The best constant in the Khintchine inequality for complex Steinhaus variables, the case $p = 1$

by

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Abstract. It is shown that

$$
\left( \frac{1}{2\pi} \right)^{2n} \prod_{i=1}^{n} \left| \sum_{j=1}^{m} a_j e^{ij} \right| dt_1 \cdots dt_n \geq \frac{\sqrt{\pi}}{2} \left( \sum_{j=1}^{m} |a_j|^2 \right)^{1/2}
$$

for arbitrary complex numbers $a_1, a_2, \ldots, a_n$ and for $n = 1, 2, \ldots$ The constant $\sqrt{\pi}/2$ is the largest possible.

1. Introduction. The main result of the present paper is

THEOREM A.

$$
\left( \frac{1}{2\pi} \right)^{2n} \prod_{i=1}^{n} \left| \sum_{j=1}^{m} a_j e^{ij} \right| dt_1 \cdots dt_n \geq \frac{\sqrt{\pi}}{2} \left( \sum_{j=1}^{m} |a_j|^2 \right)^{1/2}
$$

for arbitrary complex numbers $a_1, a_2, \ldots, a_n$ and for $n = 1, 2, \ldots$

The constant $\sqrt{\pi}/2$ is the largest possible because, by the central limit theorem for independent complex variables, we have

$$
\lim_{n \to \infty} \left( \frac{1}{2\pi} \right)^{2n} \prod_{i=1}^{n} \left| \sum_{j=1}^{m} \frac{e^{ij}}{\sqrt{n}} \right| dt_1 \cdots dt_n = \frac{\sqrt{\pi}}{2}.
$$

* The paper is a part of the Master's thesis of the author written under the supervision of Professor Stanislaw Kwapień.
Our result is analogous to that of Szarek [6] (cf. also Haagerup [2]) where it is shown that \( c = \sqrt{2/2} \) is the largest possible constant in the Khintchine inequality.

\[
\text{Average} \left| \sum_{i=1}^{n} a_i a_i \right| \geq c \left( \sum_{i=1}^{n} a_i^2 \right)^{1/2}
\]

for arbitrary real numbers \( a_1, a_2, \ldots, a_n \) and \( n = 1, 2, \ldots \) Obviously the analogue of the average over real signs,

\[
\text{Average} \left| \sum_{i=1}^{n} a_i a_i \right|
\]

i.e., the average over all complex signs, is the integral

\[
\left( \frac{1}{2\pi} \right)^{2n} \int \cdots \int \left| \sum_{j=1}^{n} a_j e^{ijt} \right|^2 dt_1 \cdots dt_n.
\]

It is convenient for us to deal with the probabilistic interpretation of the above integral. Let \( C \) (resp. \( R \)) denote the field of complex (resp. real) scalars. Let \( (\Omega, F, P) \) be a probability space. Let \( a_i : \Omega \rightarrow C \) denote complex Stekhaus variables, i.e., the sequence \( (a_i) \) of mutually independent random variables each distributed as the function \( t \mapsto e^{it} \) for \( t \in [0, 2\pi] \). We put \( Ef = \int f \, dP \) for integrable \( f : \Omega \rightarrow C \).

Theorem A is equivalent to

**Theorem B.** For arbitrary complex numbers \( z_1, \ldots, z_n \) we have

\[
E \left| \sum_{i=1}^{n} z_i a_i \right| \geq \frac{\sqrt{n}}{2} \left( \sum_{i=1}^{n} |z_i|^2 \right)^{1/2}.
\]

Since the variables \( (a_i) \) and \( (a_j a_j) \) are equidistributed for an arbitrary sequence \( (a_i) \) of scalars of modulus one, it is enough to prove Theorem B for real \( a_1, a_2, \ldots, a_n \).

In the sequel the abbreviation \( r.v. \) stands for “random variable”. If \( \xi = (\xi_1, \ldots, \xi_n) \) is an r.v. with values in \( R^n \) then by \( \varphi_q \) we denote the characteristic function of \( \xi \), i.e., the function in \( R^n \) defined by

\[
\varphi_q(t_1, \ldots, t_n) = E \exp \left\{ i \sum_{j=1}^{n} t_j \xi_j \right\}.
\]

Let \( (R^n, | \cdot |) \) denote the linear space \( R^n \) with the norm given by \( |(x_1, x_2)| = (x_1^2 + x_2^2)^{1/2} \). To investigate the quantities \( E \left| \sum_{i=1}^{n} a_i a_i \right| \) it is convenient to introduce the following notion.

Let \((S_i)\) denote the sequence of \( R^2 \)-valued mutually independent r.v. each distributed as the function \( t \mapsto (\cos t, \sin t) \) for \( t \in [0, 2\pi] \). For fixed \( n = 1, 2, \ldots \) and real \( a_1, a_2, \ldots, a_n \) we put \( S = \sum_{i=1}^{n} a_i S_i \). Clearly we have

\[
(*) \quad E \left| \sum_{i=1}^{n} a_i a_i \right| = E|S|
\]

\[
= \left( \frac{1}{2\pi} \right)^{2n} \int \cdots \int \left( \sum_{i=1}^{n} a_i \cos t_i \right)^2 + \left( \sum_{i=1}^{n} a_i \sin t_i \right)^2 dt_1 \cdots dt_n.
\]

In view of (*) Theorem B reduces to the following

**Theorem C.** For \( n = 1, 2, \ldots \) we have

\[
E \left| \sum_{i=1}^{n} a_i S_i \right| \geq \sqrt{n/2}
\]

whenever \( a_1 \geq a_2 \geq \ldots \geq a_n \geq 0 \) and \( \sum_{i=1}^{n} a_i^2 = 1 \).

The proof of Theorem C splits into two cases, each treated in a separate section.

1. **Case 1:** \( a_1 \leq \frac{5}{8} \). The argument in this case is based upon the analytic properties of the zero Bessel function,

\[
J_0(t) = \frac{1}{2\pi} \int_{0}^{\infty} \cos (r \cos \phi) \, d\phi.
\]

We shall also need the function

\[
F(s) = \frac{1}{\sqrt{2}} \int_{0}^{\infty} [1 - |J_0(t)|^2] t^{-2} \, dt = \int_{0}^{\infty} [1 - |J_0(t/\sqrt{2})|^2] t^{-2} \, dt.
\]

These functions are used via the following

**Proposition 1.** Let \( S = \sum_{i=1}^{n} a_i S_i \) and \( \psi(t) = \prod_{i=1}^{n} J_0(a_i t) \). Then

\[
E|S| = \int_{0}^{\infty} [1 - \psi(t)] t^{-2} \, dt
\]

and

\[
E|S| \geq \sum_{i=1}^{n} a_i^2 F(a_i^2).
\]
Proof. We shall first show that if $X = (X_1, X_2)$ is a rotation invariant r.v., i.e., $UX$ and $X$ are equidistributed for every rotation $U: \mathbb{R}^2 \to \mathbb{R}^2$, then

\begin{equation}
\phi_{X_1}(t) = E J_0(|X|) t,
\end{equation}

(2.3)

\begin{equation}
\phi_{X}(t_1, t_2) = E J_0(|X|) (t_1^2 + t_2^2)^{1/2},
\end{equation}

(2.4)

\begin{equation}
E|X_1| = \frac{2}{\pi} E|X|,
\end{equation}

(2.5)

\begin{equation}
E|X_2| = \frac{2}{\pi} \int_0^\infty [1 - \phi_{X_1}(t)] t^{-2} dt.
\end{equation}

(2.6)

Now we prove (2.3)-(2.5). Let us observe that if $X = (X_1, X_2)$ is a rotation invariant r.v. then $X_1$, is equidistributed with $X \cos \eta$ and $t_1 X_1 + t_2 X_2$ is equidistributed with $|X|(t_1^2 + t_2^2)^{1/2} \cos \eta$ for $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$, where $\eta$ is an r.v. uniformly distributed in $[0, 2\pi]$ and independent of $X$. Using this fact, the symmetry of the r.v.'s $X_1$, $X_2$ and the definition of $J_0$, we get

\begin{equation}
\phi_{X_1}(t) = E \cos X_1 t = E \frac{1}{2\pi} \int_0^{2\pi} \cos (|X| \cos \eta) d\eta
\end{equation}

(2.7)

\begin{equation}
= E J_0(|X|) t,
\end{equation}

and

\begin{equation}
\phi_{X}(t_1, t_2) = E \cos (t_1 X_1 + t_2 X_2)
\end{equation}

(2.8)

\begin{equation}
= E \frac{1}{2\pi} \int_0^{2\pi} \cos (|X| \sqrt{t_1^2 + t_2^2} \cos \eta) d\eta = E J_0(|X| \sqrt{t_1^2 + t_2^2}),
\end{equation}

Moreover,

\begin{equation}
E|X_1| = E \frac{1}{2\pi} \int_0^{2\pi} |X| \cos \eta d\eta = \frac{2}{\pi} E|X|.
\end{equation}

(2.9)

We have thus proved (2.3)-(2.5). For (2.6) see Haagerup [2], Lemma 1.2.

Since $S_1, \ldots, S_n$ are independent random variables, we have

\begin{equation}
\phi_{S}(t_1, t_2) = \prod_{i=1}^n \phi_{S_i}(t_1, t_2) = \prod_{i=1}^n J_0(\alpha_i (t_i^2 + t_2^2)^{1/2}).
\end{equation}

(2.10)

The random variable $S$ as the sum of rotation invariant r.v.'s is a rotation invariant r.v. If $S = (S_1, S_2)$ then, using (2.3), we get $\phi_{S_1}(t) = \phi_{S}(0, t)$

\begin{equation}
= \prod_{i=1}^n J_0(\alpha_i t).\end{equation}

From this and from (2.5), (2.6) it follows that

\begin{equation}
E|S| = \frac{\pi}{2} E|S|^2 = \int_0^\infty [1 - \phi_{S}(t)] t^{-2} dt
\end{equation}

(2.11)

\begin{equation}
= \int_0^\infty [1 - \prod_{i=1}^n J_0(\alpha_i t)] t^{-2} dt.
\end{equation}

(2.12)

This completes the proof of (2.1).

Next we shall show (2.2). It is well known that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive numbers such that $\sum_{i=1}^n \alpha_i = 1$ and if $p_1, p_2, \ldots, p_n$ are non-negative numbers then

\begin{equation}
\prod_{i=1}^n p_i^{\alpha_i} \leq \prod_{i=1}^n p_i^{\alpha_i}.
\end{equation}

(2.13)

Specifying $\alpha_i = \alpha_i^2$, $p_i = |J_0(\alpha_i t)|^{\alpha_i^2}$ for $i = 1, \ldots, n$, we get

\begin{equation}
\prod_{i=1}^n J_0(\alpha_i t) \leq \sum_{i=1}^n \alpha_i^2 |J_0(\alpha_i t)|^{\alpha_i^2}.
\end{equation}

(2.14)

Thus, taking into account (2.1), we obtain

\begin{equation}
E|S| = \int_0^\infty \left[ \prod_{i=1}^n J_0(\alpha_i t) \right] t^{-2} dt \leq \int_0^\infty \left[ \prod_{i=1}^n J_0(\alpha_i t) \right] t^{-2} dt
\end{equation}

(2.15)

\begin{equation}
\geq \int_0^\infty \left[ \prod_{i=1}^n \alpha_i^2 |J_0(\alpha_i t)|^{\alpha_i^2} \right] t^{-2} dt
\end{equation}

(2.16)

\begin{equation}
= \sum_{i=1}^n \alpha_i^2 \int_0^\infty \left[ \prod_{i=1}^n |J_0(\alpha_i t)|^{\alpha_i^2} \right] t^{-2} dt = \sum_{i=1}^n \alpha_i^2 F(\alpha_i^2).
\end{equation}

(2.17)

This completes the proof of Proposition 1.

Let $\mu = 2, 4048 \ldots$ be the first positive zero of $J_0$ (cf. Watson [3], p. 748). Next we prove some properties of $J_0$.

Lemma 2. For $0 \leq \mu \leq \mu$,

\begin{equation}
0 \leq J_0(\alpha) \leq \exp \left( -\frac{1}{2} \left( \frac{\alpha}{2} \right)^2 - \frac{1}{4} \left( \frac{\alpha}{2} \right)^4 \right).
\end{equation}

(2.18)

For $\mu < \alpha$,

\begin{equation}
|J_0(\alpha)| \leq \frac{1}{2}.
\end{equation}

(2.19)
Moreover, we have

\[ (2.9) \quad \int_{\mu}^{t} J_0^2(t) \cdot t^{-2} \, dt \leqslant \frac{1}{42}. \]

and

\[ (2.10) \quad F\left(\frac{8}{3}\right) \geqslant \frac{\sqrt{\pi}}{2} \quad \text{and} \quad F(2) = \frac{2\sqrt{2}}{\pi}. \]

Proof. We first prove that for \( 0 \leqslant t \leqslant \mu \) we have

\[ (2.11) \quad 0 \leqslant J_0(t) \leqslant \left[ 1 - \frac{1}{2} \left( \frac{t}{2} \right)^2 \right]^2. \]

Since \( 0 \leqslant J_0(t) \) for \( 0 \leqslant t \leqslant \mu \), it is enough to show the right-hand inequality. We have, for real \( x \),

\[ \cos x \leqslant 1 - \frac{x^2}{2} + \frac{x^4}{24}. \]

Hence from the definition of \( J_0 \) it follows that

\[ J_0(t) = \frac{2}{\pi} \int_{0}^{2\pi} \cos(t \cos \eta) \, d\eta \leqslant \frac{2}{\pi} \int_{0}^{2\pi} \left[ 1 - \frac{(t \cos \eta)^2}{2} \right] \frac{(t \cos \eta)^4}{24} \, d\eta \]

\[ = \left[ 1 - \frac{1}{2} \left( \frac{t}{2} \right)^2 \right]^2. \]

This proves (2.11).

Now we shall prove (2.7). Using (2.11) and the inequality \( \ln(1-x) \leqslant -x - x^2/2 \) for \( 0 < x < 1 \), we get for \( 0 \leqslant t \leqslant \mu \leqslant 2\sqrt{2} \).

\[ 0 \leqslant J_0(t) \leqslant \left[ 1 - \frac{1}{2} \left( \frac{t}{2} \right)^2 \right]^2 = \exp \left[ 2 \ln \left( 1 - \frac{1}{2} \left( \frac{t}{2} \right)^2 \right) \right] \]

\[ \leqslant \exp \left[ 2 \left( -\frac{1}{2} \left( \frac{t}{2} \right)^2 - \frac{1}{8} \left( \frac{t}{2} \right)^4 \right) \right] \]

which ends the proof of (2.7).

Now we shall prove (2.8). Since \( J_0 \) is a real analytic function which satisfies the differential equation

\[ t J_0(t) + J_0(t) + t J_0''(t) = 0 \]

and \( J_0(\mu) = \lim_{t \to \infty} J_0(t) = 0 \), it suffices to show that if \( J_0(t) = 0 \) then \( |J_0(t)| \leqslant \frac{1}{42} \) for \( t \geqslant \mu > 0 \). If \( J_0(t) = 0 \) then it follows from the differential equation that

\[ |J_0(t)| = |J_0(t)| \leqslant \frac{1}{2\pi} \int_{0}^{2\pi} \cos^2 \eta \cdot |\cos(t \cos \eta)| \, d\eta \]

\[ \leqslant \frac{1}{2\pi} \int_{0}^{2\pi} \cos^2 \eta \cdot |\cos(t \cos \eta)| \, d\eta \leqslant \frac{1}{2\pi} \int_{0}^{2\pi} \cos^2 \eta \, d\eta = \frac{1}{2}, \]

which proves (2.8).

Proof of (2.9). Since \( 2\sqrt{2} < \mu \) (cf. Watson [7], p. 748), it is enough to show that \( \int_{0}^{2\sqrt{2}} J_0^2(t) t^{-2} \, dt \leqslant 1/42 \).

We shall need the identities

\[ (2.12) \quad 1 - J_0^2(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left( \frac{1}{2} \right)^{2n} \cdot \]

\[ (2.13) \quad \int_{0}^{1} \left[ 1 - J_0^2(t) \right] t^{-2} \, dt = \sqrt{\frac{2}{F(2)} = \frac{4}{\pi}}. \]

For (2.12) cf. Watson [7], p. 32. To prove (2.13) observe that

\[ E \left[ \frac{1}{\sqrt{2} (S_1 + S_2)} \right] \leqslant \left( \frac{1}{\sqrt{2}} \right)^2 \int_{0}^{2\pi} \left( \frac{1}{\sqrt{2}} \right)^{2n} \int_{0}^{2\pi} \left( \cos \eta_1 + \cos \eta_2 \right)^{2n} + \left( \sin \eta_1 + \sin \eta_2 \right)^{2n} \, d\eta_1 \, d\eta_2 \]

\[ = 2\sqrt{2} \pi \]

Hence, applying identity (2.1) for \( n = 2 \) and for \( a_1 = a_2 = 1/\sqrt{2} \), we get (2.13). It follows from (2.12) that for \( a = 2\sqrt{2} \)

\[ \int_{0}^{a} J_0^2(t) t^{-2} \, dt = \int_{0}^{a} \left[ 1 - J_0^2(t) \right] t^{-2} \, dt + \int_{a}^{0} t^{-2} \, dt - \frac{4}{\pi} \]

\[ = \int_{0}^{a} \left[ 1 - J_0^2(t) \right] t^{-2} \, dt + \frac{4}{a - \pi}. \]
By (2.12) we get

$$\int_0^1 [1 - J_0(t)] t^{-2} dt = \frac{1}{2} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{2}\right)^{2i-2} = A + B,$$

where

$$A = \frac{1}{4} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{2}\right)^{2i-2} = \frac{7}{8} \left(23 - \sqrt{2} \left(1 + \frac{5}{144}\right)\right)$$

and

$$B = -\frac{1}{4} \sum_{i=1}^{\infty} \left[\frac{1}{(2i)!} \left(\frac{1}{2}\right)^{2i-2} - \frac{8i+2}{(2i+1)!} \left(\frac{1}{2}\right)^{2i} \right] dt < 0.$$

because for $t \leq a$ and $i \geq 3$

$$1 - \frac{8i+2}{(2i+1)!} \left(\frac{1}{2}\right)^{2i} > 0.$$

Hence

(2.14)

$$\int_0^1 J_0(t)^2 t^{-2} dt < A + \frac{1}{a} \leq \frac{1}{42}.$$

Finally we shall prove (2.10). It is sufficient to show that $F(8/5) > \sqrt{\pi} / 2$, because from (2.13) it follows that $F(2) = 2 \sqrt{\pi} / 2$. From (2.11) it follows that for $0 < t = \mu = 2.4048, \ldots$, we have $J_0(t) < (1 - \frac{1}{6} (t/2)^2)$, from which it follows that for $s > 0$ and $0 < t < 2.4$, $1 - |J_0(t)|^2 r^{-2} \geq 1 - \left(1 - \frac{1}{2}\right)^{2s} = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{2}\right)^{2i-2}.$

Since the series converges uniformly in $[0, 2.4]$, we can integrate “term-by-term”. We obtain for $s > 0$

(2.15)

$$\int_0^1 \frac{1}{s} \left[1 - J_0(t)^2 \right] t^{-2} dt \geq \frac{1}{s} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2^{i+1} i!} \left(\frac{1}{2}\right)^{2i-1} dt$$

$$= \frac{1}{s} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2^{i+1} i!} \left(\frac{1}{2}\right)^{2i-1} (1,2)^{2i-1} = A + B,$$

where

$$A = \frac{1}{s} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2^{i+1} i!} \left(\frac{1}{2}\right)^{2i-1}$$

$$B = \frac{1}{s} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2^{i+1} i!} \left(\frac{1}{2}\right)^{2i-1}.$$

For $s = 8/5$ we get by direct computation

$$A = \frac{5}{8} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{(1,2)^{2i-1}} \left(\frac{1}{2}\right)$$

$$B = \frac{8}{21} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{(1,2)^{2i-1}} \left(\frac{1}{2}\right).$$

For $s = 8/5$ and for $i \geq 5$ we have $\left|\frac{5}{2}\right| \leq \left|\frac{3}{5}\right|$, $1/(2i - 1) \leq 1/9$. Hence

$$B \geq -\frac{8}{21} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{(1,2)^{2i-1}} \frac{1}{2i - 1} \geq -\frac{1}{2500} = -0.0004.$$

Since $A + B \geq 0.5919$, it follows from (2.15) that

(2.16)

$$\int_0^1 \frac{1}{s} \left[1 - J_0(t)^2 \right] t^{-2} dt \geq 0.5919.$$

From the Hölder inequality it follows that

$$\int_0^1 \frac{1}{s} \left[1 - J_0(t)^2 \right] t^{-2} dt \leq \frac{8}{\pi} \left(\int_0^1 J_0(t)^2 t^{-2} dt\right)^{0.8} \left(\int_0^1 t^{-2} dt\right)^{0.2}.$$

Since from (2.14) it follows that

$$\int_0^1 \frac{1}{s} \left[1 - J_0(t)^2 \right] t^{-2} dt \leq \frac{1}{142},$$

we have

(2.17)

$$\int_0^1 \frac{1}{s} \left[1 - J_0(t)^2 \right] t^{-2} dt \leq \frac{8}{\pi} \left(\frac{1}{2}\right)^{0.8} \left(\frac{1}{24}\right)^{0.2} \leq 0.0334.$$
Moreover, we have
\[ \left( \frac{5}{8} \right)^{n/2} \int_0^1 t^{-2} dt = \frac{5}{8} \cdot \frac{1}{2} = 0.3294. \]  

Thus from (2.16), (2.17) and (2.18) we get
\[ F \left( \frac{5}{8} \right) = \int_0^1 \left[ 1 - \left| J_0(t) \right|^2 \right] t^{-2} dt \geq 0.5919 - 0.0334 + 0.3294 \geq \frac{\sqrt{\pi}}{2}. \]

This completes the proof of Lemma 2.

Recall that
\[ F(s) = \frac{1}{\sqrt{s}} \int_0^1 \left[ 1 - \left| J_0(t) \right|^2 \right] t^{-2} dt, \quad s > 0. \]

Let us put
\[ G(s) = \frac{1}{\sqrt{s}} \int_0^1 \left[ \exp \left( -s \left( \frac{t}{2} \right)^2 \right) - \exp \left( -s \left( \frac{t}{2} \right) - \frac{1}{4} \left( \frac{t}{2} \right)^2 \right) \right] t^{-2} dt, \quad s > 0, \]
\[ H(s) = 2 \sqrt{s} \sum_{k=0}^{n} \ln \left| J_0(t) \right| \left| J_0(t) \right|^2 t^{-2} dt, \quad s > 0, \]
\[ I(s) = 4s \sqrt{s} \sum_{k=0}^{n} \ln^2 \left| J_0(t) \right| \left| J_0(t) \right|^2 t^{-2} dt, \quad s > 0. \]

It is not difficult to show that
\[ \left( \frac{5}{8} \right)^{n/2} \int_0^1 t^{-2} dt = \frac{5}{8} \cdot \frac{1}{2} = 0.3294. \]  

\[ F(s) = \frac{1}{2s} (H(s) - F(s)). \]

\[ H(s) = \frac{1}{2s} (H(s) - I(s)). \]

The following lemma will be used in the sequel:

**Lemma 3.** For \( s \geq 2 \) we have
\[ F(s) \geq \frac{\sqrt{\pi}}{2} + G(s) - \frac{1}{40s} \left( \frac{1}{2} \right)^2, \]
\[ G(s) \geq \frac{1}{20} \sqrt{2s}. \]

For \( 1.6 \leq s \leq 2 \) we have
\[ \frac{\sqrt{\pi}}{2} + I(s) \geq H(s). \]

**Proof.** For \( a > 0 \) we have (cf. Dwight [1], p. 155)
\[ \int_0^a \left[ 1 - \exp \left( -a \left( \frac{t}{2} \right)^2 \right) \right] t^{-2} dt \geq \frac{\sqrt{\pi}}{2}. \]

From (2.7) it follows that, for \( 0 \leq t \leq \mu \), \( \left| J_0(t) \right|^2 \leq \exp \left( -s(t/2)^2 - \frac{1}{4} s(t/2)^4 \right). \)

Thus
\[ \frac{1}{\sqrt{s}} \int_0^a \left[ \exp \left( -s \left( \frac{t}{2} \right)^2 \right) - \left| J_0(t) \right|^2 \right] t^{-2} dt \geq G(s). \]

From (2.8) and (2.9) we get
\[ \frac{1}{\sqrt{s}} \int_0^a \left[ \exp \left( -s \left( \frac{t}{2} \right)^2 \right) - \left| J_0(t) \right|^2 \right] t^{-2} dt \geq \frac{1}{\sqrt{s}} \int_0^a \left\{ \frac{1}{\sqrt{s}} \left| J_0(t) \right|^2 t^{-2} dt \right\} \]
\[ = \frac{1}{\sqrt{s}} \int_0^a \left( \frac{1}{\sqrt{s}} \right)^{\frac{1}{2}} \left| J_0(t) \right|^2 t^{-2} dt \geq \frac{1}{\sqrt{s}} \int_0^a \left( \frac{1}{\sqrt{s}} \right)^{\frac{1}{2}} \left| J_0(t) \right|^2 t^{-2} dt \]
\[ \geq \frac{1}{\sqrt{s}} \left( \frac{1}{\sqrt{s}} \right)^{\frac{1}{2}} \frac{1}{42} \geq \frac{1}{40} \left( \frac{1}{\sqrt{s}} \right)^{\frac{1}{2}}. \]

Hence, using (2.24), (2.25) and (2.26), we obtain (2.21).

Now we shall show (2.22). From the definition of \( G(s) \) it follows that for \( s \geq 2 \)
\[ G(s) = \frac{1}{\sqrt{s}} \int_0^a \left[ \exp \left( -s \left( \frac{t}{2} \right)^2 \right) - \exp \left( -s \left( \frac{1}{2} \right)^2 \right) \right] t^{-2} dt \]
\[ \geq \frac{1}{\sqrt{s}} \int_0^a \left[ \exp \left( -s \left( \frac{t}{2} \right)^2 \right) - \exp \left( -s \left( \frac{1}{2} \right)^2 \right) \right] t^{-2} dt. \]
Using (2.27), and the inequalities $1 - e^{-s} \geq x - x^2/2$ and $\int_0^1 t^s \exp(-t^2/2) dt \geq 1/\sqrt{2\pi}$ for $x > 0$, we get for $s \geq 2$

$$G(s) \geq \frac{1}{\sqrt{2\pi}} \int_0^s \left[ 1 - \exp\left( -16s^2 t^2 \right) \right] \exp\left( -t^2/2 \right) t^{-2} dt \geq \frac{1}{16\sqrt{2\pi}} \left( 1 - \frac{15}{32s} s \right) \int_0^s t^2 \exp\left( -t^2/2 \right) dt.$$

Since $1 - 15/32s \geq 49/64$ for $s \geq 2$ and $\int_0^s t^2 \exp(-t^2/2) dt \geq 1.09$, we have for $s \geq 2$

$$\left( 1 - \frac{15}{32s} s \right) \int_0^s t^2 \exp\left( -t^2/2 \right) dt \geq 0.8.$$

From this and (2.28) it follows that $G(s) \geq 1/(16\sqrt{2s})$, which ends the proof of (2.22).

Now we prove (2.23). We begin by establishing the inequalities

$$4s \int_0^s \left[ \ln^2 J_0(t) \right] J_0(t)^2 t^{-2} dt \geq 2 \sqrt{s} \int_0^s \left[ \left( -\ln J_0(t) \right) J_0(t)^2 t^{-2} dt - \sqrt{2}/3, \right.$$\n
which proves (2.29). It follows from (2.8) that for $\frac{s}{2} \leq s$ and $\mu = t$ $4s \int_0^s \ln^2 J_0(t) J_0(t)^2 t^{-2} dt \geq 2 \sqrt{s} \int_0^s \left( -\ln J_0(t) \right) J_0(t)^2 t^{-2} dt.$

From (2.7) it follows that $0 \leq J_0(t) \leq \exp(-t^2/2)$ for $0 \leq t \leq \mu$. Thus $\ln^2 J_0(t) \geq (t^2/2) - \ln J_0(t)$ for $0 \leq t \leq \mu$. Hence for $8/5 \leq s \leq 2$,

$$4s \int_0^s \ln^2 J_0(t) J_0(t)^2 t^{-2} dt \geq 2 \sqrt{s} \int_0^s \left( (t^2/2) - \ln J_0(t) \right) J_0(t)^2 t^{-2} dt.$$

Now we can prove the following

**Theorem 4.** If $a_1 \leq \sqrt{5/8}$, then $E(S) \geq \sqrt{n}/2$.

**Proof.** We shall show that for $s \geq 1.6$

$$F(s) \geq \sqrt{n}/2.$$

We first prove (2.32) for $s \geq 2$. From Lemma 3, (2.21) and (2.22) we get

$$F(s) \geq \sqrt{n}/2 + G(s) \geq \frac{1}{40\sqrt{n}} \left( \frac{1}{s} \right)^{s-2} \geq \frac{1}{20\sqrt{2s}} + \frac{1}{40\sqrt{n}} \left( \frac{1}{s} \right)^{s-2}$$

$$= \frac{\sqrt{n}}{2} + \frac{1}{s} \left[ \frac{1}{\sqrt{2s}} \left( \frac{1}{s} \right)^{s-2} \right].$$

If $f(s) = 1/4 - \sqrt{1/2} (1/2)^s$, then $f(2) = 0$ and $f'(s) = (1/2)^s \sqrt{1/2} (1/2)^s$, from which it follows that $f(s) \geq f'(2) = 0$ for $s \geq 2$. Using this
and (2.33), we have for $s \geq 2$

$$F(s) \geq \frac{\sqrt{\pi}}{2} + \frac{1 - \sqrt{2}}{s} F(s) \geq \frac{\sqrt{\pi}}{2}.$$  

Now we shall show that for $s/5 \leq 2, F(s) \geq \sqrt{\pi}/2$. From Lemma 2 (2.10) we have $F(2) = 2\sqrt{\pi}$ and $F(8/5) = \sqrt{\pi}/2$. Assume to the contrary that $F(s) < \sqrt{\pi}/2$ for some $s \in (8/5, 2)$. Then, by the continuity of $F$, the set $F^{-1}(\sqrt{\pi}/2) \cap (8/5, 2)$ is non-empty. Let $s_1 = \sup \{s \in (8/5, 2) \mid F(s) = \sqrt{\pi}/2\}$. Then $F(2) - F(s_1) = 2\sqrt{\pi} - \sqrt{\pi}/2$. On the other hand, $F(2) - F(s_1) = (2 - s_1)F(\frac{\sqrt{\pi}}{2})$ for some $s \in (s_1, 2)$. Using this and (2.19), we get

$$H(s) = F(s) + \left(2\frac{\sqrt{\pi}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \frac{2s}{2 - s_1} \geq F(s) + \left(2\frac{\sqrt{\pi}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \cdot 8,$$

because $F(\frac{\sqrt{\pi}}{2}) \geq \sqrt{\pi}/2$.

Let $s_2 = \inf \{s \in (8/5, 2) \mid F(s) = \inf_{1 \leq s \leq 2} F(s)\}$. Then $F(s_2) < \sqrt{\pi}/2$ and $F(s_2) = 0$. Therefore from (2.20) it follows that

$$H(s_2) = F(s_2) < \sqrt{\pi}/2.$$  

From the definition of $s_1$, $s_2$ and $s_3$ it follows that $2 \geq s_3 > s_1 > s_2 > 8/5$. Hence from (2.34) and (2.35), using (2.20), we get

$$\left(2\frac{\sqrt{\pi}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \frac{8}{2 - s_3} < H(s) - H(s_2) = (s_3 - s_2)H'(s_3) = (s_3 - s_2) \frac{1}{2s_3} (H(s_3) - I(s_3))$$

for some $s_3$ such that $s_2 < s_3 < s$. From this we get

$$\left(2\frac{\sqrt{\pi}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \frac{64}{8/5} < \left(2\frac{\sqrt{\pi}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \frac{8}{3 - s_2} (H(s_3) - I(s_3)),$$

because $8/5 \leq s_2 < s_3 < 3$. So we have

$$\left(2\frac{\sqrt{\pi}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \frac{64}{8/5} + I(s_3) < H(s_3),$$

for some $s_3$ with $8/5 \leq s_3 < 3$, which contradicts Lemma 3 (the inequality (2.23)), because $2\sqrt{\pi} - \sqrt{\pi}/2 - 64 > \sqrt{2}/3$.

From (3.23) it follows that if $a_1 \leq \sqrt{5}/8$ then $F(a_1^{-\frac{\sqrt{\pi}}{2}}) \geq \sqrt{\pi}/2$. From Proposition 1, (2.2) it follows that if $a_1 \leq \sqrt{5}/8$ then

$$E|S| \geq \sum_{i=1}^{\infty} a_i^2 F(a_i^{-\frac{\sqrt{\pi}}{2}}) \geq \sum_{i=1}^{\infty} \frac{a_i^2 \sqrt{\pi}}{2} \geq \frac{\sqrt{\pi}}{2}.$$  

**Corollary 5.** $E|S| \geq \sqrt{5}/8$.

Proof. We have just proved that if $a_1 \leq \sqrt{5}/8$ then $E|S| \geq \sqrt{\pi}/2$. If $a_1 \geq \sqrt{5}/8$ then $E|S| \geq \sqrt{5}/8$, because

$$E|S| = \sqrt[4]{(E|a_1S_1 + \sum_{i=2}^{\infty} a_i S_i| + E|a_1 S_1 - \sum_{i=2}^{\infty} a_i S_i|) a_1} \geq a_1.$$  


$$h_a(x) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a^2 + 2a \cdot x \cdot \cos \varphi + x^2} \, d\varphi, \quad x \geq 0, a > 0.$$  

Next define $c_n(a)$ by

$$c_n(a) = \sqrt{\frac{8}{5}}, \quad c_n(a) = h_n(s_n - 1 - a^2) \quad \text{for} \quad n > 1.$$  

Clearly, $0 < c_1(a) \leq 1$, for $n = 1, 2, \ldots$

Now we prove the following:

**Lemma 6.** (a) For every $a > 0$ the function $h_a$ is strictly increasing and convex in $[0, \infty)$.

(b) If $\sqrt{5}/8 \leq a \leq 1$, then $\lim_{n \to \infty} c_n(a) > \pi/2$.

Proof. If $a > 0$ and $x \geq 0$ then

$$h_a(x) = (a + x) \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - \frac{4ax}{(a + x)^2} \sin^2 \varphi} \, d\varphi.$$  

Let $E(k) = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi$, where $|k| \leq 1$; then

$$E(k) = \frac{1}{1 + \frac{m^2}{2 \Delta}} \left(\frac{1}{2} + \frac{1}{2 \Delta} - \frac{3}{4 \Delta^2} m^4 + \frac{3}{2 \Delta^2} m^6 + \ldots\right),$$  

where $m = \sqrt{1 - k^2}$ (cf. Dwight [1]). In particular, if $k^2 = 4ax/(a + x)^2$, then

$$E(k) \geq \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi.$$  

$$E(k) = \frac{1}{1 + \frac{m^2}{2 \Delta}} \left(\frac{1}{2} + \frac{1}{2 \Delta} - \frac{3}{4 \Delta^2} m^4 + \frac{3}{2 \Delta^2} m^6 + \ldots\right),$$  

where $m = \sqrt{1 - k^2}$ (cf. Dwight [1]). In particular, if $k^2 = 4ax/(a + x)^2$, then

$$E(k) \geq \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi.$$
then \( m = \min(a, x)/\max(a, x) \). Using this and (3.1), (3.2), we get

\[
(3.3) \quad h_a(x) = \max(a, x) \left[ 1 + \frac{1}{4} \min^2(a, x) + \frac{1}{2} \min^3(a, x) + \ldots \right].
\]

It follows from (3.3) that for each \( a > 0 \) the function \( h_a \) is continuous and strictly increasing. It is easy to see that for \( x > 0 \) and for \( x \neq a \), the second derivative \( h_a''(x) \) exists, and if \( h_a''(x) > 0 \) from which it follows that \( h_a \) is a convex function on the intervals \([0, a) \) and \((a, \infty) \). Since \( h_a \) is continuous at \( a = x \) and \( h'_a(a+) \geq h'_a(a-) \), we infer that \( h_a \) is convex on the interval \([0, \infty) \), which ends the proof of the first part of Lemma 6.

For \( \sqrt{5/8} < a \leq 1 \) define \( f_a \) by \( f_a(x) = h_a(x \sqrt{1-a^2}) - x \) for \( 0 \leq x \leq 1 \).

From (3.3) it follows that if \( \sqrt{5/8} \leq a \leq 1 \) and \( 0 \leq x \leq 1 \) then

\[
(3.4) \quad f_a(x) = a \left[ 1 + 1 + \frac{1}{4} (1-a^2) + \frac{1}{2} (1-a^2)^2 + \ldots \right] - x.
\]

Thus \( f_a \) is differentiable and

\[
1 + f_a'(x) = a \left[ 1 + 1 + \frac{1}{4} (1-a^2) + \frac{1}{2} (1-a^2)^2 + \ldots \right] - x^2 + \frac{1}{2} (1-a^2)^2 + \ldots - x^2 + \ldots
\]

\[
\leq a \left[ 1 + 1 + \frac{1}{4} (1-a^2) + \frac{1}{2} (1-a^2)^2 + \ldots \right] - x^2 + \frac{1}{2} (1-a^2)^2 + \ldots - x^2 + \ldots
\]

\[
\leq a \left[ 1 + 1 + \frac{1}{4} (1-a^2) + \frac{1}{2} (1-a^2)^2 + \ldots \right] - x^2 + \frac{1}{2} (1-a^2)^2 + \ldots - x^2 + \ldots
\]

because \( 0 \leq x \leq 1 \) and \( \sqrt{5/8} \leq a \leq 1 \). Thus \( f_a \) is strictly decreasing for \( 0 \leq x \leq 1 \).

Next we show that, for \( \sqrt{5/8} \leq a \leq 1 \),

\[
(3.5) \quad f_a(\sqrt{\pi}/2) > 0,
\]

(3.6) \[
f_a(1) \leq 0.
\]

Indeed, from the definition of \( f_a \) and from the Schwarz inequality we get

\[
f_a(1) = h_a(\sqrt{1-a^2}) - 1 = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a^2 + 2a\sqrt{1-a^2} \cos \phi + 1-a^2} \, d\phi - 1
\]

\[
= \left[ \frac{1}{2\pi} \int_0^{2\pi} \left( a^2 + 2a \sqrt{1-a^2} \cos \phi + 1-a^2 \right) \, d\phi \right]^{1/2} - 1 = 0.
\]

Finally, from (3.4) it follows that, for \( \sqrt{5/8} \leq a \leq 1 \),

\[
f_a(\sqrt{\pi}/2) = a \left[ 1 + \frac{1}{4} (1-a^2) + \frac{1}{2} (1-a^2)^2 + \ldots \right] - \frac{\sqrt{\pi}}{2}
\]

\[
\leq a \left[ 1 + \frac{1}{20} (1-a^2) + \frac{\pi^2}{200} (1-a^2)^2 \right] - \frac{\sqrt{\pi}}{2}.
\]

So, if we denote by \( g \) the function defined on \([\sqrt{5/8}, 1] \) by

\[
g(a) = a \left[ 1 + \frac{1}{20} (1-a^2) + \frac{\pi^2}{200} (1-a^2)^2 \right] - \frac{\sqrt{\pi}}{2},
\]

then

\[
(3.7) \quad f_a(\sqrt{\pi}/2) = g(a) \quad \text{for} \quad \sqrt{5/8} \leq a \leq 1.
\]

For \( \sqrt{5/8} \leq a \leq 1 \), \( g'(a) > 0 \) and \( g(\sqrt{5/8}) > 0 \). Thus, for \( \sqrt{5/8} \leq a \leq 1 \), using (3.7) and these inequalities, we get \( f_a(\sqrt{\pi}/2) > g(\sqrt{5/8}) > 0 \), which ends the proof of (3.6).

Since \( f_a \) is strictly decreasing on \([0, 1]\) and continuous, it follows from (3.5) and (3.6) that if \( \sqrt{5/8} \leq a \leq 1 \) then there exists exactly one \( c(a) \) with \( \sqrt{\pi}/2 \leq c(a) \leq 1 \) such that \( f_a(c(a)) = 0 \). From the definition of \( c(a) \) it follows that \( c_1(a) = \sqrt{5/8}/c(a) \).

Now we prove that

\[
(3.8) \quad c_1(a) - c_1(b) \leq a - b \quad \text{for} \quad a, b \in [1, 2, \ldots].
\]

For \( n = 1 \) this is true. For \( n \geq 2 \) we have, from the definition of \( c_n(a) \) and \( c_n(a) = h_1(c_{n-1}(a), \sqrt{1-a^2}) \). Since \( h_1 \) is increasing, we infer that \( c_n(a) \leq c(a) \); then \( c_n(a) = h_1(c_{n-1}(a), \sqrt{1-a^2}) \leq h_2(c(a), \sqrt{1-a^2}) = c(a) \). This ends the proof of (3.8).

Finally we show

\[
(3.9) \quad \lim_{n \to \infty} c_n(a) = c(a) \gg \sqrt{\pi}/2.
\]

From the definition of \( c_n(a) \), \( f_1(a) \), \( c_2(a), (3.8) \) and the monotonicity of \( f_1 \), we get \( c_{n+1}(a) = h_1(c_n(a), \sqrt{1-a^2}) = f_1(c_n(a)) + c_n(a) \geq c_n(a) \). From this inequality and (3.8) it follows that \( \lim_{n \to \infty} c_n(a) = d(a) \) exists and \( c(a) \gg d(a) \gg \sqrt{5/8} \).

Moreover, from the definition of \( c_n(a) \), it follows that \( d(a) = h_1(d(a), \sqrt{1-a^2}) \). Thus \( f_1(d(a)) = 0 \). Hence \( d(a) = c(a) \) because for \( 0 \leq a \leq 1 \) there exists exactly one root of the equation \( f_1(x) = 0 \) - thus we have shown that \( \lim_{n \to \infty} c_n(a) = c(a) \gg \sqrt{\pi}/2. \)
Now we can prove the following:

**Theorem 7.** If $a_1 \geq \sqrt{5}/8$ then $E|S| \geq \sqrt{n}/2$.

**Proof.** Let us denote by $X = (X_1, X_2) = (\sum_{t=1}^{n} a_t \cos \xi_t, \sum_{t=1}^{n} a_t \sin \xi_t)$, where $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables uniformly distributed on $[0, 2\pi]$. Let us observe that $X = (X_1, X_2)$ and $(X/\cos \xi, |X|/\sin \xi)$ are equidistributed, where $\eta$ is an r.v. uniformly distributed in $[0, 2\pi]$ and independent of $X$. Therefore

\[
E|S| = E|\sum_{i=1}^{n} a_i S_1 + X|
\]

\[
= E\left(\frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \sqrt{|a_1 \cos \psi + |X| \cos \xi|^2 + (a_1 \sin \psi + |X| \sin \xi)^2} \, d\psi \, d\xi \right)
\]

\[
= E \frac{1}{2\pi} \int_{0}^{2\pi} \sqrt{|a_1^2 + 2a_1 |X| \cos \eta + |X|^2} \, d\eta = E(h_{a_1}(|X|)).
\]

From (3.10), Lemma 6 and from Jensen's inequality it follows that

\[
E|S| = E(h_{a_1}(|X|)) \geq h_{a_1}(E|X|).
\]

Now from Corollary 5 it follows that $E|X| \geq \sqrt{5}/8 \sqrt{E|X|^2} = c_1(a_1) \sqrt{1 - a_1^2}$. Using this and (3.11), we get for $n > 1$, $E|S| \geq h_{a_1}(c_{n-1}(a_1) \sqrt{1 - a_1^2}) = c_{n}(a_1)$ because $h_{a_1}$ is an increasing function. Thus $E|S| \geq c_{n}(a_1)$ for $n = 1, 2, \ldots$. Using this and Lemma 6, we get $E|S| \geq \lim_{n \to \infty} c_{n}(a_1) = c(a_1) \geq \sqrt{n}/2$ because $a_1 \geq \sqrt{5}/8$. This completes the proofs of Theorem 7, and Theorem A.

4. Final remarks.

4.1. Theorem A implies that the one summing norm of the natural injection of the complex $l^p$ into the complex $l^2$ equals $2/\sqrt{\pi}$ (cf. Szarek [6] for details).


**Corollary 8.** If $X$ is a complex Banach space which is isometrically isomorphic to a subspace of $E$, in particular if $X$ is a Hilbert space, then

\[
E\left|\sum_{j=1}^{n} a_j x_j\right| \geq \frac{1}{2} \sqrt{\pi} \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2}
\]

for arbitrary $x_1, x_2, \ldots, x_n$ in $X$ and for $n = 1, 2, \ldots$.

4.3. Our main result is a step towards substantiating the Haagerup conjecture (cf. Pełczyński [8], Section 4). Let us put

\[
\gamma_p = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left|\frac{x^2 + y^2}{x^2 y^2 + (x^2 + y^2)^2 \exp(-x^2 - y^2)}\right| dx \, dy = \left(\frac{\Gamma(p+2)}{\Gamma(p/2)}\right)^{1/p},
\]

\[
d_p = \frac{3^{1/2} 2^{p/2}}{2\pi} \int_{0}^{\pi} \frac{\sin x^p \, dx}{\sin x} = \sqrt{2} \left(\frac{\Gamma(p+1/2)}{\Gamma(p/2)}\right)^{1/p}.
\]

Analysing the proof of Theorem A, one can show

**Theorem 8.** There exists a $\delta > 0$ such that if $|p - l| < \delta$ then

\[
\left(E\left|\sum_{j=1}^{n} a_j x_j\right|^p\right)^{1/p} \geq \gamma_p \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2}
\]

for arbitrary complex numbers $a_1, a_2, \ldots, a_n$ and for $n = 1, 2, \ldots$.

Obviously, the constant $\gamma_p$ cannot be enlarged because, by the Central Limit Theorem,

\[
\lim_{n \to \infty} \left(\frac{E\left|\sum_{j=1}^{n} a_j x_j\right|^p}{(\sum_{j=1}^{n} |a_j|^2)^{1/2}}\right)^{1/p} = \gamma_p.
\]

On the other hand, using the asymptotic expansion for small $p > 0$ we can show that there exists a $\delta$ with $1 > \delta > 0$ such that $d_p < \gamma_p$ for $0 < p < \delta$. Note that

\[
d_p = \left(E\left|x+\frac{1}{2}(\sigma_1+\sigma_3)^p\right|^p\right)^{1/p}.
\]

Thus for $0 < p < \delta$ the best constant $c_p$ in the inequality

\[
\left(E\left|\sum_{j=1}^{n} a_j x_j\right|^p\right)^{1/p} \geq c_p \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2}
\]

is less than $\gamma_p$.

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**Added in proof.** We can show the following theorem:

- **A.** Let $p_0 \in (0, 2)$ be the unique root of the equation

\[
2^{p_0} \Gamma(2p_0) = \sqrt{\pi} \Gamma(p_0+1/2)^2;
\]

$\gamma_{p_0} = 0.47562\ldots$

Then for $2 > p > p_0$

\[
\left|\sum_{j=1}^{n} a_j x_j\right| \geq \Gamma(p+1/2)\left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2}
\]

for arbitrary complex scalars $a_1, a_2, \ldots$. The constant $\Gamma(2p_0+1/2)^{1/2}$ is the best possible.
B. If \( p \geq 2 \) then

\[
\left( \sum a_i \right)^p \leq \Gamma \left( \frac{p+2}{2} \right)^{p^{1/2}} \left( \sum |a_i|^p \right)^{p^{1/2}}
\]

with arbitrary \( a_i \). The constant \( \Gamma \left( \frac{p+2}{2} \right)^{1/2} \) is the best possible.

The proof of A is essentially a modification of the method used in the paper. The starting point to prove B is the formula

\[
E \left( \sum a_i \right)^p = C_p \left( \prod_{i=1}^n \int_0^1 J_i(a_i t - 1)^{p^{1/2}} dt \right)
\]

where \( C_p \) is a constant. The proofs will appear in Studia Math. (J. Sawa, Some remarks on the Khintchine inequality for complex Steinitz variables).

References


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