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The best constant in the Khintchine inequality for complex Steinhaus variables, the case $p = 1$ *

by

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Abstract. It is shown that

$$\left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \left|\sum_{i=1}^n a_i e^{it_i}\right| dt_1 \ldots dt_n \geq \frac{\sqrt{\pi}}{2} \left(\sum_{i=1}^n |a_i|^2\right)^{1/2}$$

for arbitrary complex numbers $a_1, a_2, \ldots, a_n$ and for $n = 1, 2, \ldots$ The constant $\sqrt{\pi}/2$ is the largest possible.

1. Introduction. The main result of the present paper is

THEOREM A.

$$\left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \left|\sum_{i=1}^n a_i e^{it_i}\right| dt_1 \ldots dt_n \geq \frac{\sqrt{\pi}}{2} \left(\sum_{i=1}^n |a_i|^2\right)^{1/2}$$

for arbitrary complex numbers $a_1, a_2, \ldots, a_n$ and for $n = 1, 2, \ldots$

The constant $\sqrt{\pi}/2$ is the largest possible because, by the central limit theorem for independent complex variables, we have

$$\lim_{n\to\infty} \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \left|\sum_{j=1}^n \frac{e^{it_j}}{\sqrt{n}}\right| dt_1 \ldots dt_n$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \sqrt{x^2+y^2}\, dx\, dy = \frac{\sqrt{\pi}}{2}.$$

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Our result is analogous to that of Szarek [6] (cf. also Haagerup [2]) where it is shown that $c=\sqrt{2}/2$ is the largest possible constant in the Khintchine inequality,

$$\text{Average}\left|\sum_{i=1}^{n}\varepsilon_i a_i\right| \geqslant c\left(\sum_{i=1}^{n}a_i^2\right)^{1/2}$$

for arbitrary real numbers $a_1, a_2, \ldots, a_n$ and $n=1, 2, \ldots$ Obviously the analogue of the average over real signs,

$$\text{Average}\left|\sum_{i=1}^{n}\varepsilon_i a_i\right|,$$

i.e., the average over all complex signs, is the integral

$$\left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\ldots\int_0^{2\pi}\left|\sum_{j=1}^{n}a_j e^{it_j}\right|dt_1\ldots dt_n.$$

It is convenient for us to deal with the probabilistic interpretation of the above integral. Let $\boldsymbol{C}$ (resp. $\boldsymbol{R}$) denote the field of complex (resp. real) scalars. Let (Ω, F, P) be a probability space. Let $\sigma_i\colon \Omega\to\boldsymbol{C}$ denote *complex Steinhaus variables*, i.e., the sequence (σ_i) of mutually independent random variables each distributed as the function $t\mapsto e^{it}$ for $t\in[0, 2\pi]$. We put $Ef=\int_\Omega f\,dP$ for integrable $f\colon \Omega\to\boldsymbol{C}$.

Clearly, Theorem A is equivalent to

THEOREM B. *For arbitrary complex numbers $z_1, \ldots, z_n$ we have*

$$E\left|\sum_{i=1}^{n}z_i\sigma_i\right| \geqslant \frac{\sqrt{\pi}}{2}\sqrt{\sum_{i=1}^{n}|z_i|^2}.$$

Since the variables (σ_j) and $(\alpha_j\sigma_j)$ are equidistributed for an arbitrary sequence (α_j) of scalars of modulus one, it is enough to prove Theorem B for real $a_1, a_2, \ldots, a_n$.

In the sequel the abbreviation r.v. stands for "random variable". If $\xi=(\xi_1, \ldots, \xi_n)$ is an r.v. with values in $\boldsymbol{R}^n$ then by φ_ξ we denote the characteristic function of ξ, i.e., the function in $\boldsymbol{R}^n$ defined by

$$\varphi_\xi(t_1, \ldots, t_n) = E\exp\{i\cdot\sum_{j=1}^{n}t_j\xi_j\}.$$

Let $(\boldsymbol{R}^2, |\ |)$ denote the linear space $\boldsymbol{R}^2$ with the norm given by $|(x_1, x_2)| = (x_1^2+x_2^2)^{1/2}$. To investigate the quantities $E\left|\sum_{i=1}^{n}a_i\sigma_i\right|$ it is convenient to introduce the following notion.

Let (S_j) denote the sequence of $\boldsymbol{R}^2$-valued mutually independent r.v. each distributed as the function $t\mapsto(\cos t, \sin t)$ for $t\in[0, 2\pi]$. For fixed $n=1, 2, \ldots$ and real $a_1, a_2, \ldots, a_n$ we put $S=\sum_{i=1}^{n}a_iS_i$. Clearly we have

$$(*)\qquad E\left|\sum_{i=1}^{n}a_i\sigma_i\right| = E|S|$$
$$= \left(\frac{1}{2\pi}\right)^n\int_0^{2\pi}\ldots\int_0^{2\pi}\sqrt{\left(\sum_{i=1}^{n}a_i\cos t_i\right)^2+\left(\sum_{i=1}^{n}a_i\sin t_i\right)^2}\,dt_1\,dt_2\ldots dt_n.$$

In view of $(*)$ Theorem B reduces to the following

THEOREM C. *For $n=1, 2, \ldots$ we have*

$$E\left|\sum_{i=1}^{n}a_iS_i\right| \geqslant \sqrt{\pi}/2$$

whenever $a_1\geqslant a_2\geqslant\ldots\geqslant a_n>0$ and $\sum_{j=1}^{n}a_j^2=1$.

The proof of Theorem C splits into two cases, each treated in a separate section.

2. Case 1: $a_1^2\leqslant 5/8$. The argument in this case is based upon the analytic properties of the zero Bessel function,

$$J_0(t)=\frac{1}{2\pi}\int_0^{2\pi}\cos(t\cos\varphi)\,d\varphi.$$

We shall also need the function

$$F(s)=\frac{1}{\sqrt{s}}\int_0^\infty[1-|J_0(t)|^s]\,t^{-2}\,dt=\int_0^\infty[1-|J_0(t/\sqrt{s})|^s]\,t^{-2}\,dt.$$

These functions are used via the following

PROPOSITION 1. *Let $S=\sum_{i=1}^{n}a_iS_i$ and $\psi_S(t)=\prod_{i=1}^{n}J_0(a_it)$. Then*

$$E|S| = \int_0^\infty[1-\psi_S(t)]\,t^{-2}\,dt \tag{2.1}$$

and

$$E|S| \geqslant \sum_{i=1}^{n}a_i^2F(a_i^{-2}). \tag{2.2}$$

Proof. We shall first show that if $X=(X_1, X_2)$ is a rotation invariant r.v., i.e., UX and X are equidistributed for every rotation $U\colon \mathbf{R}^2\to\mathbf{R}^2$, then

$$\varphi_{X_1}(t)=EJ_0(|X|t), \tag{2.3}$$

$$\varphi_X(t_1, t_2)=EJ_0\big(|X|(t_1^2+t_2^2)^{1/2}\big), \tag{2.4}$$

$$E|X_1|=\frac{2}{\pi}E|X|, \tag{2.5}$$

$$E|X_1|=\frac{2}{\pi}\int_0^\infty [1-\varphi_{X_1}(t)]\,t^{-2}\,dt. \tag{2.6}$$

Now we prove (2.3)–(2.5). Let us observe that if $X=(X_1, X_2)$ is a rotation invariant r.v. then X_1 is equidistributed with $X\cos\eta$, and $t_1X_1+t_2X_2$ is equidistributed with $|X|(t_1^2+t_2^2)^{1/2}\cos\eta$ for $t_1\in\mathbf{R}$, $t_2\in\mathbf{R}$, where η is an r.v. uniformly distributed in $[0, 2\pi]$ and independent of X. Using this fact, the symmetry of the r.v.'s X_1, $t_1X_1+t_2X_2$ and the definition of J_0, we get

$$\varphi_{X_1}(t)=Ee^{itX_1}=E\cos X_1t=E\frac{1}{2\pi}\int_0^{2\pi}\cos(|X|t\cos\eta)\,d\eta$$
$$=EJ_0(|X|t),$$

and

$$\varphi_X(t_1, t_2)=Ee^{i(t_1X_1+t_2X_2)}=E\cos(t_1X_1+t_2X_2)$$
$$=E\frac{1}{2\pi}\int_0^{2\pi}\cos\big(|X|\sqrt{t_1^2+t_2^2}\cos\eta\big)\,d\eta=EJ_0\big(|X|\sqrt{t_1^2+t_2^2}\big),$$

moreover,

$$E|X_1|=E\frac{1}{2\pi}\int_0^{2\pi}|X|\,|\cos\eta|\,d\eta=\frac{2}{\pi}E|X|.$$

We have thus proved (2.3)–(2.5).

For (2.6) see Haagerup [2], Lemma 1.2.

Since $S_1, \ldots, S_n$ are independent random variables, we have

$$\varphi_S(t_1, t_2)=\prod_{i=1}^n\varphi_{a_iS_i}(t_1, t_2)=\prod_{i=1}^n J_0\big(a_i(t_1^2+t_2^2)^{1/2}\big).$$

The random variable S as the sum of rotation invariant r.v.'s is a rotation invariant r.v. If $S=(S^1, S^2)$ then, using (2.3), we get $\varphi_{S^1}(t)=\varphi_S(0, t)$ $=\prod_{i=1}^n J_0(a_it)$. From this and from (2.5), (2.6) it follows that

$$E|S|=\frac{\pi}{2}E|S^1|=\int_0^\infty[1-\varphi_{S^1}(t)]\,t^{-2}\,dt$$
$$=\int_0^\infty\Big[1-\prod_{i=1}^n J_0(a_it)\Big]t^{-2}\,dt.$$

This completes the proof of (2.1).

Next we shall show (2.2). It is well known that if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive numbers such that $\sum_{i=1}^n\alpha_i=1$ and if $p_1, p_2, \ldots, p_n$ are non-negative numbers then

$$\prod_{i=1}^n p_i^{\alpha_i}\leqslant\sum_{i=1}^n p_i\cdot\alpha_i.$$

Specifying $\alpha_i=a_i^2$, $p_i=|J_0(a_it)|^{a_i^{-2}}$ for $i=1, \ldots, n$, we get

$$\Big|\prod_{i=1}^n J_0(a_it)\Big|\leqslant\sum_{i=1}^n a_i^2|J_0(a_it)|^{a_i^{-2}}.$$

Thus, taking into account (2.1), we obtain

$$E|S|=\int_0^\infty\Big[1-\prod_{i=1}^n J_0(a_it)\Big]t^{-2}\,dt\geqslant\int_0^\infty\Big[1-\Big|\prod_{i=1}^n J_0(a_it)\Big|\Big]t^{-2}\,dt$$
$$\geqslant\int_0^\infty\Big[1-\sum_{i=1}^n a_i^2|J_0(a_it)|^{a_i^{-2}}\Big]t^{-2}\,dt$$
$$=\sum_{i=1}^n a_i^2\int_0^\infty\big[1-|J_0(a_it)|^{a_i^{-2}}\big]t^{-2}\,dt=\sum_{i=1}^n a_i^2F(a_i^{-2}).$$

This completes the proof of Proposition 1.

Let $\mu=2{,}4048\ldots$ be the first positive zero of J_0 (cf. Watson [3], p. 748). Next we prove some properties of J_0.

Lemma 2. *For* $0\leqslant t\leqslant\mu$,

$$0\leqslant J_0(t)\leqslant\exp\Big(-\Big(\frac{t}{2}\Big)^2-\frac{1}{4}\Big(\frac{t}{2}\Big)^4\Big). \tag{2.7}$$

For $\mu\leqslant t$

$$|J_0(t)|\leqslant\frac{1}{2}. \tag{2.8}$$

Moreover, we have

$$\int_{\mu}^{\infty} J_0^2(t)\,t^{-2}\,dt \leqslant \frac{1}{42}, \tag{2.9}$$

and

$$F\left(\frac{8}{5}\right) \geqslant \frac{\sqrt{\pi}}{2} \quad \text{and} \quad F(2) = \frac{2\sqrt{2}}{\pi}. \tag{2.10}$$

Proof. We first prove that for $0 \leqslant t \leqslant \mu$ we have

$$0 \leqslant J_0(t) \leqslant \left[1-\frac{1}{2}\left(\frac{t}{2}\right)^2\right]^2. \tag{2.11}$$

Since $0 \leqslant J_0(t)$ for $0 \leqslant t \leqslant \mu$, it is enough to show the right-hand inequality. We have, for real x,

$$\cos x \leqslant 1-\frac{x^2}{2}+\frac{x^4}{24}.$$

Hence from the definition of J_0 it follows that

$$J_0(t) = \frac{1}{2\pi}\int_0^{2\pi} \cos(t\cos\eta)\,d\eta \leqslant \frac{1}{2\pi}\int_0^{2\pi} 1-\frac{(t\cos\eta)^2}{2}+\frac{(t\cos\eta)^4}{24}\,d\eta$$
$$= \left[1-\frac{1}{2}\left(\frac{t}{2}\right)^2\right]^2.$$

This proves (2.11).

Now we shall prove (2.7). Using (2.11) and the inequality $\ln(1-x) \leqslant -x-x^2/2$ for $0 \leqslant x < 1$, we get for $0 \leqslant t \leqslant \mu < 2\sqrt{2}$

$$0 \leqslant J_0(t) \leqslant \left[1-\frac{1}{2}\left(\frac{t}{2}\right)^2\right]^2 = \exp\left[2\ln\left(1-\frac{1}{2}\left(\frac{t}{2}\right)^2\right)\right]$$
$$\leqslant \exp\left(2\left(-\frac{1}{2}\left(\frac{t}{2}\right)^2-\frac{1}{8}\left(\frac{t}{2}\right)\right)^4\right) = \exp\left[-\left(\frac{t}{2}\right)^2-\frac{1}{4}\left(\frac{t}{2}\right)^4\right],$$

which ends the proof of (2.7).

Now we shall prove (2.8). Since J_0 is a real analytic function which satisfies the differential equation

$$tJ_0(t)+J_0'(t)+tJ_0''(t) = 0$$

and $J_0(\mu) = \lim_{t\to\infty} J_0(t) = 0$, it suffices to show that if $J_0'(t) = 0$ then $|J_0(t)| \leqslant \frac{1}{2}$ for $t \geqslant \mu > 0$. If $J_0'(t) = 0$ then it follows from the differential equation that

$$|J_0(t)| = |J_0''(t)| = \left|\frac{1}{2\pi}\int_0^{2\pi}\cos^2\eta\cdot\cos(t\cos\eta)\,d\eta\right|$$
$$\leqslant \frac{1}{2\pi}\int_0^{2\pi}\cos^2\eta\cdot|\cos(t\cos\eta)|\,d\eta \leqslant \frac{1}{2\pi}\int_0^{2\pi}\cos^2\eta\,d\eta = \frac{1}{2},$$

which proves (2.8).

Proof of (2.9). Since $2\sqrt[4]{2} < \mu$ (cf. Watson [7], p. 748), it is enough to show that $\int_{2\sqrt[4]{2}}^{\infty} J_0^2(t)\,t^{-2}\,dt \leqslant 1/42$.

We shall need the identities

$$1-J_0^2(t) = -\sum_{i=1}^{\infty}\frac{(-1)^i}{i!^2}\binom{2i}{i}\left(\frac{t}{2}\right)^{2i}, \tag{2.12}$$

$$\int_0^{\infty}[1-J_0^2(t)]\,t^{-2}\,dt = \sqrt{2}F(2) = \frac{4}{\pi}. \tag{2.13}$$

For (2.12) cf. Watson [7], p. 32. To prove (2.13) observe that

$$E\left|\frac{1}{\sqrt{2}}(S_1+S_2)\right| = \left(\frac{1}{2\pi}\right)^2\frac{1}{\sqrt{2}}\int_0^{2\pi}\int_0^{2\pi}\sqrt{(\cos\eta_1+\cos\eta_2)^2+(\sin\eta_1+\sin\eta_2)^2}\,d\eta_1\,d\eta_2$$
$$= \frac{2\sqrt{2}}{\pi}.$$

Hence, applying identity (2.1) for $n = 2$ and for $a_1 = a_2 = 1/\sqrt{2}$, we get (2.13). It follows from (2.12) that for $a = 2\sqrt[4]{2}$

$$\int_a^{\infty} J_0^2(t)\,t^{-2}\,dt = \int_0^a[1-J_0^2(t)]\,t^{-2}\,dt+\int_a^{\infty} t^{-2}\,dt-\frac{4}{\pi}$$
$$= \int_0^a[1-J_0^2(t)]\,t^{-2}\,dt+\frac{1}{a}-\frac{4}{\pi}.$$

By (2.12) we get

$$\int_0^a [1-J_0^2(t)]\,t^{-2}\,dt = \frac{1}{4}\int_0^a \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!\,i!}\binom{2i}{i}\left(\frac{t}{2}\right)^{2i-2} dt = A+B,$$

where

$$A = \frac{1}{4}\int_0^a \sum_{i=1}^{5} \frac{(-1)^{i+1}}{i!\,i!}\binom{2i}{i}\left(\frac{t}{2}\right)^{2i-2} dt = \frac{\sqrt[4]{2}}{2}\left(2{,}23-\sqrt{2}\left(\frac{1}{2}+\frac{5}{144}\right)\right)$$

$$< \frac{1}{42}-\frac{1}{a}+\frac{4}{\pi},$$

$$B = -\frac{1}{4}\int_0^a \sum_{i=3} \left[\frac{1}{(2i)!}\right]^2 \binom{4i}{2i}\left(\frac{t}{2}\right)^{4i-2}\left[1-\frac{8i+2}{(2i+1)^3}\left(\frac{t}{2}\right)^2\right] dt < 0$$

because for $t \leqslant a$ and $i \geqslant 3$

$$1-\frac{8i+2}{(2i+1)^3}\left(\frac{t}{2}\right)^2 > 0.$$

Hence

(2.14) $$\int_a^{\infty} J_0^2(t)\,t^{-2}\,dt < A+\frac{1}{a}-\frac{4}{\pi} \leqslant \frac{1}{42}.$$

Finally we shall prove (2.10). It is sufficient to show that $F(8/5) \geqslant \sqrt{\pi}/2$, because from (2.13) it follows that $F(2) = 2\cdot\sqrt{2/\pi}$. From (2.11) it follows that for $0 \leqslant t \leqslant 2{,}4 < \mu = 2{,}4048\ldots$, we have $J_0(t) \leqslant [1-\frac{1}{2}(t/2)^2]^2$, from which it follows that for $s > 0$ and $0 \leqslant t \leqslant 2{,}4$,

$$1-|J_0(t)|^s\,t^{-2} \geqslant 1-\left[1-\frac{1}{2}\left(\frac{t}{2}\right)^2\right]^{2s} t^{-2} = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2^{i+1}}\binom{2s}{i}\left(\frac{t}{2}\right)^{2i-2}.$$

Since the series converges uniformly in $[0;\,2{,}4]$, we can integrate "term-by-term". We obtain for $s > 0$

(2.15) $$\frac{1}{\sqrt{s}}\int_0^{2{,}4} [1-|J_0(t)|^s]\,t^{-2}\,dt \geqslant \frac{1}{\sqrt{s}}\sum_{i=1}^{\infty}\int_0^{2{,}4} \frac{(-1)^{i+1}}{2^{i+1}}\binom{2s}{i}\left(\frac{t}{2}\right)^{2i-2} dt$$

$$= \frac{1}{\sqrt{s}}\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2^{i+1}}\binom{2s}{i}\frac{(1{,}2)^{2i-1}}{2i-1} = A+B,$$

where

$$A = \frac{1}{\sqrt{s}}\sum_{i=1}^{4} \frac{(-1)^{i+1}}{2^{i+1}}\binom{2s}{i}\frac{(1{,}2)^{2i-1}}{2i-1},$$

$$B = \frac{1}{\sqrt{s}}\sum_{i=5}^{\infty} \frac{(-1)^{i+1}}{2^{i+1}}\binom{2s}{i}\frac{(1{,}2)^{2i-1}}{2i-1}.$$

For $s = 8/5$ we get by direct computation

$$A = \sqrt{\frac{5}{8}}\sum_{i=1}^{4} \frac{(-1)^{i+1}}{2^{i+1}}\binom{3{,}2}{i}\frac{(1{,}2)^{2i-1}}{2i-1} \geqslant 0{,}5923.$$

For $s = 8/5$ and for $i \geqslant 5$ we have $\left|\binom{2s}{i}\right| \leqslant \left|\binom{3{,}2}{5}\right|$, $1/(2i-1) \leqslant 1/9$. Hence

$$B \geqslant -\frac{1}{\sqrt{s}}\sum_{i=5}^{\infty} \frac{1}{2^{i+1}}\left|\binom{2s}{i}\right|\frac{(1{,}2)^{2i-1}}{2i-1}$$

$$\geqslant -\sqrt{\frac{5}{8}}\left|\binom{3{,}2}{5}\right|\frac{1}{9}\sum_{i=5}^{\infty}\frac{(1{,}2)^{2i-1}}{2^{i+1}}$$

$$= -\sqrt{\frac{5}{8}}\left|\binom{3{,}2}{5}\right|\frac{1}{9}\frac{(1{,}2)^9}{2^6}\cdot\frac{1}{1-0{,}72} \geqslant -\frac{1}{2500} = -0{,}0004.$$

Since $A+B \geqslant 0{,}5919$, it follows from (2.15) that

(2.16) $$\sqrt{\frac{5}{8}}\int_0^{2{,}4} [1-|J_0(t)|^{1{,}6}]\,t^{-2}\,dt \geqslant 0{,}5919.$$

From the Hölder inequality it follows that

$$\sqrt{\frac{5}{8}}\int_{2{,}4}^{\infty} |J_0(t)|^{1{,}6}\,t^{-2}\,dt \leqslant \sqrt{\frac{5}{8}}\left(\int_{2{,}4}^{\infty} J_0^2(t)\,t^{-2}\,dt\right)^{0{,}8}\left(\int_{2{,}4}^{\infty} t^{-2}\,dt\right)^{0{,}2}.$$

Since from (2.14) it follows that

$$\int_{2{,}4}^{\infty} J_0^2(t)\,t^{-2}\,dt < \int_{2\sqrt[4]{2}}^{\infty} J_0^2(t)\,t^{-2}\,dt \leqslant \frac{1}{42},$$

we have

(2.17) $$\sqrt{\frac{5}{8}}\int_{2{,}4}^{\infty} |J_0(t)|^{1{,}6}\,t^{-2}\,dt \leqslant \sqrt{\frac{5}{8}}\left(\frac{1}{42}\right)^{0{,}8}\left(\frac{1}{2{,}4}\right)^{0{,}2} \leqslant 0{,}0334.$$

Moreover, we have

(2.18) $$\sqrt{\frac{5}{8}}\int_{2,4}^{\infty} t^{-2}\,dt = \sqrt{\frac{5}{8}}\frac{1}{2,4} \geqslant 0,3294.$$

Thus from (2.16), (2.17) and (2.18) we get

$$F\left(\frac{5}{8}\right) = \sqrt{\frac{5}{8}}\int_{0}^{\infty} [1-|J_0(t)|^{1,6}]\,t^{-2}\,dt \geqslant 0,5919-0,0334+0,3294 \geqslant \frac{\sqrt{\pi}}{2}.$$

This completes the proof of Lemma 2.

Recall that

$$F(s) = \frac{1}{\sqrt{s}}\int_{0}^{\infty} [1-|J_0(t)|^s]\,t^{-2}\,dt, \qquad s>0.$$

Let us put

$$G(s) = \frac{1}{\sqrt{s}}\int_{0}^{\mu}\left[\exp\left(-s\left(\frac{t}{2}\right)^2\right)-\exp\left(-s\left(\frac{t}{2}\right)^2-\frac{1}{4}\left(\frac{t}{2}\right)^4\right)\right]t^{-2}\,dt, \qquad s>0,$$

$$H(s) = 2\sqrt{s}\int_{0}^{\infty} -\ln|J_0(t)|\,|J_0(t)|^s\,t^{-2}\,dt, \qquad s>0,$$

$$I(s) = 4s\sqrt{s}\int_{0}^{\infty} \ln^2|J_0(t)|\,|J_0(t)|^s\,t^{-2}\,dt, \qquad s>0.$$

It is not difficult to show that

(2.19) $$F(s) = \frac{1}{2s}(H(s)-F(s)),$$

(2.20) $$H'(s) = \frac{1}{2s}(H(s)-I(s)).$$

The following lemma will be used in the sequel:

Lemma 3. *For $s \geqslant 2$ we have*

(2.21) $$F(s) \geqslant \frac{\sqrt{\pi}}{2}+G(s)-\frac{1}{40s}\left(\frac{1}{2}\right)^{s-2},$$

(2.22) $$G(s) \geqslant \frac{1}{20\sqrt{2s}}.$$

For $1,6 \leqslant s \leqslant 2$ we have

(2.23) $$\frac{\sqrt{2}}{3}+I(s) \geqslant H(s).$$

Proof. For $a>0$ we have (cf. Dwight [1], p. 155)

(2.24) $$\frac{1}{\sqrt{a}}\int_{0}^{\infty}\left[1-\exp\left(-a\left(\frac{t}{2}\right)^2\right)\right]t^{-2}\,dt = \frac{\sqrt{\pi}}{2}.$$

From (2.7) it follows that, for $0 \leqslant t \leqslant \mu$, $|J_0(t)|^s \leqslant \exp(-s(t/2)^2-\frac{1}{4}s(t/2)^4)$. Thus

(2.25) $$\frac{1}{\sqrt{s}}\int_{0}^{\mu}\left[\exp\left(-s\left(\frac{t}{2}\right)^2\right)-|J_0(t)|^2\right]t^{-2}\,dt \geqslant G(s).$$

From (2.8) and (2.9) we get

(2.26) $$\frac{1}{\sqrt{s}}\int_{\mu}^{\infty}\left[\exp\left(-s\left(\frac{t}{2}\right)^2\right)-|J_0(t)|^s\right]t^{-2}\,dt \geqslant \frac{1}{\sqrt{s}}\int_{\mu}^{\infty} -|J_0(t)|^s\,t^{-2}\,dt$$

$$= -\frac{1}{\sqrt{s}}\int_{\mu}^{\infty} |J_0(t)|^{s-2}J_0^2(t)\,t^{-2}\,dt \geqslant -\frac{1}{\sqrt{s}}\int_{\mu}^{\infty}\left(\frac{1}{2}\right)^{s-2}J_0^2(t)\,t^{-2}\,dt$$

$$\geqslant -\frac{1}{\sqrt{s}}\left(\frac{1}{2}\right)^{s-2}\cdot\frac{1}{42} \geqslant -\frac{1}{40\sqrt{s}}\left(\frac{1}{2}\right)^{s-2}.$$

Hence, using (2.24), (2.25) and (2.26), we obtain (2.21).

Now we shall show (2.22). From the definition of $G(s)$ it follows that for $s \geqslant 2$

(2.27) $$G(s) = \frac{1}{\sqrt{2}}\int_{0}^{\mu\sqrt{s/2}}\left[\exp\left(-\frac{t^2}{2}\right)-\exp\left(-\frac{t^2}{2}-\frac{1}{16s}t^4\right)\right]t^{-2}\,dt$$

$$\geqslant \frac{1}{\sqrt{2}}\int_{0}^{\mu}\left[\exp\left(-\frac{t^2}{2}\right)-\exp\left(-\frac{t^2}{2}-\frac{1}{16s}t^4\right)\right]t^{-2}\,dt.$$

Using (2.27), and the inequalities $1-e^{-x} \geqslant x - x^2/2$ and $\int_0^x t^6 \exp(-t^2/2)\,dt \geqslant 15 \int_0^x t^2 \exp(-t^2/2)\,dt$ for $x \geqslant 0$, we get for $s \geqslant 2$

$$G(s) \geqslant \frac{1}{\sqrt{2}} \int_0^\mu \left[1-\exp\left(-\frac{1}{16s}t^4\right)\right] \exp\left(-\frac{t^2}{2}\right) t^{-2}\,dt \tag{2.28}$$

$$\geqslant \frac{1}{16\sqrt{2s}}\left(1-\frac{15}{32s}\right)\int_0^\mu t^2 \exp\left(-\frac{t^2}{2}\right) dt.$$

Since $1-15/(32s) \geqslant 49/64$ for $s \geqslant 2$ and $\int_0^\mu t^2 \exp(-t^2/2)\,dt \geqslant 1{,}09$, we have for $s \geqslant 2$

$$\left(1-\frac{15}{32s}\right)\int_0^\mu t^2 \exp\left(-\frac{t^2}{2}\right) dt \geqslant 0{,}8.$$

From this and (2.28) it follows that $G(s) \geqslant 1/(16\sqrt{2}s)$, which ends the proof of (2.22).

Now we prove (2.23). We begin by establishing the inequalities

$$4s\sqrt{s}\int_0^\mu \ln^2|J_0(t)|\,|J_0(t)|^s t^{-2}\,dt \tag{2.29}$$

$$\geqslant 2\sqrt{s}\int_0^\mu (-\ln|J_0(t)|)\,|J_0(t)|^s t^{-2}\,dt - \sqrt{2}/3,$$

$$4s\sqrt{s}\int_\mu^\infty \ln^2|J_0(t)|\,|J_0(t)|^s t^{-2}\,dt \geqslant 2\sqrt{s}\int_\mu^\infty -\ln|J_0(t)|\,|J_0(t)|^s t^{-2}\,dt. \tag{2.30}$$

From (2.7) it follows that $0 \leqslant J_0(t) \leqslant \exp(-(t/2)^2)$ for $0 \leqslant t \leqslant \mu$. Thus $\ln^2|J_0(t)| \geqslant (t/2)^2(-\ln|J_0(t)|)$ for $0 \leqslant t \leqslant \mu$. Hence for $8/5 \leqslant s \leqslant 2$,

$$4s\sqrt{s}\int_0^\mu \ln^2|J_0(t)|\,|J_0(t)|^s t^{-2}\,dt \tag{2.31}$$

$$\geqslant 4s\sqrt{s}\int_0^\mu (t/2)^2(-\ln|J_0(t)|)\,|J_0(t)|^s t^{-2}\,dt$$

$$= 2\sqrt{s}\int_0^\mu [s(t/2)^2-1](-\ln|J_0(t)|)\,|J_0(t)|^s t^{-2}\,dt +$$

$$+2\sqrt{s}\int_0^\mu -\ln|J_0(t)|\,|J_0(t)|^s t^{-2}\,dt$$

$$\geqslant 2\sqrt{s}\int_0^{\sqrt{2/s}} [s(t/2)^2-1](-\ln|J_0(t)|)\,|J_0(t)|^s t^{-2}\,dt +$$

$$+2\sqrt{s}\int_0^\mu (-\ln|J_0(t)|)\,|J_0(t)|^s t^{-2}\,dt.$$

Let us put $f(x) = -x^s \ln x$ for $0 < x$. Then f is an increasing function for $0 < x \leqslant \exp(-1/s)$ and $f(x) \leqslant 1/s$ for $0 < x$. Since $0 \leqslant J_0(t) \leqslant \exp(-(t/2)^2)$ for $0 \leqslant t \leqslant \mu$, thus $f(J_0(t)) \leqslant \frac{1}{4}t^2$ for $0 \leqslant t \leqslant \sqrt{2/s}$ and $f(J_0(t)) \leqslant 1/s \leqslant \frac{1}{4}t^2$ for $\sqrt{2/s} < t$. Hence $-\ln J_0(t)\,|J_0(t)|^s t^{-2} \leqslant \frac{1}{4}$ for $0 \leqslant t \leqslant \mu$. Using this and (2.31), we get

$$4s\sqrt{s}\int_0^\mu \ln^2|J_0(t)|\,|J_0(t)|^s t^{-2}\,dt$$

$$\geqslant 2\sqrt{s}\int_0^{\sqrt{2/s}} (st^2/2-1)\tfrac{1}{4}\,dt + 2\sqrt{s}\int_0^\mu (-\ln|J_0(t)|\,|J_0(t)|^s)\,t^{-2}\,dt$$

$$= -\sqrt{2}/3 + 2\sqrt{s}\int_0^\mu (-\ln|J_0(t)|\,|J_0(t)|^s)\,t^{-2}\,dt,$$

which proves (2.29). It follows from (2.8) that for $\frac{8}{5} \leqslant s$ and $\mu \leqslant t$ $4s\sqrt{s}\ln^2|J_0(t)| \geqslant 2\sqrt{s}(-\ln|J_0(t)|)$. Thus we get (2.30). Adding (2.29) to (2.30), we get (2.23).

Now we can prove the following

Theorem 4. *If $a_1 \leqslant \sqrt{5/8}$, then $E|S| \geqslant \sqrt{\pi}/2$.*

Proof. We shall show that for $s \geqslant 1{,}6$

$$F(s) \geqslant \sqrt{\pi}/2. \tag{2.32}$$

We first prove (2.32) for $s \geqslant 2$. From Lemma 3, (2.21) and (2.22) we get

$$F(s) \geqslant \frac{\sqrt{\pi}}{2} + G(s) - \frac{1}{40\sqrt{s}}\left(\frac{1}{2}\right)^{s-2} \geqslant \frac{\sqrt{\pi}}{2} + \frac{1}{20\sqrt{2}s} - \frac{1}{40\sqrt{s}}\left(\frac{1}{2}\right)^{s-2} \tag{2.33}$$

$$= \frac{\sqrt{\pi}}{2} + \frac{1}{5\sqrt{2}s}\left[\frac{1}{4} - \sqrt{\frac{s}{2}}\left(\frac{1}{2}\right)^s\right].$$

If $f(s) = 1/4 - \sqrt{s/2}\,(1/2)^s$, then $f(2) = 0$ and $f'(s) = (1/2)^s \sqrt{s/2}\,(\ln 2 - 1/2s) > 0$, from which it follows that $f(s) \geqslant f(2) = 0$ for $s \geqslant 2$. Using this and (2.33), we have for $s \geq 2$

$$F(s) \geq \frac{\sqrt{\pi}}{2} + \frac{1}{5\sqrt{2s}} \cdot f(s) \geq \frac{\sqrt{\pi}}{2}.$$

Now we shall show that for $8s/5 \leq 2$, $F(s) \geq \sqrt{\pi}/2$. From Lemma 2 (2.10) we have $F(2) = 2\sqrt{2/\pi}$ and $F(8/5) \geq \sqrt{\pi}/2$. Assume to the contrary that $F(s) < \sqrt{\pi}/2$ for some $s \in (8/5, 2)$. Then, by the continuity of F, the set $F^{-1}(\sqrt{\pi}/2) \cap (8/5, 2)$ is non-empty. Let $s_1 = \sup\{s \in (8/5, 2) |\, F(s) = \sqrt{\pi}/2\}$. Then $F(2) - F(s_1) = 2\sqrt{2/\pi} - \sqrt{\pi}/2$. On the other hand, $F(2) - F(s_1) = (2 - s_1) F'(\tilde{s})$ for some $\tilde{s} \in (s_1, 2)$. Using this and (2.19), we get

$$(2.34) \quad H(\tilde{s}) = F(\tilde{s}) + \left(\frac{2\sqrt{2}}{\pi} - \frac{\sqrt{\pi}}{2}\right)\frac{2\tilde{s}}{2 - s_1} \geq F(\tilde{s}) + \left(\frac{2\sqrt{2}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \cdot 8$$
$$\geq \frac{\sqrt{\pi}}{2} + \left(\frac{2\sqrt{2}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \cdot 8,$$

because $F(\tilde{s}) \geq \sqrt{\pi}/2$.

Let $s_2 = \inf\{s \in [8/5, 2] |\, F(s) = \inf_{1,6 \leq t \leq 2} F(t)\}$. Then $F(s_2) < \sqrt{\pi}/2$ and $F'(s_2) = 0$. Therefore from (2.20) it follows that

$$(2.35) \quad H(s_2) = F(s_2) < \sqrt{\pi}/2.$$

From the definition of s_1, $\tilde{s}$ and s_2 it follows that $2 \geq \tilde{s} > s_1 > s_2 \geq 8/5$. Hence from (2.34) and (2.35), using (2.20), we get

$$\left(\frac{2\sqrt{2}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \cdot 8 < H(\tilde{s}) - H(s_2) = (\tilde{s} - s_2) H'(s_3)$$
$$= (\tilde{s} - s_2)\frac{1}{2s_3}(H(s_3) - I(s_3))$$

for some s_3 such that $s_2 < s_3 < \tilde{s}$. From this we get

$$\left(\frac{2\sqrt{2}}{\pi} - \frac{\sqrt{\pi}}{2}\right) 64 \leq \left(\frac{2\sqrt{2}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \cdot 8 \cdot \frac{2s_3}{\tilde{s} - s_2} \cdot (H(s_3) - I(s_3)),$$

because $8/5 \leq s_2 < s_3 < \tilde{s} \leq 2$. So we have

$$\left(\frac{2\sqrt{2}}{\pi} - \frac{\sqrt{\pi}}{2}\right) \cdot 64 + I(s_3) < H(s_3)$$

for some s_3 with $8/5 \leq s_3 \leq 2$, which contradicts Lemma 3 (the inequality (2.23)), because $(2\sqrt{2/\pi} - \sqrt{\pi}/2) \cdot 64 > \sqrt{2}/3$.

From (2.32) it follows that if $a_1 \leq \sqrt{5/8}$ then $F(a_i^{-2}) \geq \sqrt{\pi}/2$. From

Proposition 1, (2.2) it follows that if $a_1 \leq \sqrt{5/8}$ then

$$E|S| \geq \sum_{i=1}^{n} a_i^2 F(a_i^{-2}) \geq \sum_{i=1}^{n} a_i^2 \frac{\sqrt{\pi}}{2} \geq \frac{\sqrt{\pi}}{2}.$$

COROLLARY 5. $E|S| \geq \sqrt{5/8}$.

Proof. We have just proved that if $a_1 \leq \sqrt{5/8}$ then $E|S| \geq \sqrt{\pi}/2$. If $a_1 \geq \sqrt{5/8}$ then $E|S| \geq \sqrt{5/8}$, because

$$E|S| = \tfrac{1}{2}\Big(E\Big|a_1 S_1 + \sum_{i=2}^{n} a_i S_i\Big| + E\Big|a_1 S_1 - \sum_{i=2}^{n} a_i S_i\Big|\Big) \geq a_1.$$

3. Case 2: $a_1^2 \geq 5/8$. Let us put

$$h_a(x) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{a^2 + 2a \cdot x\cos\varphi + x^2}\, d\varphi, \qquad x \geq 0,\ a > 0.$$

Next define $c_n(a)$ by

$$c_1(a) = \sqrt{5/8}, \qquad c_n(a) = h_a\big(c_{n-1}(a)\sqrt{1 - a^2}\big) \quad \text{for} \quad n > 1.$$

Clearly, $0 < c_n(a) \leq 1$, for $n = 1, 2, \ldots$

Now we prove the following:

LEMMA 6. (a) *For every $a > 0$ the function h_a is strictly increasing and convex in $[0, \infty)$.*

(b) *If $\sqrt{5/8} \leq a \leq 1$ then $\lim_{n \to \infty} c_n(a) \geq \pi/2$.*

Proof. If $a > 0$ and $x \geq 0$ then

$$(3.1) \quad h_a(x) = (a + x)\frac{1}{2\pi}\int_0^{2\pi} \sqrt{1 - \frac{4ax}{(a + x)^2}\sin^2\varphi}\, d\varphi.$$

Let $E(k) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{1 - k^2\sin^2\varphi}\, d\varphi$, where $|k| \leq 1$; then

$$(3.2) \quad E(k) = \frac{1}{1 + m}\left(1 + \frac{m^2}{2^2} + \frac{1^2}{2^2 4^2} m^4 + \frac{1^2 3^2}{2^2 4^2 6^2} m^6 + \ldots\right),$$

where $m = \frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}$ (cf. Dwight [1]). In particular, if $k^2 = 4ax/(a + x)^2$, then $m = \min(a, x)/\max(a, x)$. Using this and (3.1), (3.2), we get

$$(3.3)\qquad h_a(x) = \max(a, x)\left[1+\frac{1}{4}\frac{\min^2(a, x)}{\max^2(a, x)}+\frac{1^2}{2^2 4^2}\frac{\min^4(a, x)}{\max^4(a, x)}+\ldots\right].$$

It follows from (3.3) that for each $a > 0$ the function h_a is continuous and strictly increasing. It is easy to see that for $x \geqslant 0$ and for $x \neq a$ the second derivative $h_a''(x)$ exists, $h_a''(x) > 0$, from which it follows that h_a is a convex function on the intervals $[0, a)$ and (a, ∞). Since h_a is continuous at $a = x$ and $h_a'(a+) \geqslant h_a'(a-)$, we infer that h_a is convex on the interval $[0, \infty)$, which ends the proof of the first part of Lemma 6.

For $\sqrt{5/8} \leqslant a \leqslant 1$ define f_a by $f_a(x) = h_a(x\sqrt{1-a^2})-x$ for $0 \leqslant x \leqslant 1$. From (3.3) it follows that if $\sqrt{5/8} \leqslant a \leqslant 1$ and $0 \leqslant x \leqslant 1$ then

$$(3.4)\qquad f_a(x) = a\left[1+\frac{1}{4}\left(\frac{1-a^2}{a^2}\right)x^2+\frac{1^2}{2^2 4^2}\left(\frac{1-a^2}{a^2}\right)^2 x^4+\ldots\right]-x.$$

Thus f_a is differentiable and

$$1+f_a'(x) = \frac{1}{2}\left(\frac{1-a^2}{a^2}\right)x+\frac{1^2}{2^2 4}\left(\frac{1-a^2}{a^2}\right)^2 x^3+\frac{1^2 3^2}{2^2 4^2 6^2}6\left(\frac{1-a^2}{a^2}\right)^3 x^5+\ldots$$

$$\leqslant \frac{1}{2}\left[\left(\frac{1-a^2}{a^2}\right)x+\left(\frac{1-a^2}{a^2}\right)^2 x^3+\left(\frac{1-a^2}{a^2}\right)^3 x^5\ldots\right]$$

$$\leqslant \frac{1}{2}\left[\left(\frac{1-a^2}{a^2}\right)+\left(\frac{1-a^2}{a^2}\right)^2+\left(\frac{1-a^2}{a^2}\right)^3\right] = \frac{1}{2}\frac{1-a^2}{2a^2-1} \leqslant \frac{3}{4},$$

because $0 \leqslant x \leqslant 1$ and $\sqrt{5/8} \leqslant a \leqslant 1$. Thus f_a is strictly decreasing for $0 \leqslant x \leqslant 1$.

Next we show that, for $\sqrt{5/8} \leqslant a \leqslant 1$,

$$(3.5)\qquad f_a(\sqrt{\pi}/2) > 0,$$

$$(3.6)\qquad f_a(1) \leqslant 0.$$

Indeed, from the definition of f_a and from the Schwarz inequality we get

$$f_a(1) = h_a(\sqrt{1-a^2})-1 = \frac{1}{2\pi}\int_0^{2\pi}\sqrt{a^2+2a\sqrt{1-a^2}\cos\varphi+1-a^2}\,d\varphi-1$$

$$= \left[\frac{1}{2\pi}\int_0^{2\pi}(a^2+2a\sqrt{1-a^2}\cos\varphi+1-a^2)\,d\varphi\right]^{1/2}-1 = 0.$$

Finally, from (3.4) it follows that, for $\sqrt{5/8} \leqslant a \leqslant 1$,

$$f_a\left(\frac{\sqrt{\pi}}{2}\right) = a\left[1+\frac{1}{4}\left(\frac{1-a^2}{a^2}\right)\frac{\pi}{4}+\frac{1^2}{2^2 4^2}\left(\frac{1-a^2}{a^2}\right)^2\frac{\pi^2}{4}+\ldots\right]-\frac{\sqrt{\pi}}{2}$$

$$\leqslant a\left[1+\frac{\pi}{20}(1-a^2)+\frac{\pi^2}{100}(1-a^2)^2\right]-\frac{\sqrt{\pi}}{2}.$$

So, if we denote by g the function defined on $[\sqrt{5/8}, 1]$ by

$$g(a) = a\left[1+\frac{\pi}{20}(1-a^2)+\frac{\pi^2}{100}(1-a^2)^2\right]-\frac{\sqrt{\pi}}{2},$$

then

$$(3.7)\qquad f_a(\sqrt{\pi}/2) \geqslant g(a)\quad\text{for}\quad\sqrt{5/8} \leqslant a \leqslant 1.$$

For $\sqrt{5/8} \leqslant a \leqslant 1$, $g'(a) > 0$ and $g(\sqrt{5/8}) > 0$. Thus, for $\sqrt{5/8} \leqslant a \leqslant 1$, using (3.7) and these inequalities, we get $f_a(\sqrt{\pi}/2) \geqslant g(\sqrt{5/8}) > 0$, which ends the proof of (3.6).

Since f_a is strictly decreasing on $[0, 1]$ and continuous, it follows from (3.5) and (3.6) that if $\sqrt{5/8} \leqslant a \leqslant 1$ then there exists exactly one $c(a)$ with $\sqrt{\pi/2} \leqslant c(a) \leqslant 1$ such that $f_a(c(a)) = 0$. From the definition of $c_n(a)$ it follows that $c_1(a) = \sqrt{5/8} < \sqrt{\pi}/2$.

Now we prove that

$$(3.8)\qquad c_n(a) \leqslant c(a)\quad\text{for}\quad n = 1, 2, \ldots$$

For $n = 1$ this is true. For $n \geqslant 2$ we have, from the definition of $c_n(a)$, $c_n(a) = h_a(c_{n-1}(a)\sqrt{1-a^2})$. Since h_a is increasing, we infer that $c_{n-1}(a) \leqslant c(a)$; then $c_n(a) = h_a(c_{n-1}(a)\sqrt{1-a^2}) \leqslant h_a(c(a)\sqrt{1-a^2}) = c(a)$. This ends the proof of (3.8).

Finally we show

$$(3.9)\qquad \lim_{n\to\infty} c_n(a) = c(a) \geqslant \sqrt{\pi}/2.$$

From the definition of $c_n(a)$, f_a, $c(a)$, (3.8) and the monotonicity of f_a we get $c_{n+1}(a) = h_a(c_n(a)\sqrt{1-a^2}) = f_a(c_n(a))+c_n(a) \geqslant c_n(a)$. From this inequality and (3.8) it follows that $\lim_{n\to\infty} c_n(a) = d(a)$ exists and $c(a) \geqslant d(a) \geqslant \sqrt{5/8}$. Moreover, from the definition of $c_n(a)$ it follows that $d(a) = h_a(d(a)\sqrt{1-a^2})$. Thus $f_a(d(a)) = 0$. Hence $d(a) = c(a)$ because for $0 \leqslant x \leqslant 1$ there exists exactly one root of the equation $f_a(x) = 0$. Thus we have shown that $\lim_{n\to\infty} c_n(a) = c(a) \geqslant \sqrt{\pi}/2$.

Now we can prove the following:

THEOREM 7. *If* $a_1 \geq \sqrt{5/8}$ *then* $E|S| \geq \sqrt{\pi}/2$.

Proof. Let us denote by $X = (X_1, X_2) = (\sum_{i=2}^{n} a_i \cos\xi_i, \sum_{i=2}^{n} a_i \sin\xi_i)$, where $\xi_2, \xi_3, \ldots, \xi_n$ are independent random variables uniformly distributed on $[0, 2\pi]$. Let us observe that $X = (X_1, X_2)$ and $(|X|\cos\xi, |X|\sin\xi)$ are equidistributed, where η is an r.v. uniformly distributed in $[0, 2\pi]$ and independent of X. Therefore

$$(3.10)\quad E|S| = E|a_1 S_1 + X|$$
$$= E\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \sqrt{(a_1\cos\psi + |X|\cos\xi)^2 + (a_1\sin\psi + |X|\sin\xi)^2}\, d\psi\, d\xi$$
$$= E\frac{1}{2\pi}\int_0^{2\pi} \sqrt{a_1^2 + 2a_1|X|\cos\eta + |X|^2}\, d\eta = E(h_{a_1}(|X|)).$$

From (3.10), Lemma 6 and from Jensen's inequality it follows that

$$(3.11)\quad E|S| = E(h_{a_1}(|X|)) \geq h_{a_1}(E|X|).$$

Now from Corollary 5 it follows that $E|X| \geq \sqrt{5/8}\sqrt{E|X|^2} = c_1(a)\sqrt{1-a_1^2}$. Using this and (3.11), we get for $n > 1$, $E|S| \geq h_{a_1}(c_{n-1}(a_1)\sqrt{1-a_1^2}) = c_n(a_1)$, because h_{a_1} is an increasing function. Thus $E|S| \geq c_n(a_1)$ for $n = 1, 2, \ldots$ Using this and Lemma 6, we get $E|S| \geq \lim_{n\to\infty} c_n(a_1) = c(a_1) \geq \sqrt{\pi}/2$ because $a_1 \geq \sqrt{5/8}$. This completes the proofs of Theorem 7, and Theorem A.

4. Final remarks.

4.1. Theorem A implies that the one summing norm of the natural injection of the complex l^1 into the complex l^2 equals $2/\sqrt{\pi}$ (cf. Szarek [6] for details).

4.2. Combining Theorem A with an argument of Orlicz [3] (cf. Szarek [6] for details), one gets

COROLLARY 8. *If X is a complex Banach space which is isometrically isomorphic to a subspace of L^1, in particular if X is a Hilbert space, then*

$$E\|\sum_{j=1}^{n} \sigma_j x_j\|_X \geq \tfrac{1}{2}\sqrt{\pi}(\sum_{j=1}^{n} \sigma_j \|x_j\|^2)^{1/2}$$

for arbitrary $x_1, x_2, \ldots, x_n$ in X and for $n = 1, 2, \ldots$

4.3. Our main result is a step towards substantiating the Haagerup conjecture (cf. Pełczyński [8], Section 4). Let us put

$$\gamma_p = \left(\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2)^{p/2}\exp(-(x^2+y^2))\,dx\,dy\right)^{1/p} = \left(\Gamma\left(\frac{p+2}{2}\right)\right)^{1/p},$$

$$d_p = \left(\frac{2^{p/2}}{2\pi}\int_0^{2\pi} |\sin x|^p\, dx\right)^{1/p} = \sqrt{2}\left(\frac{\Gamma((p+1)/2)}{\sqrt{\pi}\,\Gamma((p+2)/2)}\right)^{1/p}.$$

Analysing the proof of Theorem A, one can show

THEOREM 8. *There exists a $\delta > 0$ such that if $|1-p| < \delta$ then*

$$(E|\sum_{j=1}^{n} a_j\sigma_j|^p)^{1/p} \geq \gamma_p(\sum_{j=1}^{n} |a_j|^2)^{1/2}$$

for arbitrary complex numbers $a_1, a_2, \ldots, a_n$ and for $n = 1, 2, \ldots$

Obviously, the constant γ_p cannot be enlarged because, by the Central Limit Theorem,

$$\lim_{n\to\infty} (E|\sum_{j=1}^{n} \sigma_j/\sqrt{n}|^p)^{1/p} = \gamma_p.$$

On the other hand, using the asymptotic expansion for small $p > 0$, one can show that there exists a δ with $1 > \delta > 0$ such that $d_p < \gamma_p$ for $0 < p < \delta$. Note that

$$d_p = \left(E\left|\frac{1}{\sqrt{2}}(\sigma_1+\sigma_2)\right|^p\right)^{1/p}.$$

Thus for $0 < p < \delta$ the best constant c_p in the inequality

$$(E|\sum_{j=1}^{n} a_j\sigma_j|^p)^{1/p} \geq c_p(\sum_{j=1}^{n} |a_j|^2)^{1/2}$$

is less than γ_p.

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Added in proof. We can show the following theorem:

A. *Let $p_0 \in (0, 2)$ be the unique root of the equation*

$$2^{p/2}\Gamma((p+1)/2) = \sqrt{\pi}\,\Gamma((p+2)/2)^2; \quad p_0 = 0{,}47562\ldots$$

Then for $2 \geq p > p_0$

$$\|\sum a_j\sigma_j\|_p \geq \Gamma((p+2)/2)^{1/p}(\sum |a_j|^2)^{1/2}$$

for arbitrary complex scalars $a_1, a_2, \ldots$ The constant $\Gamma((p+2)/2)^{1/p}$ is the best possible.

B. *If* $p \geqslant 2$ *then*

$$\left\|\sum a_j \sigma_j\right\|_p \leqslant \Gamma\left((p+2)/2\right)^{1/p}\left(\sum |a_j|^2\right)^{1/2}$$

with arbitrary a_j. *The constant* $\Gamma\left((p+2)/2\right)^{1/p}$ *is the best possible.*

The proof of A is essentially a modification of the method used in the paper. The starting point to prove B is the formula

$$E\left|\sum a_i \sigma_i\right|^p = C_p \int_0^{\infty} \left(\prod_{i=1}^{n} J_0(a_i t) - 1\right) t^{-p-1}\, dt$$

where C_p is some constant. The proofs will appear in Studia Math. (J. Sawa, *Some remarks on the Khintchine inequality for complex Steinhaus variables*).

References

[1] G. B. Dwight, *Tables of Integrals and Other Mathematical Data* (in Russian), Nauka, Moscow 1982.

[2] U. Haagerup, *The best constants in the Khintchine inequality*, Studia Math. 70 (1982), 231–283.

[3] W. Orlicz, *Über unbedingte Konvergenz in Funktionenräumen I*, ibid. 4 (1933), 33–37.

[4] A. Pietsch, *Nukleare lokalkonvexe Räume*, Akademie-Verlag, Berlin 1965.

[5] S. B. Stechkin, *On the best lacunary systems of functions* (in Russian), Izv. Akad. Nauk SSSR 25 (1961), 357–366.

[6] S. J. Szarek, *On the best constants in the Khinchin inequality*, Studia Math. 58 (1976), 197–208.

[7] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Univ. Press, Cambridge 1944.

[8] A. Pełczyński, *Norms of classical operators in function spaces*, Astérisque, 1985, Colloque L. Schwartz, to appear.

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