \leqslant CLM_q\big(\mathfrak{g}(f)\big)(0) + CL^{-q/2} B_\varepsilon.$$

Hence

$$B_\varepsilon \leqslant CLM_q\big(\mathfrak{g}(f)\big)(0) + CL^{-q/2} B_\varepsilon.$$

Since $B_\varepsilon < +\infty$ by $f \in L^2(R^n)$, we get

$$B_\varepsilon \leqslant CLM_q\big(\mathfrak{g}(f)\big)(0)$$

by taking L large enough. Since $\varepsilon > 0$ is arbitrary, we get

$$B_0 \leqslant CLM_q\big(\mathfrak{g}(f)\big)(0),$$

which means the desired result.

Remark 5.1. We add the explanation of what we mean by "by dilation" in the proof of Lemma 4.4 (and by "by the argument of dilation" at the final stage of the proof of Lemma 4.1). Let $s > 0$ and fix it. Put

$$\tilde{E} = \{t/s:\ t \in E\}, \quad \tilde{\mu}(A) = \mu(\{ts:\ t \in A\}) \quad \text{for} \quad A \subset \tilde{E}$$

and

$$\tilde{\varphi}(x, t) = s^n \varphi(sx, st).$$

Then these satisfy (1.6)–(1.9). Put

$$\tilde{\mathfrak{g}}(f)(x)^2 = \int_{\tilde{E}} |f * \tilde{\varphi}(x, t)|^2\, d\tilde{\mu}(t).$$

Then

$$\int_E |(\eta)_s * f * \varphi(0, t)|^2\, d\mu(t) = \int_{\tilde{E}} |\eta * f(s\,\cdot) * \tilde{\varphi}(0, t)|^2\, d\tilde{\mu}(t)$$

and

$$M_q\big(\mathfrak{g}(f)\big)(0) = M_q\big(\tilde{\mathfrak{g}}(f(s\,\cdot))\big)(0).$$

Thus in order to compare $\{\int_E |(\eta)_s * f * \varphi(0, t)|^2\, d\mu(t)\}^{1/2}$ and $M_q\big(\mathfrak{g}(f)\big)(0)$ in the proof of Lemma 4.4 we may assume $s = 1$ by considering $\tilde{E}$, $\tilde{\mu}$, $\tilde{\varphi}$ and $f(s\,\cdot)$ instead of E, μ, φ and f, which we mean by "dilation".

Remark 5.2. E and φ_t in (1.6)–(1.9) are "Borel" measurable.

References

[1] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Adv. in Math. 16 (1975), 1–63.
[2] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. 76 (1962), 547–559.
[3] R. Coifman, *A real variable characterization of H^p*, Studia Math. 51 (1974), 269–274.
[4] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–646.
[5] P. Duren, *Extension of a theorem of Carleson*, ibid. 75 (1969), 143–146.
[6] P. Duren, B. Romberg and A. Schields, *Linear functionals on H^p spaces with $0 < p < 1$*, J. Reine Angew. Math. 238 (1969), 32–60.
[7] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.
[8] R. Fefferman and E. Stein, *Singular integrals on product spaces*, Adv. in Math. 45 (1982), 117–143.
[9] L. Hörmander, *L^p estimates for (pluri-) subharmonic functions*, Math. Scand. 20 (1967), 65–78.
[10] R. Latter, *A characterization of $H^p(R^n)$ in terms of atoms*, Studia Math. 62 (1978), 93–101.
[11] R. Latter and A. Uchiyama, *The atomic decomposition for parabolic H^p spaces*, Trans. Amer. Math. Soc. 253 (1979), 391–398.
[12] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton 1970.
[13] T. Walsh, *The dual of $H^p(R_+^{n+1})$ for $p < 1$*, Canad. J. Math. 25 (1973), 567–577.

Received May 24, 1983 (1893)

STUDIA MATHEMATICA, T. LXXXI. (1985)

Banach spaces which are proper M-ideals

by

EHRHARD BEHRENDS and PETER HARMAND (Berlin)

Abstract. In the theory of Banach spaces certain subspaces J of Banach spaces X, the *M-ideals*, have been investigated in great detail. *M-summands*, i.e. subspaces J for which there exists a subspace $J^\perp$ such that $X = J \oplus J^\perp$ and $\|j+j^\perp\| = \max\{\|j\|, \|j^\perp\|\}$ for $j \in J$, $j^\perp \in J^\perp$, are special examples of M-ideals, but there is an abundance of M-ideals which are not of this simple form. They will be called *proper M-ideals*.

The more interesting examples of M-ideals are proper, and in the development of M-structure theory it turned out that all these examples share some geometric properties. This motivated the present investigations to give conditions concerning the geometry of a Banach space J such that J can be a proper M-ideal in a suitable space X. The main results are the following:

- if J can be a proper M-ideal, then J contains a copy of c_0;
- if J satisfies a certain intersection property then J is never a proper M-ideal;
- J can be a proper M-ideal iff J contains a pseudoball which is not a ball (a pseudoball is a closed convex subset B of diameter two such that for every finite collection $x_1, \ldots, x_n$ of elements with $\|x_i\| < 1$ there is an $x \in B$ such that $x + x_i \in B$ for every i).

1. Introduction. At first we recall some basic definitions from M-structure theory.

1.1. Definition. Let X be a (real or complex) Banach space, J a closed subspace of X.

(i) J is called an *L-summand* (resp. *M-summand*) if there exists a subspace $J^\perp$ such that $X = J \oplus J^\perp$ and $\|j+j^\perp\| = \|j\| + \|j^\perp\|$ (resp. $\|j+j^\perp\| = \max\{\|j\|, \|j^\perp\|\}$) for $j \in J$, $j^\perp \in J^\perp$.

(ii) J is called an *M-ideal* if the annihilator J^π of J in X' is an L-summand.

Note. It is easy to see that every M-summand is an M-ideal. M-ideals which are not of this simple form will be called *proper M-ideals* in the sequel.

These notions play an important rôle in the applications of M-structure to approximation theory and the theory of L^1-preduals (for references see [1] or [9]).

If X is a given space it is often important to determine the collection of M-ideals and M-summands of X. Here we are interested in the *converse problem*: Given a Banach space J, can J be a proper M-ideal in a suitable