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Preassigning eigenvalues and zeros of nuclear operators

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Abstract. We solve problems of Saphar (Problem 17 and 18, Studia Mathematica 35 (1970), p. 472) and a problem of Pełczyński. Every non-zero square summable sequence is the eigenvalue sequence of some nuclear operator. Also, there are nuclear operators with certain preassigned finite-dimensional kernels and arbitrarily assigned square-summable eigenvalues.

0. Introduction. The classical Weyl inequality [13] asserts (among other things) that for a trace class operator T on Hilbert space with eigenvalues $(\lambda_n(T))$

(1) $$\sum_{n=1}^{\infty} |\lambda_n(T)| < +\infty.$$

Of course, a trace class operator T has a representation

(2) $$T = \sum_{n=1}^{\infty} T_n$$

where rank $T_n \leqslant 1$ and

(3) $$\sum_{n=1}^{\infty} \|T_n\| < +\infty.$$

Since (2) and (3) make sense for arbitrary Banach spaces, it is natural to extend the trace class operators to arbitrary Banach spaces by using these defining properties. This was done by Grothendieck ([3], [4]) who called such operators *nuclear*.

While this definition produces a useful (and well studied) class of operators, in general (1) is lost. Indeed, Grothendieck [4] proved the following result: *If X is a Banach space and $T\colon X \to X$ a nuclear operator with eigenvalues $(\lambda_n(T))$ then*

(4) $$\sum_{n=1}^{\infty} |\lambda_n(T)|^2 < +\infty.$$

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Moreover, two is the best possible exponent. This fundamental result of Grothendieck motivated the problems of Saphar and Pełczyński discussed below.

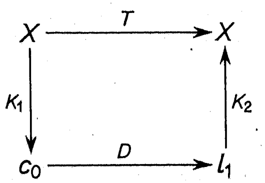
1. Grothendieck factorization. Since we are concerned with spectral properties of operators, we only consider operators $T\colon X \to X$, X a complex Banach space. We write $T \in N(X)$ to mean that T satisfies (2) and (3). Observe that this means that $T \in N(X)$ can be written

$$T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes x_n \tag{5}$$

where $\sum_{n=1}^{\infty} |\lambda_n| < +\infty$, $f_n \in X^*$, $x_n \in X$ and $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|f_n\| = 0$.

This observation yields *Grothendieck's factorization theorem* for $T \in N(X)$: Such a T factors

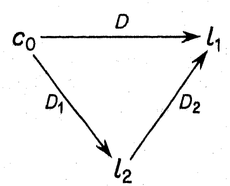
(6)



where K_1 and K_2 are norm limits of finite rank operators and D is a diagonal mapping corresponding to (λ_n), i.e. $D(\xi_n) = (\lambda_n \xi_n)$. In particular, D is nuclear.

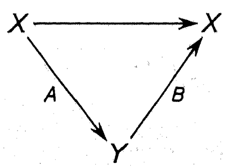
2. Related operators. Observe that the operator D in diagram (6) further factors:

(7)



producing, finally, a factorization of the form

(8)

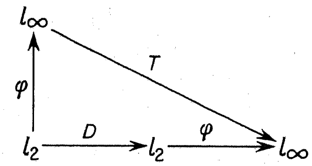


Such pairs (A, B) were called by Pietsch ([8], p. 375) *related operators*. Clearly, AB and BA have the same eigenvalues. This simple fact yields a truly elementary proof (discovered by Pietsch) of (4): $D_2 D_1$ (from (7)) is a Hilbert–Schmidt operator.

3. Two is best possible. Grothendieck's original example to show two is the best possible exponent for eigenvalue summability of nuclear operators was a convolution operator on $L_1[-\pi, \pi]$. We give here an alternative example which shows more and, in particular, motivates the main result of this work: Let $(\lambda_n) \in l_2$ have no non-zero terms and let D be the diagonal mapping $l_2 \to l_2$ given by (λ_n). Then D is in the Hilbert–Schmidt class and $(\lambda_n(D)) = (\lambda_n)$. Let $\varphi\colon l_2 \to l_\infty$ be an isometric embedding. Then φD is nuclear [since $D^* \varphi^*$ is the composition of a Hilbert–Schmidt operator and an operator $l_1 \to l_2$ and the spaces in question have the metric approximation property].

Clearly, a nuclear operator can be extended to a superspace by replacing the functionals in the representation (5) by Hahn–Banach extensions. We thus obtain the diagram

(9)



with T nuclear. Among the eigenvalues of T is the sequence (λ_n). Since $(\lambda_n) \in l_2$ was arbitrary (except all terms different from zero), two is the best possible eigenvalue summability exponent for nuclear operators.

4. The problems of Saphar and Pełczyński: The result (4) of Grothendieck led Saphar (1969) to ask the following questions ([11], prob. 17 and 18): "Soit (λ_n) une suite de nombres complexes telle que $\sum |\lambda_n|^2 < +\infty$. Existe-t-il un opérateur nucléaire T, tel que le spectre de T soit la suite (λ_n)?" "Si la réponse est non, caractériser les suites (λ_n) qui sont spectre d'un opérateur nucléaire."

Pełczyński elaborated on these problems by asking if there are nuclear operators with prescribed eigenvalues and a preassigned number of linearly independent zeros.

The answer to Saphar's first question and to that of Pełczyński is affirmative.

5. The main lemma. The first three parts of the following lemma require only trivial modifications of the construction given in the deservedly neglected paper [6]. The last two parts are a bit more delicate.

MAIN LEMMA. *Let X be an infinite-dimensional Banach space and $(\alpha_n)\in c_0$, with $1>\alpha_1>\alpha_2>\ldots>0$. Then there are sequences (x_n) in X and (f_n) in X^* such that*

(10) *(x_n) is a Schauder basis for its closed linear span $[x_n]$;*

(11) $$\operatorname{Sup}\Big\{\sum_{n=1}^{\infty}|f(x_n)|^2 \colon \|f\|\leqslant 1\Big\}<+\infty;$$

(12) $$f_n(x_m)=\alpha_n\delta_{mn};$$

(13) $$\|f_n\|\to 0.$$

Moreover, if X is separable then

(14) *the (f_n) can also be chosen to be total, i.e. if $f_n(x)=0$ for all n then $x=0$.*

A quick perusal of [6] will show how to vary the weights in the proof there to achieve (10), (11), (12), (13). However, the f_n's there obtained are via Hahn–Banach extensions and that method cannot work to achieve (14). Instead, one takes the "Dvoretzky" elements obtained in [6] *before weighting* but after *blocking* so that the elements remain bounded and bounded away from zero. Then apply the remark of Singer ([12], p. 213), (see also remark 7.6, [12], p. 214-215). After the total extension is obtained apply the weights to force (11), (12) and (13).

We should mention that the full strength of the Dvoretzky ε-isometry theorem [1] (as used in [6]) is not needed for the above construction (unless one is interested in precise bounds). Indeed, the relatively easy Dvoretzky–Rogers theorem [2] suffices for the lemma.

6. The main theorem. Our principal result is a simple consequence of the above lemma.

MAIN THEOREM. *Let X be an infinite-dimensional Banach space and $(\lambda_n)\in l_2$, $\lambda_n\neq 0$ for each n. Then there is an operator $T\colon X\to X$ such that T is the norm limit of finite rank operators and the eigenvalue sequence of T is precisely (λ_n). Moreover, if X is separable, T can be constructed to be one-to-one.*

Proof. Given such a $(\lambda_n)\in l_2$, there are sequences $(\alpha_n)\in c_0$, $(\beta_n)\in l_2$, $1>\alpha_1>\alpha_2>\ldots>0$, $\beta_n\neq 0$ for each n and $\alpha_n\beta_n=\lambda_n$. Choose (x_n), (f_n) satisfying (10), (11), (12), (13), (and also (14) if X is separable) with respect to (α_n). Define the following operators:

$$A\colon X\to c_0,\qquad Ax=(f_n(x));$$

$$B\colon c_0\to l_2,\qquad B(\xi_n)=(\beta_n\xi_n);\quad\text{and,}$$

$$C\colon l_2\to X,\qquad C(\xi_n)=\sum_{n=1}^{\infty}\xi_n x_n.$$

Let $T=CBA$. Since $\beta_n\neq 0$ for each n, B is one-to-one and C is one-to-one since (x_n) is a basis for its closed span. When (f_n) is total A is also one-to-one. A routine calculation using (10) and (12) shows that the eigenvalue sequence of T is precisely (λ_n).

Actually we have proved a bit more than asserted. The operator B above is clearly absolutely two-summing (even two-nuclear). See [8] for appropriate definitions. Thus, *on every infinite-dimensional Banach space there is an absolutely two-summing operator (two-nuclear operator) with preassigned eigenvalues.*

7. Concerning zeros. It is now relatively easy to construct, in the separable case, an operator T such that T is the limit of finite rank operators, has preassigned eigenvalues, and preassigned finite-dimensional kernel.

Indeed, if $(\lambda_n)\in l_2$ has finitely many zeros and X is separable construct (α_n), (β_n) as in the main theorem except that $\beta_n=0$ if and only if $\lambda_n=0$. Let (λ_n^*) be (λ_n) with the zeros removed, (α_n^*) and (β_n^*) are appropriate subsequences of (α_n) and (β_n), and construct (x_n) and total (f_n) according to the main lemma (with respect to α_n^*) and let $\tilde{T}=\sum_{n=1}^{\infty}\beta_n^* f_n\otimes x_n$. If $\{n_j\}_{j=1}^{k}$ denotes the set of indices for which $\beta_n=0$ let $P=\sum_{j=1}^{k}g_{n_j}\otimes x_{n_j}$ where (g_j) is biorthogonal to (x_j) (i.e. (g_j) are suitable multiples of (f_j)). Then P is a bounded projection (whose norm may be quite large). We claim that $T=\tilde{T}(I-P)$, I the identity operator, has the desired properties.

Indeed, since $\tilde{T}$ is one-to-one,

$$\operatorname{Ker}T=\operatorname{Ker}(I-P)=\operatorname{Range}P=[x_{n_j}\colon j=1,\ldots,k],$$

so the zeros are preassigned.

Suppose $x\neq 0$, $\lambda\neq 0$ and $Tx=\lambda x$. Then

$$\sum_{n=1}^{\infty}\beta_n^* f_n(x)x_n-\sum_{j=1}^{k}\beta_{n_j}^*\alpha_{n_j}^* g_{n_j}(x)x_{n_j}=\lambda x.$$

If $m\in\{n_j\}_{j=1}^{k}$, applying g_m implies

$$\beta_m^* f_m(x)-\lambda_m^* g_m(x)=\lambda g_m(x)\quad\text{or}\quad \lambda g_m(x)=0,\ \text{ i.e. }\ g_m(x)=0.$$

If $m\notin\{n_j\}_{j=1}^{k}$, again applying g_m yields $\beta_m^*\alpha_m^* g_m(x)=\lambda g_m(x)$ (for all m), so there is an N such that $\lambda=\lambda_N^*$. On the other hand, $T(x_n)=\tilde{T}(I-P)(x_n)=\tilde{T}x_n$ if $n\notin\{n_j\}_{j=1}^{k}$ and so the eigenvalues are precisely (λ_n^*) and the eigenvalues are preassigned. Let us remark that if $\tilde{T}$ is constructed to be one-to-one with preassigned eigenvalues and if Y is an *arbitrary* Banach space then T defined on $X \oplus Y$ (any Banach space norm) by $T(x, y) = (\tilde{T}x, 0)$ then the eigenvalues of T are the same as those of $\tilde{T}$ and $\operatorname{Ker} T = \{(0, y): y \in Y\}$. That is, with obvious identification, $\operatorname{Ker} T = Y$. In this sense we can arbitrarily preassign the zeros.

8. The Pisier space. Recently Pisier [10] has constructed a Banach space with many remarkable properties. His example solves a long outstanding problem of Grothendieck.

THEOREM. *There is an infinite-dimensional, separable Banach space P such that*:

(15) *the injective completion $P \otimes_\varepsilon P$ of $P \otimes P$ is isomorphic to the projective completion $P \otimes_\pi P$*;

(16) *$N(P)$ coincides with the (operator) norm closure of the finite rank operators on P*;

(17) *P fails the approximation property*;

(18) *there is a constant C such that if $X_n \subset P$, $\dim X_n = n$ and if $q: P \to X_n$ is a projection, then $\|q\| \geqslant C\sqrt{n}$*;

(19) *both P and P^* are of cotype 2.*

Obviously, by applying the main theorem to P, (16) solves the problems of Saphar. Property (18) shows that in the construction for n-zeros (Pełczyński's problem) the operator norms of the constructed maps are $O(\sqrt{n})$.

Concluding remarks. Knowing P exists, we obtain from the main theorem, related operators, and Grothendieck factorization (6) that the nuclear operators we desire with preassigned eigenvalues and zeros exist on the "nice" spaces c_0 and l_1. Since K_1 (or K_2) in diagram (6) is compact and thus factors through a reflexive space [7], again, using related operators, these operators also exist on *reflexive spaces.*

Let us call a *Pisier space* any Banach space satisfying (15), (16) and (19). ((19) is crucial for the construction of P!) It is unknown if reflexive Pisier spaces exist. Our results easily show that *uniformly convex Pisier spaces cannot exist.*

Remark. *If X has cotype 2 and is uniformly convex there is a $p < 2$ such that the eigenvalue exponent for any nuclear operator on X is $\leqslant p$.*

Proof. If X is uniformly convex, by yet another result of Pisier there is an $r > 1$ such that X has type r [9]. By [5], p. 116, the eigenvalues of any nuclear operator on X are in the Lorentz space $l_{s,\infty}$ where $1/s = 1-(1/r-\frac{1}{2}) > \frac{1}{2}$, so the eigenvalues of any nuclear operator on such an X are in $l_{s+\varepsilon}$ for any ε satisfying $s+\varepsilon < 2$.

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