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Monotonic mod one transformations

by

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Abstract. The aim of the paper is the investigation of the topological structure of monotonic mod one transformations on the interval. First, the nonwandering set of the shift space obtained by f-expansion is determined by using an oriented graph which we call Markov diagram. It reflects the orbit structure of this shift space. Then we consider the intervals, with the same f-expansion elements, and get a characterization of the nonwandering sets of monotonic mod one transformations.

§ 0. Introduction. Let $f\colon [0, 1) \to \boldsymbol{R}$ be continuous and increasing such that $f(0) \in [0, 1)$. We call then $T\colon [0, 1) \to [0, 1)$ defined by $T(x) = f(x)$ (mod 1) a monotonic mod 1 transformation. Our goal is to investigate the nonwandering set of these transformations. To this end we use the methods developed in [2] und used in [6] to prove some results about the nonwandering set of a more general class of interval transformations. Here we give a complete classification of sets which can occur as nonwandering sets for monotonic mod 1 transformations.

In § 1 we define a one-sided shift space Σ_T^+ we get from $([0, 1), T)$ by f-expansion, and an oriented graph which we shall call the Markov diagram of T, the one-sided paths of which represent the elements of Σ_T^+. In § 2 and § 3 we classify the oriented graphs which can occur as Markov diagrams of T. As the Markov diagram determines the nonwandering set Ω of Σ_T^+, we obtain from this a classification of all possible Ω in § 4. This is used in § 5 to find all possible subsets of $[0, 1)$ which can occur as nonwandering sets L of T. Let $\mathfrak{I}$ be the set of all open intervals $I \subset [0, 1)$ such that $T^k|I$ is monotone for all $k \geqslant 0$. Then we have for some $n \leqslant \infty$

$$L = \bigcup_{0 \leqslant i \leqslant n} L_i \cup Y \cup P$$

where the L_i are ω-limit sets, pairwise disjoint up to finite sets, P is the set of periodic points contained in $\bigcup_{I \in \mathfrak{I}} \bar{I}$, and Y is an empty, finite or countable set, contained in $\bigcup_{I \in \mathfrak{I}} \operatorname{bd} I$ and wandering in $(L, T|L)$.

Such a classification of the nonwandering set was given in [8] for continuous transformations on $[0, 1]$ with a unique turning point. In this paper the points contained in an $I \in \mathfrak{J}$ are not distinguished. It is shown in [3] how the transformations of [8] can be identified with those monotonic mod 1 transformations which satisfy $f(\frac{1}{2}-x)+f(\frac{1}{2}+x) = 2$. Hence the results of [8] can be viewed as a special case of this paper (cf. also [7]). I do not believe that the methods of [8] can be applied to monotonic mod 1 transformations in general.

§ 1. The Markov diagram. Let T be a monotonic mod 1 transformation on $[0, 1)$, and $J_1, J_2, \ldots, J_N$ the subintervals of $[0, 1)$ with $\bigcup J_i = [0, 1)$, such that $T|J_i$ is monotone. To avoid trivial cases, we assume that $N \geqslant 2$, i.e., $\lim_{t\uparrow 1} f(t) > 1$. Instead of T, in §§ 2, 3 and 4 we investigate a shift space which we now define.

We set $\Sigma_N^+ = \{1, 2, \ldots, N\}^{\mathbf{N}}$. Let σ denote the shift transformation on Σ_N^+ and $\leqslant$ the lexicographic ordering on Σ_N^+. Let $0 = c_0 < c_1 < \ldots < c_N = 1$ be the points where T is discontinuous such that $J_i = [c_{i-1}, c_i)$ for $1 \leqslant i \leqslant N$. We define the *f-expansion* $\varphi\colon [0, 1) \to \Sigma_N^+$ by

$$\varphi(x) = \boldsymbol{x} = x_0 x_1 x_2 \ldots \tag{1.1}$$

where x_i is such that $T^i x \in J_{x_i}$. One easily checks that $\varphi \circ T = \sigma \circ \varphi$ and that $x < y$ implies $\varphi(x) \leqslant \varphi(y)$, i.e., φ is order preserving (cf. Lemma 1 of [2]). Set $\boldsymbol{a} = \varphi(0) = \lim_{t\downarrow 0} \varphi(t)$, $\boldsymbol{b} = \lim_{t\uparrow 1} \varphi(t)$ and

$$\Sigma_T^+ = \{\boldsymbol{x} \in \Sigma_N^+\colon \boldsymbol{a} \leqslant \sigma^k \boldsymbol{x} = x_k x_{k+1} \ldots \leqslant \boldsymbol{b} \quad \text{for} \quad k \geqslant 0\}. \tag{1.2}$$

We introduce in Σ_T^+ the product topology generated by the cylinder sets $[x_0 x_1 \ldots x_{k-1}] = \{\boldsymbol{y} \in \Sigma_T^+\colon y_i = x_i \text{ for } 0 \leqslant i \leqslant k-1\}$. The following lemma is a special case of the results in [2].

LEMMA 1. (i) $\overline{\varphi([0, 1))} = \Sigma_T^+$.

(ii) $\Sigma_T^+ \setminus \varphi([0, 1)) = \{\boldsymbol{x} \in \Sigma_T^+\colon \sigma^k \boldsymbol{x} = \boldsymbol{b} \text{ for some } k \geqslant 0\}$.

The *Markov diagram* of T is defined as an oriented graph in which every arrow has one of the numbers $1, 2, \ldots, N$ and whose vertices are closed subintervals of Σ_T^+ with respect to the lexicographic ordering $\leqslant$. We denote the set of these intervals by $\mathfrak{D}$. If D is a closed subinterval of Σ_T^+, then we call the nonempty sets among $\sigma([i] \cap D)$ for $1 \leqslant i \leqslant N$ the *successors* of D. Remark that the sets $[i] = \overline{\varphi(J_i)}$ are those on which σ is monotone. Hence the successors of D are again closed subintervals of Σ_T^+. We let $\mathfrak{D}$ contain $\sigma([i])$ for $1 \leqslant i \leqslant N$ and if $D \in \mathfrak{D}$, then all successors of D are also in $\mathfrak{D}$. To get the oriented graph, which we call the Markov diagram, we insert an arrow from D to all its successors. Furthermore, the arrow $D \to \sigma([i] \cap D)$ obtains the number i.

In [2] and [6], a slightly different definition is used. The elements of $\mathfrak{D}$ here are the images under σ of the vertices of the Markov diagram in [2] and [6]. This makes no essential difference.

To get a better picture of the Markov diagram, we compute it explicitly. To this end we define integers $r_1, r_2, \ldots$ and $s_1, s_2, \ldots$ with $r_k \geqslant 1$, $s_k \geqslant 1$ in the following way. Choose r_1 such that

$$a_i = b_{i-1} \quad \text{for} \quad 1 \leqslant i \leqslant r_1 - 1, \quad a_{r_1} \neq b_{r_1 - 1}.$$

If $r_1, \ldots, r_k$ are defined, set $R_k = r_1 + \ldots + r_k$ and define r_{k+1} by

$$a_{R_k+i} = b_{i-1} \quad \text{for} \quad 1 \leqslant i \leqslant r_{k+1} - 1, \quad a_{R_k + r_{k+1}} \neq b_{r_{k+1}-1}. \tag{1.3}$$

Similarly, by writing S_j for $s_1 + \ldots + s_j$, we define s_k inductively by

$$b_{S_k+i} = a_{i-1} \quad \text{for} \quad 1 \leqslant i \leqslant s_{k+1} - 1, \quad b_{S_k + s_{k+1}} \neq a_{s_{k+1}-1}. \tag{1.4}$$

By Lemma 1 we have $\boldsymbol{a}, \boldsymbol{b} \in \Sigma_T^+$, hence $\sigma^{R_k+1} \boldsymbol{a} \leqslant \boldsymbol{b}$, which implies, by (1.3), that

$$a_{R_k + r_{k+1}} = a_{R_{k+1}} < b_{r_{k+1}-1}. \tag{1.5}$$

Similarly from (1.4) we get

$$b_{S_k + s_{k+1}} = b_{S_{k+1}} > a_{s_{k+1}-1}. \tag{1.6}$$

The following lemma is proved in [4].

LEMMA 2. *Setting $R_0 = S_0 = 0$ and $R_\infty = S_\infty = \infty$, for every $n \geqslant 1$ there are a $P(n)$ and a $Q(n)$, $0 \leqslant P(n), Q(n) \leqslant \infty$, such that*

$$r_n = 1 + S_{P(n)}, \quad s_n = 1 + R_{Q(n)}.$$

Now we can describe the Markov diagram. For $m \geqslant 1$, we define the following closed subintervals of Σ_T^+:

$$\begin{aligned} A_m &= [\sigma^m \boldsymbol{a}, \sigma^{m-R_k-1} \boldsymbol{b}], \quad && k \text{ such that } R_k < m \leqslant R_{k+1}, \\ B_m &= [\sigma^{m-S_k-1} \boldsymbol{a}, \sigma^m \boldsymbol{b}], \quad && k \text{ such that } S_k < m \leqslant S_{k+1}. \end{aligned} \tag{1.7}$$

Furthermore, let E_m for $2 \leqslant m \leqslant N-1$ be different copies of $\Sigma_T^+ = [\boldsymbol{a}, \boldsymbol{b}]$ and set $\mathfrak{E} = \{E_m\colon 2 \leqslant m \leqslant N-1\}$ for $N \geqslant 3$ and $\mathfrak{E} = \emptyset$ for $N = 2$. Now we can prove

THEOREM 1. $\mathfrak{D} = \mathfrak{E} \cup \{A_m, B_m\colon m \geqslant 1\}$ *and the Markov diagram has the following arrows:*

$$\begin{array}{llllll} A_m \to A_{m+1}, & B_m \to B_{m+1} & & \textit{for} & m \geqslant 1, \\ A_{R_k} \to B_{r_k}, & A_{R_k} \to E_l & (a_{R_k} < l < b_{r_k - 1}) & \textit{for} & k \geqslant 1, \\ B_{S_k} \to A_{s_k}, & B_{S_k} \to E_l & (a_{s_k - 1} < l < b_{S_k}) & \textit{for} & k \geqslant 1, \\ E_k \to E_m & (2 \leqslant m \leqslant N-1), & E_k \to A_1, B_1 & \textit{for} & 2 \leqslant k \leqslant N-1. \end{array}$$

All arrows ending at A_m have the number a_{m-1}, all arrows ending at B_m have the number b_{m-1}, all arrows ending at E_m have the number m.

Proof. We have $\sigma[i]=E_i$ for $2\leqslant i\leqslant N-1$, $\sigma[1]=A_1$ and $\sigma[N]=B_1$. Hence $\mathfrak{D}$ contains A_1, B_1 and the elements of $\mathfrak{E}$. As $E_i=\Sigma_T^+$, the successors of E_i are $E_j=\sigma[j]$ for $2\leqslant j\leqslant N-1$ where the arrow has the number j, $A_1=\sigma[1]$ where the arrow has the number $1=a_0$, and $B_1=\sigma[N]$ where the arrow has the number $N=b_0$.

Now we determine the successors of an A_m. If we have $R_k<m<R_{k+1}$ for some k, it follows from (1.7) that $A_m\subset[a_m]$, because $a_m=b_{m-R_k-1}$ by (1.3). Hence A_m has only the successor $\sigma A_m=A_{m+1}$ and the corresponding arrow has the number a_m. Now suppose $m=R_k$. The initial point of A_m begins with a_{R_k} and the endpoint with b_{r_k-1}. Hence $A_m\cap[i]\neq\emptyset$ for $a_{R_k}\leqslant i\leqslant b_{r_k-1}$. The successors of A_m are $\sigma(A_m\cap[a_{R_k}])=A_{m+1}$ with arrow $a_{R_k}=a_m$, $\sigma(A_m\cap[i])=E_i$ for $a_{R_k}<i<b_{r_k-1}$ with arrow i and $\sigma(A_m\cap[b_{r_k-1}])=[\boldsymbol{a},\sigma^{r_k}\boldsymbol{b}]=B_{r_k}$ by (1.7), because $r_k=1+S_{P(k)}$ by Lemma 2, with arrow b_{r_k-1}.

The proof will be done if one computes also the successors of B_m. We omit this, because it is similar to the computation carried out for A_m.

The importance of the Markov diagram consists in the possibility of representing the elements of Σ_T^+ as one-sided paths. We say that $\boldsymbol{x}=x_0x_1\ldots$ *is represented by the path* $\to D_0\to D_1\to D_2\to\ldots (D_i\in\mathfrak{D})$ which begins at D_0 if the arrow ending at D_0 has the number x_0 and the arrow $D_{i-1}\to D_i$ has the number x_i. The following important property of a $D\in\mathfrak{D}$ is proved in Lemma 3 of [6]:

(1.8) $\quad D=\{\sigma\boldsymbol{x}\colon \boldsymbol{x}$ can be represented as a path in the Markov diagram, which begins at $D\}$.

Furthermore, Lemma 4 of [6] shows that this representation is in some sense unique.

The following result about $P(n)$ and $Q(n)$ defined in Lemma 2, which we shall need later, is a special case of Lemma 2 of [5].

LEMMA 3. *If $m\geqslant 1$ is such that $P(m)\geqslant 1$ and that there are no arrows $A_{R_m}\to E_j$ and $B_{S_{P(m)}}\to E_j$, then $r_{m+1}\geqslant r_{Q(P(m))+1}$. If $m\geqslant 1$ is such that $Q(m)\geqslant 1$ and that there are no arrows $B_{S_m}\to E_j$ and $A_{R_{Q(m)}}\to E_j$, then $s_{m+1}\geqslant s_{P(Q(m))+1}$.*

Remark that $s_m>r_{Q(m)}$ and $r_{Q(m)}>S_{P(Q(m))}$ by Lemma 2 and hence $s_m>S_{P(Q(m))}$. Similarly one gets $r_m>R_{Q(P(m))}$. This implies

(1.9) $$P(Q(m))\leqslant m-1,\qquad Q(P(m))\leqslant m-1.$$

§ **2. Closed subsets of $\mathfrak{D}$.** A subset $\mathfrak{H}$ of $\mathfrak{D}$ is called *closed* if $D\in\mathfrak{H}$ and $D\to C$ imply $C\in\mathfrak{H}$. Closed subsets are important, because they give rise to σ-invariant subsets of Σ_T^+. We define a decreasing sequence of closed subsets in $\mathfrak{D}$.

Suppose $\mathfrak{H}_j=\{A_l, B_m\colon l>R_p,\ m>S_q\}\subset\mathfrak{D}$ is a closed subset of $\mathfrak{D}$. By Theorem 1, we then have $r_t\geqslant S_q+1$ for $t\geqslant p+1$ and $s_t\geqslant R_p+1$ for $t\geqslant q+1$. Otherwise the arrow $A_{R_t}\to B_{r_t}$ or $B_{S_t}\to A_{s_t}$ would imply $\mathfrak{H}_j$ not closed. In order to define a closed set $\mathfrak{H}_{j+1}\subset\mathfrak{H}_j$ we consider the following four cases:

(2.1) $$r_{p+1}=S_q+1,\qquad s_{q+1}=R_p+1,$$

(2.2) $$r_{p+1}=S_q+1,\qquad s_{q+1}>R_p+1,$$

(2.3) $$r_{p+1}>S_q+1,\qquad s_{q+1}=R_p+1,$$

(2.4) $$r_{p+1}>S_q+1,\qquad s_{q+1}>R_p+1.$$

In case (2.1) we have the following arrows in the Markov diagram, which form a closed path (cf. Theorem 1):

(2.5) $$A_{R_p+1}\to\ldots\to A_{R_{p+1}}\to B_{S_q+1}\to\ldots\to B_{S_{q+1}}\to A_{R_p+1}.$$

We say $\mathfrak{C}=\{A_l, B_m\colon R_p<l\leqslant R_{p+1},\ S_q<m\leqslant S_{q+1}\}$ is a *cycle* in the Markov diagram. In case (2.1) we define no $\mathfrak{H}_{j+1}$.

In case (2.4) we get also a cycle $\mathfrak{C}$, as the following lemma shows, and again we define no $\mathfrak{H}_{j+1}$.

LEMMA 4. *In case* (2.4) *we have*:

(i) $r_{p+1}=s_{q+1}=\infty$, *i.e.*, $\sigma^{R_p+1}\boldsymbol{a}=\boldsymbol{b}$, $\sigma^{S_q+1}\boldsymbol{b}=\boldsymbol{a}$.

(ii) $A_t=\{\sigma^t\boldsymbol{a}\}$ *for* $t\geqslant R_p+1$, $B_t=\{\sigma^t\boldsymbol{b}\}$ *for* $t\geqslant S_q+1$.

(iii) $A_{R_p+S_q+2}=B_{S_q+1}$, $B_{R_p+S_q+2}=A_{R_p+1}$ *and*

$$\mathfrak{C}=\{A_l, B_m\colon R_p<l\leqslant R_p+S_q+1,\ S_q<m\leqslant R_p+S_q+1\}$$

is a cycle in the Markov diagram.

Proof. (i): We show $r_{p+1}=\infty$. The proof for s_{q+1} is similar. Suppose $r_{p+1}<\infty$. By Lemma 2, there is a $k=P(p+1)$ with $r_{p+1}=1+S_k$. By (2.4) we have $k>q$. Hence $s_{q+1}<r_{p+1}$. In particular, $s_{q+1}<\infty$. The same argument gives that $s_{q+1}<\infty$ implies $r_{p+1}<s_{q+1}$, a contradiction. Hence $r_{p+1}=\infty$. Now (1.3) implies $\sigma^{R_p+1}\boldsymbol{a}=\boldsymbol{b}$ and (1.4) implies $\sigma^{S_q+1}\boldsymbol{b}=\boldsymbol{a}$.

(ii): This follows immediately from (i) and (1.7).

(iii): This is a consequence of (i) and (ii).

In the remaining cases we set

(2.6) $$\mathfrak{H}_{j+1}=\{A_l, B_m\colon l>R_{p+1},\ m>S_q\}\quad\text{in case (2.2)},$$

(2.7) $$\mathfrak{H}_{j+1}=\{A_l, B_m\colon l>R_p,\ m>S_{q+1}\}\quad\text{in case (2.3)}.$$

LEMMA 5. *$\mathfrak{H}_{j+1}$ is a closed subset of $\mathfrak{D}$.*

Proof. We suppose that $\mathfrak{H}_{j+1}$ is defined by (2.6). Assume that $\mathfrak{H}_{j+1}$ is not closed. As $\mathfrak{H}_j$ is closed, this can happen only if there is an arrow from some B_{s_k} with $k>q$ to $A_{s_k}\in\mathfrak{H}_{j+1}\setminus\mathfrak{H}_j=\{A_{R_p+1},\dots,A_{R_{p+1}}\}$. By Lemma 2, s_k must be R_p+1. By (2.2), $s_{q+1}>R_p+1$, hence $k\geq q+2$. Choose $k\geq q+2$ such that

(2.8) $$s_k=R_p+1,\qquad s_m>R_p+1\qquad\text{for}\qquad q+1\leq m\leq k-1.$$

By Lemma 2, this gives

(2.9) $$Q(k)=p,\qquad Q(m)\geq p+1\qquad\text{for}\qquad q+1\leq m\leq k-1.$$

Because $\mathfrak{H}_j$ is closed, we have $r_t\geq S_q+1$ for $t\geq p+1$, or by Lemma 2

(2.10) $$P(t)\geq q\qquad\text{for}\qquad t\geq p+1.$$

It follows from (2.9) and (2.10) that $P\big(Q(k-1)\big)+1\geq q+1$ and from (1.9) that $P\big(Q(k-1)\big)+1\leq k-1$. Hence it follows from (2.8) that $s_{P(Q(k-1))+1}>R_p+1$. Because $\mathfrak{H}_j$ is closed, the requirements of Lemma 3 for $m=k-1$ are satisfied. Hence $s_k\geq s_{P(Q(k-1))+1}$, which implies $s_k>R_p+1$; a contradiction to (2.8). Hence $\mathfrak{H}_{j+1}$ is closed.

We conclude § 2 with three lemmas we shall need later.

LEMMA 6. *If $\mathfrak{H}_{j+1}$ is defined by (2.6), we have*

$$a_{R_{p+1}}=b_{s_q}-1,\qquad b_{R_{p+1}}=a_{R_p}\qquad\text{and}\qquad a_i=b_i,$$
$$\text{for}\qquad R_{p+1}<i\leq R_{p+1}+r_{p+1}-1=R_{p+1}+S_q.$$

If $\mathfrak{H}_{j+1}$ is defined by (2.7), we have $b_{S_{q+1}}=a_{R_p}+1$, $a_{S_{q+1}}=b_{s_q}$ and $a_i=b_i$ for $S_{q+1}<i\leq S_{q+1}+s_{q+1}-1=S_{q+1}+R_p$.

Proof. We give the proof only if $\mathfrak{H}_{j+1}$ is defined by (2.6). Let X be the block $a_0\dots a_{R_p-1}$ and Y the block $b_0\dots b_{S_q-1}$. By (2.2) and (1.3) we have $a_{R_p+1}\dots a_{R_{p+1}-1}=Y$. Since $s_{q+1}\geq R_{p+1}+1$ (cf. (2.2) and Lemma 2), we get

(2.11) $$b_{S_q+1}\dots b_{S_q+R_{p+1}}=a_0\dots a_{R_{p+1}-1}=Xa_{R_p}Y.$$

In particular, $b_{R_{p+1}}=a_{R_p}$ (remark that $S_q+1=r_{p+1}$ by (2.2)), one of the three required results.

As $\mathfrak{H}_{j+1}$ is closed, we have $r_{p+2}\geq S_q+1$, hence $a_{R_{p+1}+1}\dots a_{R_{p+1}+S_q}=Y$ by (1.3) which implies, together with (2.11), that

$$a_{R_{p+1}+1}\dots a_{R_{p+1}+S_q}=b_{R_{p+1}+1}\dots b_{R_{p+1}+S_q},$$

the third required result.

By Theorem 1, we have the arrows $A_{R_{p+1}}\to E_l$ for $a_{R_{p+1}}<l<b_{r_{p+1}-1}$ in the Markov diagram. Since $\mathfrak{H}_j$ is closed, no such arrow exists; hence $b_{r_{p+1}-1}-a_{R_{p+1}}\leq 1$. By (1.5) this gives $a_{R_{p+1}}=b_{r_{p+1}-1}-1$, which is $b_{S_q}-1$, completing the proof.

LEMMA 7. *Suppose that, continuing the definition of the $\mathfrak{H}_k$'s, we have got an infinite sequence $\mathfrak{H}_j\supset\mathfrak{H}_{j+1}\supset\dots$ of closed sets, such that each $\mathfrak{H}_k\,(k>j)$ is*

defined by (2.6) (by (2.7)), i.e., $r_{p+1}=\infty$ and $s_t=R_p+1$ for $t\geq q+1$ ($s_{q+1}=\infty$ and $r_t=S_q+1$ for $t\geq p+1$). Then

$$B_{S_q+1}=B_{S_{q+1}+1}\qquad(A_{R_p+1}=A_{R_{p+1}+1}).$$

Hence we have a cycle $\mathfrak{C}=\{B_m\colon S_q<m\leq S_{q+1}\}$ ($\mathfrak{C}=\{A_l\colon R_p<l\leq R_{p+1}\}$).

Proof. Because $s_t=R_p+1$ for $t\geq q+1$, it follows from (1.4) that $b_{S_{t-1}+1}\dots b_{S_t-1}=a_0\dots a_{R_p-1}$ for all $t\geq q+1$. As $\mathfrak{H}_j$ is closed, there is no arrow from B_{S_t} to some E_l and hence $b_{S_t}=a_{s_t-1}+1=a_{R_p}+1$ for $t\geq q+1$ by Theorem 1. This gives $\sigma^{S_q+1}b=\sigma^{S_{q+1}+1}b$ and $B_{S_q+1}=B_{S_{q+1}+1}$ follows from (1.7).

LEMMA 8. *If $r_{p+1}=\infty$, then $[a_0\dots a_{R_p}]=\{a\}$. If $s_{q+1}=\infty$, then $[b_0\dots b_{S_q}]=\{b\}$.*

Proof. We show only the first assertion. Suppose $x\in[a_0\dots a_{R_p}]$. By (1.2) we have $x\geq a$ and $\sigma^{R_p+1}x\leq b$ which is $\sigma^{R_p+1}a$ by (1.3), since $r_{p+1}=\infty$. Hence $x\leq a_0\dots a_{R_p}b=a$. This gives $x=a$.

§ 3. Irreducible subsets of $\mathfrak{D}$. We say that there is a *path from C to D* in the Markov diagram if there are $C=C_0, C_1,\dots,C_k=D$, $C_i\in\mathfrak{D}$, with $C_{i-1}\to C_i$ for $1\leq i\leq k$, and denote it by $C\rightsquigarrow D$. For subsets $\mathfrak{J}$, $\mathfrak{C}$ of $\mathfrak{D}$ we write $\mathfrak{J}\rightsquigarrow\mathfrak{C}$ if there are $C\in\mathfrak{J}$ and $D\in\mathfrak{C}$ with $C\rightsquigarrow D$.

A subset $\mathfrak{J}$ of $\mathfrak{D}$ is called *irreducible* if for all $C, D\in\mathfrak{J}$ one has $C\rightsquigarrow D$ and $D\rightsquigarrow C$ and if every subset of $\mathfrak{D}$ which contains $\mathfrak{J}$ strictly does not have this property. We want to find all irreducible subsets of $\mathfrak{D}$. Because of the arrows $E_m\to E_l$ for $2\leq l, m\leq N-1$ (cf. Theorem 1), an irreducible subset $\mathfrak{J}$ satisfies either $\mathfrak{E}\subset\mathfrak{J}$ or $\mathfrak{J}\cap\mathfrak{E}=\emptyset$. Because of the arrows $A_k\to A_{k+1}$ and $B_k\to B_{k+1}$ (cf. Theorem 1), we have $\mathfrak{J}\setminus\mathfrak{E}=\{A_l, B_m\colon T<l\leq V, U<m\leq W\}$ where $0\leq T\leq V\leq\infty$ and $0\leq U\leq W\leq\infty$.

LEMMA 9. *Suppose $\mathfrak{J}\setminus\mathfrak{E}=\{A_l, B_m\colon T<l\leq V, U<m\leq W\}$, $\mathfrak{J}$ is irreducible, and $T<V<\infty$, $U<W<\infty$. Then $V=R_v$ and $W=S_w$ for some v and w.*

Proof. Suppose $R_i<V<R_{i+1}$ for some i. $A_V\in\mathfrak{J}$ and as $\mathfrak{J}$ is irreducible, $A_V\rightsquigarrow A_V$ holds, i.e., there are $C_0=A_V, C_1,\dots,C_k=A_V$ with $C_{i-1}\to C_i$ and all C_i belong to $\mathfrak{J}$ by our definition of irreducibility. By Theorem 1, the only arrow which begins at A_V, ends at A_{V+1}, hence $C_1=A_{V+1}$. Therefore $A_{V+1}\in\mathfrak{J}$, contradicting the definition of V. This shows $V=R_v$ for some v.

Set $R_0=S_0=0$ and $R_\infty=S_\infty=\infty$. Set $V_0=\max\{m\colon A_m\rightsquigarrow\mathfrak{E}\}$ ($=0$ if this set is empty) and $W_0=\max\{m\colon B_m\rightsquigarrow\mathfrak{E}\}$. Then $\mathfrak{D}_0=\mathfrak{E}\cup\{A_l, B_m\colon l\leq V_0 \text{ and } m\leq W_0\}$ is irreducible and by Lemma 9, $V_0=R_{v_0}$ and $W_0=S_{w_0}$ for some v_0, w_0. In case $N=2$ we have $\mathfrak{E}=\emptyset$, hence $V_0=W_0=0$ and $\mathfrak{D}_0=\emptyset$.

Because $\mathfrak{D}_0$ is irreducible and because of the arrows $A_k\to A_{k+1}$ and $B_k\to B_{k+1}$ in the Markov diagram, the set $\mathfrak{D}\setminus\mathfrak{D}_0=:\tilde{\mathfrak{D}}_0$ is closed.

Suppose we have already found an irreducible subset $\mathfrak{D}_i$ of $\mathfrak{D}$ with $R_{v_i} = \max\{m\colon A_m \in \mathfrak{D}_i\}$ and $S_{w_i} = \max\{m\colon B_m \in \mathfrak{D}_i\}$. As above for $\mathfrak{D}_0$, the set $\tilde{\mathfrak{D}}_i = \{A_l, B_m\colon l > R_{v_i}, m > S_{w_i}\}$ is closed. If $R_{v_i} < \infty$ and $S_{w_i} < \infty$, we use the results of § 2 to define a decreasing sequence of closed sets. Set $\mathfrak{H}_0 = \tilde{\mathfrak{D}}_i$. If (2.2) or (2.3) occurs for $\mathfrak{H}_0$, we define an $\mathfrak{H}_1$ by (2.6) or (2.7), respectively. If then for $\mathfrak{H}_1$ again (2.2) or (2.3) occurs, we define an $\mathfrak{H}_2$ by (2.6) or (2.7). We continue this procedure. Either we get an infinite sequence $\mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \ldots$ of closed sets, or we reach an $\mathfrak{H}_j = \{A_l, B_m\colon l > R_{t_{i+1}}, m > S_{u_{i+1}}\}$ $(j \geqslant 0)$ where (2.1) or (2.4) occurs. In these cases

$$\mathfrak{C}_{i+1} = \{A_l, B_m\colon R_{t_{i+1}} < l \leqslant R_{t_{i+1}} + S_{u_{i+1}} + 1,\ S_{u_{i+1}} < m \leqslant S_{u_{i+1}} + R_{t_{i+1}} + 1\}$$

is a cycle (cf. (2.5) and Lemma 4) which must be contained in an irreducible subset of $\mathfrak{D}$. Set

$$V_{i+1} = \max\{m\colon A_m \rightsquigarrow \mathfrak{C}_{i+1}\} \quad \text{and} \quad W_{i+1} = \max\{m\colon B_m \rightsquigarrow \mathfrak{C}_{i+1}\}.$$

Then

$$\mathfrak{D}_{i+1} = \{A_l, B_m\colon R_{t_{i+1}} < l \leqslant V_{i+1},\ S_{u_{i+1}} < m \leqslant W_{i+1}\}$$

is an irreducible subset of $\mathfrak{D}$. By Lemma 9, we get $V_{i+1} = R_{v_{i+1}}$ and $W_{i+1} = S_{w_{i+1}}$. The set $\tilde{\mathfrak{D}}_{i+1} = \{A_l, B_m\colon l > R_{v_{i+1}}, m > S_{w_{i+1}}\}$ is closed and we can start the same procedure as above if $R_{v_{i+1}} < \infty$ and $S_{w_{i+1}} < \infty$. In case (2.4) we have $\mathfrak{D}_{i+1} = \mathfrak{C}_{i+1}$ and $\tilde{\mathfrak{D}}_{i+1} = \emptyset$ by Lemma 4.

We consider six different cases. Either there are infinitely many $\mathfrak{D}_i$ (cf. (f) below) or there is an irreducible subset, which we denote $\mathfrak{D}_{n-1}$, after which the last sequence $\mathfrak{H}_0 = \tilde{\mathfrak{D}}_{n-1} \supset \mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \ldots$ of closed sets occurring in the above procedure begins. The behaviour of this sequence gives the other five cases. In cases (a) and (b) it ends, because (2.1) occurs for some $\mathfrak{H}_j$, which gives rise to a $\mathfrak{D}_n$, after which no sequence of closed sets is defined.

(a) The irreducible subsets are $\mathfrak{D}_0, \ldots, \mathfrak{D}_n$ and $v_n = w_n = \infty$.

(b) The irreducible subsets are $\mathfrak{D}_0, \ldots, \mathfrak{D}_n$ and either $v_n = \infty$, $w_n < \infty$ or $v_n < \infty$, $w_n = \infty$. Then $s_{w_n+1} = \infty$ or $r_{v_n+1} = \infty$, respectively, by the definition of v_n and w_n.

(c) The sequence $\tilde{\mathfrak{D}}_{n-1} = \mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \ldots$ reaches an $\mathfrak{H}_j$ for which (2.4) occurs. $\mathfrak{D}_n$ is then the cycle given by Lemma 4.

(d) The sequence $\tilde{\mathfrak{D}}_{n-1} = \mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \ldots$ is infinite and for some $\mathfrak{H}_j$ the situation of Lemma 7 occurs. $\mathfrak{D}_n$ is defined as the cycle given by that lemma.

(e) The sequence $\tilde{\mathfrak{D}}_{n-1} = \mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \ldots$ is infinite and $r_m < \infty$, $s_m < \infty$ hold for all m.

(f) There are infinitely many $\mathfrak{D}_i$. We set $n = \infty$.

In cases (b) and (d) we have two subcases. We shall consider only the case where

$$r_{p+1} = \infty \quad \text{and} \quad s_t < \infty \quad \text{for all } t. \tag{3.1}$$

In case (d) p is given by Lemma 7 and in case (b) $p = v_n$. The proofs in the other case where

$$s_{q+1} = \infty \quad \text{and} \quad r_t < \infty \quad \text{for all } t \tag{3.2}$$

are similar and omitted ($q = w_n$ in case (b)).

The following lemma is needed for Proposition 2.

LEMMA 10. (i) *For* $0 \leqslant i < n-1$, *and in cases* (a) *and* (b) *also for* $i = n-1$, *we have* $\sigma^k \boldsymbol{a} < \sigma^k \boldsymbol{b}$ *for* $\min\{R_{v_i+1}, S_{w_i+1}\} < k \leqslant R_{t_{i+1}} + S_{u_{i+1}} + 1$.

(ii) *In cases* (d) *and* (e) *we have* $\sigma^k \boldsymbol{a} = \sigma^k \boldsymbol{b}$ *for* $k > \min\{R_{v_{n-1}+1}, S_{w_{n-1}+1}\}$.

(iii) *In case* (c) *for* $\min\{R_{v_{n-1}+1}, S_{w_{n-1}+1}\} < k \leqslant 1 + \max\{R_{t_n}, S_{u_n}\}$ *we have* $\sigma^k \boldsymbol{a} > \sigma^k \boldsymbol{b}$, *but there is no* $\boldsymbol{x} \in \Sigma_T^+$ *with* $\sigma^k \boldsymbol{a} > \boldsymbol{x} > \sigma^k \boldsymbol{b}$.

Proof. For some fixed $i < n$ we consider the sequence $\mathfrak{H}_0 = \tilde{\mathfrak{D}}_i \supset \mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \ldots$ as defined above. We apply Lemma 6 for $j = 0$, $p = v_i$ and $q = w_i$. If $\mathfrak{H}_1$ is defined by (2.6), we get $a_j = b_j$ for $R_{v_i+1} < j \leqslant R_{v_i+1} + S_{w_i}$ and $R_{v_i+1} = \min\{R_{v_i+1}, S_{w_i+1}\}$ by (2.2). If $\mathfrak{H}_1$ is defined by (2.7), we get $a_j = b_j$ for $S_{w_i+1} < j \leqslant S_{w_i+1} + R_{v_i}$ and $S_{w_i+1} = \min\{R_{v_i+1}, S_{w_i+1}\}$ by (2.3). Furthermore, if $\mathfrak{H}_1$ is defined by (2.6) one has

$$a_{R_{p+1}} = b_{S_q} - 1 \quad \text{where} \quad p = v_i \quad \text{and} \quad q = w_i. \tag{3.3}$$

Now we apply Lemma 6 for $j = 1$. If $\mathfrak{H}_1$ is defined by (2.6), then $p = v_i + 1$, $q = w_i$. If $\mathfrak{H}_1$ is defined by (2.7), then $p = v_i$ and $q = w_i + 1$. We suppose that $\mathfrak{H}_1$ is defined by (2.6) and omit the proof in the other case. We already know that

$$a_j = b_j \quad \text{for} \quad \min\{R_{v_i+1}, S_{w_i+1}\} < j \leqslant R_{v_i+1} + S_{w_i}. \tag{3.4}$$

If now $\mathfrak{H}_2$ is defined by (2.6), we have $R_{v_i+1} + S_{w_i} = R_{v_i+2} - 1$ by (2.2) and $a_{R_{p+1}} = b_{S_q} - 1$, $b_{R_{p+1}} = a_{R_p}$ by Lemma 6 ($p = v_i + 1$, $q = w_i$), which implies by (3.3) that

$$a_j = b_j \quad \text{for} \quad j = R_{v_i+2}. \tag{3.5}$$

Again by Lemma 6, we get

$$a_j = b_j \quad \text{for} \quad R_{v_i+2} < j \leqslant R_{v_i+2} + S_{w_i}. \tag{3.6}$$

One gets from (3.4), (3.5) and (3.6) that

$$a_j = b_j \quad \text{for} \quad \min\{R_{v_i+1}, S_{w_i+1}\} < j \leqslant R_{v_i+2} + S_{w_i}. \tag{3.7}$$

If $\mathfrak{H}_2$ is defined by (2.7), one gets by a similar proof

$$a_j = b_j \quad \text{for} \quad \min\{R_{v_i+1}, S_{w_i+1}\} < j \leqslant R_{v_i+1} + S_{w_i+1}. \tag{3.8}$$

We can continue in this way. (3.5) and (3.6) together can be considered as the induction step. For $i = n-1$ in cases (d) and (e) the sequence $\mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \ldots$ is infinite. Hence we get $a_j = b_j$ for all $j > \min\{R_{v_i+1}, S_{w_i+1}\}$ proving (ii).

In all cases considered in (i), the sequence $\mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \ldots$ ends with an $\mathfrak{H}_k$ for which (2.1) holds, i.e., $\mathfrak{H}_k = \{A_l, B_m\colon l > R_{t_{i+1}}, m > S_{u_{i+1}}\}$ and (3.7) or (3.8) extends to

(3.9) $$a_j = b_j \quad \text{for} \quad \min\{R_{v_i+1}, S_{w_i+1}\} < j \leqslant R_{t_{i+1}} + S_{u_{i+1}}.$$

By (2.1) we have $r_{t_{i+1}+1} = 1 + S_{u_{i+1}}$ and $s_{u_{i+1}+1} = 1 + R_{t_{i+1}}$. Hence

$$R_{t_{i+1}} + S_{u_{i+1}} = R_{t_{i+1}+1} - 1 = S_{u_{i+1}+1} - 1.$$

We set $W = R_{t_{i+1}+1} = S_{u_{i+1}+1}$. As A_W and B_W are in $\mathfrak{D}_{i+1} \subset \tilde{\mathfrak{D}}_0$, which is closed and disjoint from $\mathfrak{E}$, we have $A_W \not\leftrightarrow \mathfrak{E}$ and $B_W \not\leftrightarrow \mathfrak{E}$. As in the proof of Lemma 6, it follows from Theorem 1, (1.5) and (1.6) that $a_W = b_m - 1$ where $m = r_{t_{i+1}+1} - 1$ and $b_W = a_l + 1$ where $l = s_{u_{i+1}+1} - 1$. Now we apply Lemma 6 for $j+1 = k$. No matter whether $\mathfrak{H}_k$ is defined by (2.6) or by (2.7), we always get $a_l = b_m - 1$, as $l = R_{t_{i+1}}$ and $m = S_{u_{i+1}}$ by (2.1). Hence we have

(3.10) $$a_W < b_W, \quad W = R_{t_{i+1}} + S_{u_{i+1}} + 1.$$

Now (i) follows from (3.9) and (3.10). If $k = 0$, then

$$R_{v_i+1} = S_{w_i+1} = R_{t_{i+1}} + S_{u_{i+1}} + 1$$

and there is nothing to show.

For $i = n-1$ in case (c) the sequence $\mathfrak{H}_0 \supset \mathfrak{H}_1 \supset \ldots$ ends with an

$$\mathfrak{H}_k = \{A_l, B_m\colon l > R_p, m > S_q\}$$

for which (2.4) occurs. Here we have written p for t_n and q for u_n. As above (3.7) or (3.8) extends to

(3.11) $$a_j = b_j \quad \text{for} \quad \min\{R_{v_{n-1}+1}, S_{w_{n-1}+1}\} < j \leqslant R_p + S_q.$$

By Lemma 4 we have $\sigma^{R_p+1} \boldsymbol{a} = \boldsymbol{b}$ and $\sigma^{S_q+1} \boldsymbol{b} = \boldsymbol{a}$. In particular, $a_{R_p+S_q+1} = b_{S_q}$ and $b_{R_p+S_q+1} = a_{R_p}$. We apply Lemma 6 for $j+1 = k$ and get in both cases, (2.6) and (2.7), that $a_{R_p} = b_{S_q} - 1$. Hence

(3.12) $$a_{R_p+S_q+1} = b_{R_p+S_q+1} + 1.$$

Together with (3.11) this gives the first assertion of (iii).

Now suppose $\sigma^k \boldsymbol{a} > \boldsymbol{x} > \sigma^k \boldsymbol{b}$ for $\min\{R_{v_{n-1}+1}, S_{w_{n-1}+1}\} < k \leqslant K := \max\{R_p, R_q\} + 1$. By (3.11), $\boldsymbol{y} = \sigma^{K-k} \boldsymbol{x} \in \Sigma_T^+$ satisfies

(3.13) $$\sigma^K \boldsymbol{a} > \boldsymbol{y} > \sigma^K \boldsymbol{b}.$$

It follows from (3.11) and (3.12) that

(3.14) $$\boldsymbol{y} \in [a_K \ldots a_{R_p+S_q+1}] \quad \text{or} \quad \boldsymbol{y} \in [b_K \ldots b_{R_p+S_q+1}].$$

Suppose $K = R_p + 1$. We omit the proof for $K = R_q + 1$. It follows from Lemma 4 that $\sigma^K \boldsymbol{a} = \boldsymbol{b}$. If the first statement of (3.14) is true, then $\boldsymbol{y} \in [b_0 \ldots b_{S_q}]$ (cf. (1.3)), which is $\{\boldsymbol{b}\}$ by Lemma 8. Hence $\boldsymbol{y} = \boldsymbol{b} = \sigma^K \boldsymbol{a}$, a contradiction to (3.13). If the second statement of (3.14) is true, then $\boldsymbol{y} \leqslant b_K \ldots b_{K+S_q} \boldsymbol{b}$, because $\boldsymbol{y} \in \Sigma_T^+$ implies $\sigma^{S_q+1} \boldsymbol{y} \leqslant \boldsymbol{b}$ by (1.2). By Lemma 4, $\sigma^{K+S_q+1} \boldsymbol{b} = \boldsymbol{b}$, hence $b_K \ldots b_{K+S_q} \boldsymbol{b} = \sigma^K \boldsymbol{b}$ and we get $\boldsymbol{y} \leqslant \sigma^K \boldsymbol{b}$, again a contradiction to (3.13). Hence (3.13) cannot hold, proving the second assertion of (iii).

Next we define subsets of Σ_T^+. Set

$$\tilde{\mathfrak{D}}_i = \mathfrak{D}_i \cup \tilde{\mathfrak{D}}_i = \{A_l, B_m\colon l > R_{t_i}, m > S_{u_i}\}$$

$$\text{and} \quad F_i = \bigcup\{D\colon D \in \tilde{\mathfrak{D}}_i\}, \quad G_i = \bigcup\{D\colon D \in \tilde{\mathfrak{D}}_i\}.$$

Proposition 1. (i) $F_i \supset G_i \supset F_{i+1}$.

(ii) $\sigma(F_i) \subset F_i$.

Proof. (i) follows, because $\tilde{\mathfrak{D}}_i \supset \tilde{\mathfrak{D}}_i \supset \tilde{\mathfrak{D}}_{i+1}$. (ii) follows, because $\tilde{\mathfrak{D}}_i$ is a closed set and, if $D \in \mathfrak{D}$, then $\sigma(D) = \bigcup\{C \in \mathfrak{D}\colon C \to D\}$ by the definition of a successor.

Proposition 2. (i) *For $0 \leqslant i < n$, and in cases* (d), (e) *for $i < n-1$, we have $G_i = F_{i+1}$. If $N = 2$, then $\tilde{\mathfrak{D}}_0 = \mathfrak{D}$ and $F_1 = G_0 = \Sigma_T^+$. If $N \geqslant 3$, then $\mathfrak{E} \subset \mathfrak{D}_0$, which gives $F_0 = \Sigma_T^+$.*

(ii) *In cases* (d), (e) *we consider the infinite sequence $\mathfrak{H}_0 = \tilde{\mathfrak{D}}_{n-1} \supset \mathfrak{H}_1 \supset \ldots$ of closed sets and set $H_k = \bigcup\{D\colon D \in \mathfrak{H}_k\}$. Then $H_k = H_0$ for $k \geqslant 0$.*

Proof. We consider the sequence $\mathfrak{H}_0 = \tilde{\mathfrak{D}}_i \supset \mathfrak{H}_1 \supset \ldots$ which ends with $\mathfrak{H}_k = \tilde{\mathfrak{D}}_{i+1}$ in the cases considered in (i) and is infinite in (ii). Both (i) and (ii) will be proved if we show $H_j = H_{j+1}$ where $H_j = \bigcup\{D\colon D \in \mathfrak{H}_j\}$. Because $\mathfrak{H}_j \supset \mathfrak{H}_{j+1}$, we have $H_j \supset H_{j+1}$. We shall show $H_j \subset H_{j+1}$. If $k = 0$, we have $\tilde{\mathfrak{D}}_i = \tilde{\mathfrak{D}}_{i+1}$ and there is nothing to show.

Suppose $\mathfrak{H}_j = \{A_l, B_m\colon l > R_p, m > S_q\}$ and $\mathfrak{H}_{j+1}$ is defined by (2.6). We omit the proof for (2.7). We have $\mathfrak{H}_j \backslash \mathfrak{H}_{j+1} = \{A_l\colon R_p < l \leqslant R_{p+1}\}$ and

$$A_l = \sigma^{l-R_p-1} A_{R_p+1} \quad \text{for} \quad R_p < l \leqslant R_{p+1}.$$

As $\mathfrak{H}_{j+1}$ is a closed subset of $\mathfrak{D}$, we have $\sigma H_{j+1} \subset H_{j+1}$ (cf. the proof of (ii) of Proposition 1) and it suffices to show $A_{R_p+1} \subset H_{j+1}$.

We have $A_{R_p+1} = [\sigma^{R_p+1} \boldsymbol{a}, \boldsymbol{b}]$ by (1.7). Furthermore,

$$A_{R_{p+1}+1} = [\sigma^{R_{p+1}+1} \boldsymbol{a}, \boldsymbol{b}] \in \mathfrak{H}_{j+1} \quad \text{and} \quad B_{R_{p+1}+1} \in \mathfrak{H}_{j+1}$$

since $R_{p+1} > S_q$ by (2.2). By (1.7), $B_{R_{p+1}+1} = [\sigma^{R_p+1} \boldsymbol{a}, \sigma^{R_{p+1}+1} \boldsymbol{b}]$ since

$S_q+1 < R_{p+1}+1 < S_{q+1}+1$ by (2.2) and $r_{p+1} = S_q+1$. In any case it follows from Lemma 10 that there is no $x \in \Sigma_T^+$ with

$$\sigma^{R_{p+1}+1} b < x < \sigma^{R_{p+1}+1} a$$

since $p \geqslant v_i$ and $p+1 \leqslant t_{i+1}$. This implies $A_{R_p+1} \subset B_{R_{p+1}+1} \cup A_{R_{p+1}+1} \subset H_{j+1}$ and the lemma is proved.

Remark. If $T(x) = f(x)$ (mod 1) satisfies $f(\frac{1}{2}-x)+f(\frac{1}{2}+x) = 2$ (cf. § 0) the above results are much easier because one can show that $\tilde{\mathfrak{D}}_{i+1} = \tilde{\mathfrak{D}}_i$ for all i. Also the cases (b), (d) and (e), which are the difficult ones, cannot occur.

We now show that the sets F_i are finite unions of intervals. Recall the cycle $\mathfrak{C}_i = \{A_l, B_m\colon R_{t_i} < l \leqslant R_{t_i}+S_{u_i}+1,\ S_{u_i} < m \leqslant R_{t_i}+S_{u_i}+1\} \subset \mathfrak{D}_i$ defined above. If $i < n$ or if $i = n$ in case (a) or (b), then $\mathfrak{C}_i$ is also $\{A_l, B_m\colon R_{t_i} < l \leqslant R_{t_i+1}, S_{u_i} < m \leqslant S_{u_i+1}\}$ by (2.1).

Proposition 3. (i) *For $i > 0$ we have $F_i = \bigcup \{D\colon D \in \mathfrak{C}_i\}$.*

(ii) *In case* (e) *we set $F_n = G_{n-1} = \bigcup \{A_l, B_m\colon l > R_p, m > S_q\}$ where $p = v_{n-1}$ and $q = w_{n-1}$. If $r_{p+1} = 1+S_q$ and $s_{q+1} > 1+R_p$* (cf. (2.2)), *then set $p' = Q(q+1)$, i.e., $R_{p'}+1 = s_{q+1}$, and $q' = q+1$. If $r_{p+1} > 1+S_q$ and $s_{q+1} = 1+R_p$* (cf. (2.3)), *then set $p' = p+1$ and $q' = P(p+1)$, i.e., $S_{q'}+1 = r_{p+1}$. Then we have $F_n = \bigcup \{A_l, B_m\colon R_p < l \leqslant R_{p'}, S_q < m \leqslant S_{q'}\}$.*

(iii) *In case* (d) *the set $\tilde{\mathfrak{D}}_{n-1}$ is finite and hence $F_n = G_{n-1} = \bigcup \{D\colon D \in \tilde{\mathfrak{D}}_{n-1}\}$ is trivially a finite union of intervals.*

Proof. In case (d), $\tilde{\mathfrak{D}}_{n-1}$ is finite, because $r_{p+1} = \infty$, i.e., $A_{R_p+1} = \{b\}$, $\sigma^{S_q+1} b = \sigma^{S_{q+1}+1} b$, and $B_{S_q+1} = B_{S_{q+1}+1}$ (cf. Lemma 7) proving (iii). If $i = n$ in case (c) then $\mathfrak{D}_i = \tilde{\mathfrak{D}}_i = \mathfrak{C}_i$ and there is nothing to prove. In all other cases of (i) we set $p = t_i$, $q = u_i$, $p' = t_i+1$ and $q' = u_i+1$. For $i = n$ in case (e), p, q, p' and q' are defined in (ii). Now we can show

$$\sigma^{R_{p'}+1} a \geqslant \sigma^{R_p+1} a, \qquad \sigma^{S_{q'}+1} b \leqslant \sigma^{S_q+1} b. \tag{3.15}$$

We show only the first inequality. Since the set $\{A_l, B_m\colon l > R_p, m > S_q\}$ is a closed subset of $\mathfrak{D}$, we get $r_j \geqslant r_{p+1}$ for $j \geqslant p+1$. Choose $k \leqslant \infty$ such that $r_j = r_{p+1}$ for $p+1 \leqslant j < k$ and $r_k > r_{p+1}$. By the definition of p', we have $k \geqslant p'+1$. It follows from (1.3) that

$$a_{R_{j-1}+1} \ldots a_{R_j-1} = b_0 \ldots b_{r_{p+1}-2},$$

and since $\{A_l, B_m\colon l > R_p, m > S_q\}$ is closed, that $a_{R_j} = b_{r_{p+1}-1}-1$ (cf. the proof of Lemma 6) for $p' \leqslant j < k$. If $k < \infty$, then $a_{R_{k-1}+1} \ldots a_{R_{k-1}+r_{p+1}} = b_0 \ldots b_{r_{p+1}-1}$. This implies

$$\sigma^{R_{p'}+1} a = \sigma^{R_p+1} a \text{ if } k = \infty \quad \text{and} \quad \sigma^{R_{p'}+1} a > \sigma^{R_p+1} a \text{ if } k < \infty,$$

proving (3.15).

From (1.7) and (3.15) one gets

$$A_{R_{p'}+1} \subset A_{R_p+1}, \qquad B_{S_{q'}+1} \subset B_{S_q+1}. \tag{3.16}$$

Set $K_j = \bigcup \{A_l, B_m\colon R_p < l \leqslant R_{p'}+j, S_q < m \leqslant S_{q'}+j\}$. We have to show that $F_i = K_0$. Since $\bigcup_{j=0}^{\infty} K_j = F_i$, it suffices to show $K_j = K_{j+1}$. For $j = 0$, this follows from (3.16). We proceed by induction. For $D \in \mathfrak{D}$, $\sigma D = \bigcup \{C\colon D \to C\}$, hence $\sigma K_j = K_{j+1}$ (cf. Theorem 1). The induction step is as follows: $K_{j+1} = \sigma K_j = \sigma K_{j+1} = K_{j+2}$. This completes the proof.

§ 4. The nonwandering set Ω of Σ_T^+.

The following subsets Ω_i of Σ_T^+ are proved in Theorem 2 to be topologically transitive. For $i < n$, and in cases (a), (b) and (c) also for $i = n$, we set

$$\Omega_i = \bigcap_{k=0}^{\infty} \overline{\sigma^{-k}(F_i \setminus G_i)}, \qquad W_i = (F_i \setminus G_i) \setminus \Omega_i. \tag{4.1}$$

If $\tilde{\mathfrak{D}}_i = \emptyset$ and hence $G_i = \emptyset$, then $\Omega_i = F_i$, because $\sigma F_i \subset F_i$ by Proposition 1. In case (d) the situation of Lemma 7 occurs, which says that we have a cycle $\mathfrak{C} = \{B_m\colon S_q < m \leqslant S_{q+1}\} \subset \tilde{\mathfrak{D}}_{n-1}$ and $r_{p+1} = \infty$ if (3.1) occurs. We set

$$\Omega_n = \{\sigma^i b\colon S_q \leqslant i < S_{q+1}\} \tag{4.2}$$

which is a periodic orbit. If (3.2) occurs, we set

$$\Omega_n = \{\sigma^i a\colon R_p \leqslant i < R_{p+1}\}$$

where p is as in Lemma 7 (statement in brackets). Furthermore,

$$\Omega_n = F_n = G_{n-1} \quad \text{in case (e)}, \tag{4.3}$$

$$\Omega_\infty = \bigcap_{k=1}^{\infty} F_k \quad \text{in case (f)}. \tag{4.4}$$

We need the following lemma for Theorem 2.

Lemma 11. *Suppose $r_i < \infty$, $s_i < \infty$ for all i and that*

$$\mathfrak{F}_j = \{A_l, B_m\colon l > R_{p_j}, m > S_{q_j}\}$$

is a closed subset of $\mathfrak{D}$. Set $X_j = \bigcup \{D\colon D \in \mathfrak{F}_j\}$. Then $\{\sigma^i a, \sigma^i b\colon i \geqslant 0\} \subset \bigcap_{j=1}^{\infty} X_j$. If $p_j \to \infty$, $q_j \to \infty$ for $j \to \infty$, then $\bigcap_{j=1}^{\infty} X_j$ is the set of limit points of $\{\sigma^i a\colon i \geqslant 0\}$ and also of $\{\sigma^i b\colon i \geqslant 0\}$.

Proof. That $\sigma^i a \in \bigcap X_j$ follows, because $a \in B_m \subset X_j$ where $m = S_{q_j}+1$, and $\sigma X_j \subset X_j$ since $\mathfrak{F}_j$ is closed (cf. the proof of Proposition 1).

Now let $x = x_0 x_1 \ldots \in \bigcap X_j$. For every k we shall find an i with $\sigma^i a \in [x_0 \ldots x_{k-1}]$. To this end choose j so large that $R_{p_j} > k$ and $S_{q_j} > k$. As $\mathfrak{F}_j$ is closed, it follows that

(4.5) $\quad r_t \geqslant S_{q_j}+1 \geqslant k \quad$ for $\quad t > p_j, \quad s_t \geqslant R_{p_j}+1 \geqslant k \quad$ for $\quad t > q_j$.

As $[x_0 \dots x_{k-1}] \cap X_j \neq \emptyset$, we find a $D \in \mathfrak{F}_j$ with $[x_0 \dots x_{k-1}] \cap D \neq \emptyset$. Suppose $D = A_l$ where $R_{i-1} < l \leqslant R_i$ and $i \geqslant p_j+1$. We have the following paths in the Markov diagram:

$$A_l \to \dots \to A_{R_i} \begin{array}{l} \nearrow \\ \searrow \end{array} \begin{array}{l} A_{R_i+1} \to \dots \to A_{R_{i+1}} \begin{array}{l} \nearrow \\ \searrow \end{array} \\ B_{r_i} \to \dots \to B_{S_{Q(i)+1}} \begin{array}{l} \nearrow \\ \searrow \end{array} \end{array}$$

Since $\mathfrak{F}_j$ is closed, we have $A_{R_i} \not\rightsquigarrow \mathfrak{E}$. By (1.8), $x_0 \dots x_{k-1}$ can be represented as a path of length k in the Markov diagram which begins at A_{l+1}. As $\mathfrak{F}_j$ is closed, $B_{r_i} = B_{S_{Q(i)}+1} \in \mathfrak{F}_j$ and $r_{i+1} \geqslant k$, $s_{Q(i)+1} \geqslant k$ by (4.5). This gives that $x_0 \dots x_{k-1}$ is either

$$a_l \dots a_{R_i-1} a_{R_i} \dots a_{k+l-1} \quad \text{or} \quad a_l \dots a_{R_i-1} b_{r_i-1} \dots b_{k-R_{i-1}+l-2}$$

(Theorem 1 states what numbers the arrows have). In the first case, we have $\sigma^l \boldsymbol{a} \in [x_0 \dots x_{k-1}]$. In the second case, we get by $a_{R_{i-1}+1} \dots a_{R_i-1} = b_0 \dots b_{r_i-2}$ (cf. (1.3)) that

$$x_0 \dots x_{k-1} = b_{l-R_{i-1}-1} \dots b_{k-R_{i-1}+l-2}.$$

It follows from (4.5) and $q_j \to \infty$ that there is an m with $r_m > k-R_{i-1}+l-2$. By (1.3), this implies that

$$b_{l-R_{i-1}-1} \dots b_{k-R_{i-1}+l-2} = a_{R_{m-1}+l-R_{i-1}} \dots a_{R_{m-1}+k+l-R_{i-1}-1}$$

and hence $\sigma^p \boldsymbol{a} \in [x_0 \dots x_{k-1}]$ for $p = R_{m-1}+l-R_{i-1}$. This proves the lemma.

THEOREM 2. *For $i \leqslant n$, $\sigma|\Omega_i$ is topologically transitive. Ω_i is the set of limit points of $\{\sigma^k y\colon k \geqslant 0\}$ for some $y \in \Omega_i$.*

Proof. If Ω_i is defined by (4.1), this is shown in Lemma 7 of [6]. If Ω_i is only a periodic orbit, the result is trivial. If Ω_i is defined by (4.3), then it follows from (ii) of Proposition 2 that $\Omega_n = H_j$ for $j \geqslant 0$, i.e., $\Omega_n = \bigcap H_j$ (for the definition of H_j see Proposition 2). As $X_j = H_j$ satisfies the requirements of Lemma 11, the assertion of Theorem 2 holds for $\boldsymbol{y} = \boldsymbol{a}$ or $\boldsymbol{b}$. If Ω_i is defined by (4.4), i.e., $\Omega_\infty = \bigcap F_k$, we can also apply Lemma 11, because $X_k = F_k$ satisfies the requirements of that lemma.

In order to show that W_i is wandering, we need

LEMMA 12. (i) $\operatorname{bd} F_i \subset \{\sigma^l \boldsymbol{a}, \sigma^l \boldsymbol{b}\colon l \geqslant 0\}$.

(ii) *For $i < n$,* $\operatorname{bd} F_i \subset \operatorname{bd} F_{i+1}$.

Proof. (i): By definition, we have $F_i = \bigcup \{A_l, B_m\colon l > R_{t_i}, m > S_{u_i}\}$. Hence the result follows from (1.7).

(ii): Let $x \in \operatorname{bd} F_i$. By (i), $\boldsymbol{x} = \sigma^l \boldsymbol{a}$ for some $l \geqslant 0$ or $\boldsymbol{x} = \sigma^m \boldsymbol{b}$ for some $m \geqslant 0$. By definition, $F_{i+1} = \bigcup \{A_l, B_m\colon l > R_{t_{i+1}}, m > S_{u_{i+1}}\}$. Hence it follows from (1.7) that $x \in F_{i+1}$. If $x \in \operatorname{int} F_{i+1}$, then $x \in \operatorname{int} F_i$, because $F_i \supset F_{i+1}$ by Proposition 1. Hence $x \in \operatorname{bd} F_{i+1}$.

THEOREM 3. *For $i < n$, W_i is a wandering set.*

Proof. Let $x \in W_i$, i.e., $x \in F_i \setminus G_i$ and $x \notin \Omega_i$. Hence there is a k with $x \notin \overline{\sigma^{-k}(F_i \setminus G_i)}$. By Proposition 2, we have $G_i = F_{i+1}$ and hence it follows from Lemma 12 that $\operatorname{bd}(F_i \setminus G_i) \subset \operatorname{bd} G_i$, which is in G_i, because $G_i = F_{i+1}$ is closed by Proposition 3. Hence $F_i \setminus G_i$ is open. As $x \in F_i \setminus G_i$ and $x \notin \overline{\sigma^{-k}(F_i \setminus G_i)}$, x has a neighbourhood $V \subset F_i \setminus G_i$ with $V \cap \sigma^{-k}(F_i \setminus G_i) = \emptyset$. As $\sigma(F_i) \subset F_i$ by Proposition 1, this gives $\sigma^k V \subset G_i$. Since $\sigma(G_i) \subset G_i$ ($G_i = F_{i+1}$), we have $\sigma^j V \cap V = \emptyset$ for $j \geqslant k$. By making V smaller, if necessary, we get also $\sigma^j V \cap V = \emptyset$ for $1 \leqslant j < k$. Hence x is wandering.

The next result deals with the disjointness of the Ω_i.

PROPOSITION 4. *For $i < j$, $\Omega_i \cap F_j \neq \emptyset$ implies that we have case* (a), (b), (c) *or* (d) *and that $j = i+1 = n$. Then $\Omega_i \cap F_j$ is finite and is either $\{\sigma^l \boldsymbol{a}\colon l \geqslant k\}$ for some k or $\{\sigma^l \boldsymbol{b}\colon l \geqslant m\}$ for some m or the union of these two sets.*

Proof. As $\Omega_i \subset \overline{F_i \setminus G_i}$ and $G_i \supset F_j$ by Proposition 1, an $x \in \Omega_i \cap F_j$ has to be on $\operatorname{bd} G_i$, which equals $\operatorname{bd} F_{i+1}$ by Proposition 2. As Ω_i and F_j are σ-invariant, $\Omega_i \cap F_j$ is also σ-invariant, hence $\sigma^p x \in \Omega_i \cap F_j \subset \operatorname{bd} F_{i+1}$ for all $p \geqslant 1$. By (i) of Lemma 12 we have $\boldsymbol{x} = \sigma^l \boldsymbol{a}$ or $\boldsymbol{x} = \sigma^l \boldsymbol{b}$. Suppose $\boldsymbol{x} = \sigma^l \boldsymbol{a}$ and $R_{k-1} < l \leqslant R_k$. As in the proof or Proposition 3, one can show that

$$A_{R_k+1} = [\sigma^{R_k+1} \boldsymbol{a}, \boldsymbol{b}] \subset A_{R_p+1} = [\sigma^{R_p+1} \boldsymbol{a}, \boldsymbol{b}] \quad \text{where} \quad p = t_{i+1}.$$

Hence $\sigma^{R_k+1} \boldsymbol{a} = \sigma^{R_p+1} \boldsymbol{a}$, because otherwise $\sigma^{R_k+1-l} \boldsymbol{x} = \sigma^{R_k+1} \boldsymbol{a} \in \operatorname{int} F_{i+1}$. By (1.3) this implies $r_{k+i} = r_{p+i}$ for all $i \geqslant 1$ or $r_{p+1} = \infty$. This says that we have case (a), (b), (c) or (d) and that $i+1 = n$. Because $i < j \leqslant n$, this implies $j = n$.

Furthermore, if $x \in \Omega_i \cap F_j$, then $\boldsymbol{x} = \sigma^l \boldsymbol{a}$ or $\boldsymbol{x} = \sigma^l \boldsymbol{b}$ and $\sigma^{R_k+1} \boldsymbol{a} = \sigma^{R_p+1} \boldsymbol{a}$ or a similar equation for $\boldsymbol{b}$ holds. This implies the second assertion.

The next theorem is the main result about the nonwandering set Ω of Σ_T^+. Before stating it, we need some results about F_n in cases (b) and (d). We suppose that (3.1) occurs. Then we have in case (b) that $\mathfrak{D}_n = \{A_l, B_m\colon R_{t_n} < l \leqslant R_{v_n}, m > S_{u_n}\}$, $\tilde{\mathfrak{D}}_n = \{A_l\colon l > R_{v_n}\}$ and $r_{v_n+1} = \infty$. In case (d), we set $u_n = q$ and $t_n = v_n = p$ where q and p are as in Lemma 7. Then we set $\mathfrak{D}_n = \{B_m\colon m > S_{u_n}\} = \{B_m\colon S_{u_n} < m \leqslant S_{u_n+1}\}$, $\tilde{\mathfrak{D}}_n = \{A_l\colon l > R_{v_n}\}$ and we have $r_{v_n+1} = \infty$. If (3.2) occurs, one has similar definitions.

LEMMA 13. *Suppose we have case* (b) *or* (d) *and* (3.1) *occurs.*

(i) *By* (1.8) *an $x \in \Sigma_T^+$ can be represented as a path in the Markov diagram which begins at $A_1 = \sigma[1]$, $E_j = \sigma[j]$ or $B_1 = \sigma[N]$. If this path enters $\tilde{\mathfrak{D}}_n$, then $\sigma^k x = \boldsymbol{a}$ for some $k \geqslant 0$.*

(ii) *If $y \in \Sigma_T^+$ is represented by a path in the Markov diagram which enters or is contained in $\mathfrak{D}_n$, then $y \in \bigcup_{k=-\infty}^{\infty} \sigma^k\{\boldsymbol{a}\}$.*

Proof. (i): By Theorem 1 two cases can occur. The first one is that the path representing x ends with

$$(4.6)\qquad \to A_1 \to A_2 \to \ldots \to A_{R_p} \to A_{R_p+1} \to \ldots$$

where we set $p = v_n$. As the path (4.6) corresponds to $a \in \Sigma_T^+$, this gives $\sigma^k x = a$ for some $k \geqslant 0$. The second case is that the path representing x ends with

$$(4.7)\qquad \to B_{S_q+2} \to \ldots \to B_{S_{q+1}} \to A_{s_{q+1}} \to \ldots \to A_{R_p} \to A_{R_p+1} \to \ldots$$

where $p = v_n$, $q \geqslant 0$ and $s_{q+1} \leqslant R_p+1$. The path (4.7) represents $b_{S_q+1} \ldots b_{S_{q+1}-1}\, a_{s_{q+1}-1}\, a_{s_{q+1}} \ldots$, which is a by (1.4). Hence there is again a $k \geqslant 0$ with $\sigma^k x = a$.

(ii): The path representing y either ends with (4.6) or (4.7) or is contained in (4.6) or (4.7). This gives the desired result.

LEMMA 14. *Suppose we have case* (b) *or* (d) *and that* (3.1) *occurs. Then there is an m with $\sigma^{m+1} a \in \Omega_n$ and $F_n \setminus \Omega_n \subset \bigcup_{k=-\infty}^{m} \sigma^k \{a\}$.*

Proof. Suppose $x \in F_n \setminus \Omega_n$. As $x \in F_n$, there is a $D \in \mathfrak{D}_n$ with $x \in D$. By (1.8), x can be represented as a path $\to D_0 \to D_1 \to \ldots$ in the Markov diagram where D_0 is a successor of D. We first show

$$(4.8)\qquad D_i \in \mathfrak{D}_n \text{ for all } i \geqslant 0 \Rightarrow x \in \Omega_n.$$

In case (b) this follows from Lemma 5 of [6]. In case (d) we have $\mathfrak{D}_n = \{B_m\colon S_u < m \leqslant S_{u_n+1}\} = \{B_m\colon m > S_{u_n}\}$ which is a cycle. Hence $x = \sigma^l b$ for some $i \geqslant S_{u_n}$, which is in Ω_n (cf. (4.2)).

As $x \notin \Omega_n$, it follows from (4.8) that $D_i \notin \mathfrak{D}_n$ for some i. Since $\bar{\mathfrak{D}}_n$ is closed, we have $D_i \in \bar{\mathfrak{D}}_n$ and hence $D_i \in \bar{\mathfrak{D}}_n$. It now follows from (ii) of Lemma 13 that $x \in \bigcup_{k=-\infty}^{\infty} \sigma^k \{a\}$.

It remains to show that $\sigma^{m+1} a \in \Omega_n$ for some m. In case (d) we have $\sigma^i b \in \Omega_n$ for $i = S_{u_n}$ (cf. (4.2)), which is $\sigma^{m+1} a$ for $m = S_{u_n} + R_{v_n}$ by (1.3), as $r_{v_n+1} = \infty$. In case (b) the path

$$\to A_{R_p+1} \to \ldots \to A_{R_{p+1}} \to B_{r_p} \to B_{r_p+1} \to \ldots$$

with $p = t_n$ is contained in $\mathfrak{D}_n$ and represents b. Hence it follows from (4.8) that $b = \sigma^{m+1} a \in \Omega_n$ where $m = R_{v_n}$.

LEMMA 15. *Suppose we have case* (b) *or* (d) *and that* (3.1) *occurs. Then the set $\bigcup_{k=-\infty}^{0} \sigma^k \{a\}$ is wandering.*

Proof. As $r_{v_n+1} = \infty$, it follows from Lemma 8 that $\{a\}$ is an open set and hence $\{x\}$ is an open set where x is an inverse image of a under σ. It suffices to show that a is not periodic.

Suppose $\sigma^t a = a$ for some t. As $\sigma^j a = b$ for $j = R_{v_n}+1$, we get $\sigma^m b = a$, where $m = kt - j > 0$. It follows from (1.4) and Lemma 4 of [2] (cf. (1.8) of [2]) that $s_i = \infty$ for some i, a contradiction to $s_k < \infty$ for all k (cf. (3.1)). Hence a is not periodic.

It follows from Lemma 15 that the m defined by Lemma 14 satisfies $m \geqslant 0$, as $\Omega_n \subset \Omega$ by Theorem 2. We choose this m minimal and set $Z = \{\sigma^i a\colon 1 \leqslant i \leqslant m\}$ if (3.1) occurs. In the case of (3.2), we set $Z = \{\sigma^i b\colon 1 \leqslant i \leqslant m\}$ where m is defined in an analogous way.

LEMMA 16. *Suppose we have case* (b) *or* (d) *and that* (3.1) *occurs. Then $Z \subset \Omega$.*

Proof. As $\sigma(\Omega) \subset \Omega$, it suffices to show $\sigma a \in \Omega$. We apply (i) of Lemma 13 to $x = \sigma a$. Let $\to D_0 \to D_1 \to \ldots$ be a path in the Markov diagram with $D_0 = A_1, B_1$ or E_j, which represents x. We have $D_i \notin \bar{\mathfrak{D}}_n$ for all i, because otherwise $\sigma^k x = \sigma^{k+1} a$ equals a for some $k \geqslant 0$ by Lemma 13, a contradiction to Lemma 15. As $D_i \notin \bar{\mathfrak{D}}_n$ for $i \geqslant 0$ and as $\mathfrak{D}_n$ is irreducible, we find for every m an $l \geqslant m$ and $C_{m+1}, \ldots, C_l \in \mathfrak{D}$ such that

$$\to D_0 \to \ldots \to D_m \to C_{m+1} \to \ldots \to C_l \to A_{R_p+1} \to A_{R_p+2} \to \ldots$$

is a path in the Markov diagram where $p = v_n$. The $y \in \Sigma_T^+$ corresponding to this path by (1.8) then satisfies $y \in [x_0 \ldots x_m]$ and $\sigma^k y = a$ for some k by (i) of Lemma 13. Hence $\sigma^{k+1}[x_0 \ldots x_m] \cap [x_0 \ldots x_m] \neq \emptyset$, which says that $x = \sigma a \in \Omega$.

Now we can prove

THEOREM 4. *In cases* (a), (c), (e) *and* (f) *we have*

$$\Omega = \bigcup_{0 \leqslant i \leqslant n} \Omega_i \qquad (n \leqslant \infty).$$

In cases (b) *and* (d) *we have*

$$\Omega = \bigcup_{0 \leqslant i \leqslant n} \Omega_i \cup Z \qquad (n < \infty)$$

where Z is wandering in $(\Omega, \sigma|\Omega)$.

Proof. In case (f) we have $\Sigma_T^+ = \bigcup_{i=0}^{\infty} (F_i \setminus F_{i+1}) \cup \Omega_\infty$, in all other cases we have $\Sigma_T^+ = \bigcup_{i=0}^{n-1} (F_i \setminus F_{i+1}) \cup F_n$ (cf. Proposition 2). If $x \in F_i \setminus F_{i+1}$ for some $i < n$, then $x \in \Omega$ if and only if $x \in \Omega_i$ by Proposition 2 and Theorems 2 and 3. In cases (a), (c) and (e) we have $F_n = \Omega_n$. If $x \in F_n$ in case (b) or (d) then $x \in \Omega$ if and only if $x \in Z \cup \Omega_n$ by Lemmas 14, 15, 16 and Theorem 2.

In cases (b) and (d) the set $\bigcup \Omega_i$ is closed and Z is finite. Hence Z is isolated in Ω. As Z contains no periodic point (they are contained in $\bigcup \Omega_i$ by Lemma 7 of [6]), Z is wandering in $(\Omega, \sigma|\Omega)$.

Remark. Further results about Ω are proved in [6] and [4]. Lemma 4 of [6] says that Ω_i is either a periodic orbit, a Cantor set or a finite union of intervals. Theorem 2 of [6] says that $\sigma|\Omega_i$ has the same period as has the oriented graph one gets if one restricts the Markov diagram to $\mathfrak{D}_i$. It follows from the results of [4] that $h_{\mathrm{top}}(\Omega_i) \geqslant h_{\mathrm{top}}(\Omega_{i+1})$ if Ω_i is not only a periodic orbit.

§ 5. The nonwandering set L of $([0, 1), T)$. We denote the nonwandering set of $([0, 1), T)$ by L. We set $C = \bigcup_{i=1}^{\infty} T^{-i}\{0\}$, $\tilde{C} = \{x \in [0, 1)$: x is a limit point of $C\}$ and $\bar{C} = C \cup \tilde{C}$, the closure of C. As $\bar{C}$ is closed, $[0, 1)\setminus\bar{C}$ is a disjoint union of open subintervals I of $[0, 1)$. We denote the set of these intervals I by $\mathfrak{J}$.

LEMMA 17. *$I \in \mathfrak{J} \Rightarrow TI \in \mathfrak{J}$ unless I has 0 or 1 as an endpoint; then $TI \subset J$ for some $J \in \mathfrak{J}$. If $I \in \mathfrak{J}$ satisfies $T^k I \subset I$ and x is an endpoint of I, then $T^k x = x$ or $x \in C$.*

Proof. If $I \in \mathfrak{J}$ we have $I \cap \bar{C} = \emptyset$, hence $TI \cap \bar{C} = \emptyset$. This implies that $TI \subset J$ for some $J \in \mathfrak{J}$. Let x be an endpoint of I. If $x \in \tilde{C}$, then clearly $Tx \in \tilde{C}$. If $x \in C$, then $Tx \in C$ or $Tx = 0$. This gives that $TI = J$ if the endpoints of I are not 0 and 1. If $T^k I \subset I$, then $T^k x = x$ or $T^j x = 0$ or $\lim_{y \uparrow x} T^j y = 1$ for some $j < k$, which implies $x \in C$.

LEMMA 18. *For $\boldsymbol{x} \in \varphi([0, 1))$, $\varphi^{-1}(\{\boldsymbol{x}\})$ is an interval or a single point.* $\mathfrak{J} = \{\operatorname{int} \varphi^{-1}(\{\boldsymbol{x}\})$: $\varphi^{-1}(\{\boldsymbol{x}\})$ is an interval$\}$.

Proof. Suppose that $x, y \in [0, 1)$ satisfy $x < y$ and $\varphi(x) = \varphi(y)$. If $x < z < y$, then $\varphi(x) \leqslant \varphi(z) \leqslant \varphi(y)$ because φ is order-preserving (cf. § 1), hence $\varphi(z) = \varphi(x)$, proving the first assertion. The elements of $\mathfrak{J}$ are maximal subintervals I of $[0, 1)$ with $T^k I$ contained in some J_i (cf. § 1) for $k \geqslant 0$. This implies the second assertion.

LEMMA 19. *If $x \notin C$, then φ is continuous at x.*

Proof. Suppose y_k converges to x in $[0, 1)$. As $x \notin C$, $T^m x$ is in the interior of some J_i (cf. § 1) for $m \geqslant 0$. In particular, T is continuous at $T^m x$. Hence $T^i(y_k)$ converges to $T^i x$ for all $i \geqslant 0$. By (1.1), we get that $\varphi(y_k)$ converges to $\varphi(x)$.

Now we consider the sets $F_i \subset \Sigma_T^+$ defined in § 3. We set $K_i = \varphi^{-1}(F_i) \subset [0, 1)$. As φ satisfies $\varphi \circ T = \sigma \circ \varphi$, Proposition 1 implies that $K_i \supset K_{i+1}$ and $T(K_i) \subset K_i$.

LEMMA 20. (i) *$r_{p+1} = \infty$ (or $s_{p+1} = \infty$) for some p implies that $\varphi^{-1}\{\boldsymbol{a}\}$ and $\varphi^{-1}\{\boldsymbol{b}\}$ are nontrivial intervals.*

(ii) *For all K_i there is an $\varepsilon > 0$ such that $[0, \varepsilon) \cup (1-\varepsilon, 1) \subset K_i$.*

Proof. (i): By (1.3) we have $\sigma^{R_p+1} \boldsymbol{a} = \boldsymbol{b}$, that is $\varphi(T^{R_p+1} 0) = \lim_{t \uparrow 1} \varphi(t)$. As φ is order-preserving, all $x \in [T^{R_p+1} 0, 1)$ have the same image under φ,

i.e., $\varphi([T^{R_p+1} 0, 1)) = \{\boldsymbol{b}\}$. Now one has also an $\varepsilon > 0$ with T^{R_p+1} monotone on $[0, \varepsilon)$ and $T^{R_p+1}([0, \varepsilon)) \subset [T^{R_p+1} 0, 1)$. This gives $\varphi([0, \varepsilon)) = \{\boldsymbol{a}\}$.

(ii): By the definition of F_i we have $A_{R_t+1} = [\sigma^{R_t+1} \boldsymbol{a}, \boldsymbol{b}] \subset F_i$ and $B_{S_u+1} = [\boldsymbol{a}, \sigma^{S_u+1} \boldsymbol{b}] \subset F_i$ where $t = t_i$ and $u = u_i$. It follows that $\varphi^{-1}(A_{R_t+1})$ and $\varphi^{-1}(B_{S_u+1})$ are nontrivial intervals. If $r_{t+1} = \infty$ or $t_{u+1} = \infty$, this follows from (i) since then $A_{R_t+1} = \{\boldsymbol{a}\}$ or $B_{S_u+1} = \{\boldsymbol{b}\}$, respectively.

We now begin the investigation whether an $x \in [0, 1)$ is wandering or not. First we consider $x \in C$.

PROPOSITION 5. *Suppose $M \subset [0, 1)$ satisfies $TM \subset M$ and contains $[0, \varepsilon) \cup (1-\varepsilon, 1)$ for some $\varepsilon > 0$. If $x \in C\setminus\bar{M}$, then $x \notin L$. In particular, an $x \in C$ is either not in L or in $\bar{K}_i$ for all i.*

Proof. As $x \in C$, we have $T^m x = 0$ for some $m \geqslant 1$. We find a $\delta > 0$ with $(x-\delta, x+\delta) \cap M = \emptyset$, such that $T^m|(x-\delta, x)$ and $T^m|[x, x+\delta)$ are monotone, and such that $T^m(x-\delta, x) \subset (1-\varepsilon, 1) \subset M$, $T^m[x, x+\delta) \subset [0, \varepsilon) \subset M$. Since $TM \subset M$, we get $T^k(x-\delta, x+\delta) \cap (x-\delta, x+\delta) = \emptyset$ for $k \geqslant m$. Upon making δ smaller if necessary, this holds for all $k \geqslant 1$. Hence $x \notin L$. The second assertion follows, because one can take every K_i for M by (ii) of Lemma 20.

PROPOSITION 6. *If $x \notin C$ is an inverse image under φ of a wandering $\boldsymbol{x} \in \Sigma_T^+$, then $x \notin L$.*

Proof. As $\boldsymbol{x}$ is wandering, there is a neighbourhood U of $\boldsymbol{x}$ in Σ_T^+ with $\sigma^k U \cap U = \emptyset$ for $k \geqslant 1$. As $x \notin C$, it follows from Lemma 19 that $V = \varphi^{-1}(U)$ is a neighbourhood of x in $[0, 1)$. Because $\varphi \circ T = \sigma \circ \varphi$, we get $T^k V \cap V = \emptyset$ for $k \geqslant 1$, i.e., $x \notin L$.

Now we consider an Ω_i. We define

$$L_i = \bigcap_{I \in \mathfrak{J}} \overline{\varphi^{-1}(\Omega_i)\setminus\bar{I}}. \tag{5.1}$$

PROPOSITION 7. *L_i is T-invariant and is the set of limit points of $\{T^k y$: $k \geqslant 0\}$ for some y, which gives $L_i \subset L$.*

Proof. By Theorem 2 we find an $\boldsymbol{y} \in \Omega_i$ such that every $\boldsymbol{x} \in \Omega_i$ is a limit point of $\{\sigma^k \boldsymbol{y}$: $k \geqslant 0\}$. If $\boldsymbol{y}$ is an inverse image of $\boldsymbol{b}$, we take $\sigma\boldsymbol{b}$ for $\boldsymbol{y}$. Then we find a $y \in \varphi^{-1}\{\boldsymbol{y}\}$. It follows from (5.1) and Lemma 18 that every $x \in L_i$ is a limit point of $\{T^k y$: $k \geqslant 0\}$ since φ is order-preserving. L_i is T-invariant, because it is the set of limit points of an orbit.

Lemmas 21, 22 and 25 investigate an $x \in \bar{I}$ where $I \in \mathfrak{J}$.

LEMMA 21. *If $x \in I$ for some $I \in \mathfrak{J}$, then $x \in L$ if and only if x is periodic.*

Proof. By Lemma 17, we have $T^k I \subset J$ for some $J \in \mathfrak{J}$. If $T^k I \cap I = \emptyset$ for $k \geqslant 1$, then $x \notin L$ since I is open. If $T^k I \subset I$ for some k, then $x \in L$ if and only if $T^k x = x$ since T^k is increasing on I.

LEMMA 22. *Suppose $I, J \in \mathfrak{J}$ and x is the endpoint of J and the initial point of I. Then $x \in L$ if and only if $T^k x = x$ or $\lim_{y \uparrow x} T^k y = x$ for some k.*

Proof. Suppose first that $T^k I \subset I$. If $T^k x = x$, then $x \in L$. If $T^k x \neq x$, then $y = T^k x > x$, as $y \in I$ and $T^k I \subset I \cap (y, 1)$. This gives $T^m[x, y) \cap [x, y) = \emptyset$ for all $m \geq 1$. Now suppose that $T^j I \subset J$. Then we find an $\varepsilon > 0$ with $T^j[x, x+\varepsilon) \cap (x-\varepsilon, x) = \emptyset$, because T^j is increasing. Hence $T^m[x, x+\varepsilon) \cap (x-\varepsilon, x) = \emptyset$ for $m \geq 1$, unless $T^l J \subset J$ and $\lim_{y \uparrow x} T^l y = x$. One easily finishes the proof considering the different cases: $T^k I \subset I$ for some k or not, $T^k I \subset J$ for some k or not, $T^k J \subset I$ for some k or not, $T^k J \subset J$ for some k or not.

We collect the nonwandering points found in Lemmas 21 and 22 and set $P = \{x \in \bar{I}\colon I \in \mathfrak{J},\ T^k x = x \text{ or } \lim_{y \uparrow x} T^k y = x \text{ for some } k\}$. The next two lemmas are needed for Lemma 25 where we shall find the nonwandering set $Y = \{x \in \bigcup_{i=1}^{n} \operatorname{bd} K_i\colon x \in \operatorname{bd} I$ for some $I \in \mathfrak{J}$ with $\varphi(I) \subset \bigcap_{i=1}^{n} F_i$ and $x \in \tilde{C}\}$. If $\mathfrak{J} = \emptyset$, i.e., $\bar{C} = [0, 1)$, then $P = Y = \emptyset$. If we have case (f), then Y may be countable. In all other cases, Y is at most finite.

LEMMA 23. *Suppose we have case* (b) *or* (d) *and that* (3.1) *occurs. Let* $\boldsymbol{x}, \boldsymbol{y} \in \Sigma_T^+$, $\boldsymbol{x} < \boldsymbol{y}$, *with* $\sigma^k \boldsymbol{x} = \boldsymbol{a}$, $\sigma^l \boldsymbol{y} = \boldsymbol{a}$ *for some* k, l *and such that there is no* $\boldsymbol{z} \in \Sigma_T^+$ *with* $\boldsymbol{x} < \boldsymbol{z} < \boldsymbol{y}$. *Then* $k < l$.

Proof. Choose m such that $x_i = y_i$ for $0 \leq i \leq m-1$ and $x_m < y_m$. By (1.1) we get that $T^m|\varphi^{-1}[\boldsymbol{x}, \boldsymbol{y}]$ is continuous. Hence $\sigma^m \boldsymbol{x}$ must be the endpoint of $[x_m]$ and $\sigma^m \boldsymbol{y}$ must be the initial point of $[y_m]$, because otherwise $T^m(\varphi^{-1}(\boldsymbol{x}, \boldsymbol{y})) \neq \emptyset$ and there is a $z \in \varphi^{-1}(\boldsymbol{x}, \boldsymbol{y})$ by the mean value theorem giving rise to a $\boldsymbol{z} = \varphi(z) \in (\boldsymbol{x}, \boldsymbol{y})$, a contradiction. Therefore $\sigma^{m+1} \boldsymbol{x} = \boldsymbol{b}$ and $\sigma^{m+1} \boldsymbol{y} = \boldsymbol{a}$. As $\boldsymbol{a}$ is not periodic (cf. Lemma 15), we have $m+1 = l$. If $k \geq m+1$, then $\boldsymbol{a} = \sigma^i \boldsymbol{b}$ for $i = k-m-1 \geq 0$ and it follows from $\sigma^{R_p+1} \boldsymbol{a} = \boldsymbol{b}$ (cf. (3.1)) that $\sigma^{i+R_p+1} \boldsymbol{a} = \boldsymbol{a}$, a contradiction to the nonperiodicity of $\boldsymbol{a}$ (cf. Lemma 15). Hence $k < m+1 = l$.

LEMMA 24. (i) *Suppose we have case* (b) *and* (3.1) *occurs. Then* $\sigma^k \boldsymbol{a}$ *for* $k \geq 1$ *is a limit point from below of* $\Sigma_T^+ \setminus \bigcup_{k=-\infty}^{0} \sigma^k\{\boldsymbol{a}\}$.

(ii) *Suppose we have case* (d) *and* (3.1) *occurs. If* $\boldsymbol{x}$ *is a limit point from below of* $F_n \cap \bigcup_{k=-\infty}^{0} \sigma^k\{\boldsymbol{a}\}$, *then* $\boldsymbol{x} \in \Omega_n$.

Proof. (i): As in the proof of Lemma 16, we use Lemma 13 and the fact that $\boldsymbol{a}$ is not periodic (Lemma 15) to represent $\sigma\boldsymbol{a}$ as a path in the Markov diagram which does not enter $\mathfrak{D}_n$. Changing this path after the mth coordinate, we get for every m an $\boldsymbol{y}_m \in [a_1 \ldots a_m]$, with $\boldsymbol{y}_m \notin \bigcup_{k=-\infty}^{0} \sigma^k\{\boldsymbol{a}\}$. Remark that $\mathfrak{D}_n$ is irreducible and not only a cycle. Because of $\sigma^{R_p+1} \boldsymbol{a} = \boldsymbol{b}$ (cf. (3.1)), $\sigma\boldsymbol{a}$ is an inverse image of $\boldsymbol{b}$ and cannot be a limit from above. Hence $\boldsymbol{y}_m \uparrow \sigma\boldsymbol{a}$, and $\sigma^{k-1} \boldsymbol{y}_m \uparrow \sigma^k \boldsymbol{a}$ proving (i).

(ii): By (1.8) and Proposition 2, F_n is the set of all $\boldsymbol{x}$ which can be represented in the following part of the Markov diagram

$$\begin{tikzcd} {} \arrow[r] & B_{S_q+1} \arrow[r] & \ldots \arrow[r] & B_{S_{q+1}} \arrow[r] \arrow[ll, bend right] & A_{s_q} \arrow[r] & A_{s_q+1} \arrow[r] & \ldots \end{tikzcd}$$

Hence we have for an $\boldsymbol{x} \in F_n \cap \bigcup_{k=-\infty}^{0} \sigma^k\{\boldsymbol{a}\}$ that $\boldsymbol{x} = b_k \ldots b_{S_{q+1}-1}\, b_{S_q} \ldots \ldots b_{S_{q+1}-1} \ldots b_{S_q} \ldots b_{S_{q+1}-1}\, \boldsymbol{a}$ where $S_q \leq k \leq S_{q+1}-1$ and the block $b_{S_q} \ldots b_{S_{q+1}-1}$ is repeated l times ($l \geq 0$). The only limit point of such $\boldsymbol{x}$ is $\sigma^k \boldsymbol{b} \in \Omega_n$ (cf. (4.2)).

LEMMA 25. *Let* x *be an endpoint of some* $I \in \mathfrak{J}$ *with* $\boldsymbol{x} := \varphi(I) \in \Omega$ *and* $x \in \tilde{C}$. *Suppose* $x \in L$ *and* $x \notin \bigcup_{0 \leq j \leq n} L_j \cup P$. *Then* $x \in Y$.

Proof. As $x \in \tilde{C}$, we find a sequence (c_j), $c_j \in C$, which converges to x monotonically from the side that does not belong to I. Let V_j be the open interval with endpoints c_j and x. Then $U_j = \varphi(V_j)$ is an open interval in Σ_T^+. We show

$$U_t \subset W_k \quad \text{for some } t \text{ and some } k \leq n. \tag{5.2}$$

In cases (b) and (d) we set $W_n = \bigcup_{k=-\infty}^{0} \sigma^k\{\boldsymbol{a}\}$, in all other cases we have $W_n = \emptyset$. We first show

$$U_t \subset F_k \setminus F_{k+1} \quad \text{for some } t \text{ and } k. \tag{5.3}$$

In case (f), we have $\bigcup_{i=0}^{\infty} \operatorname{bd} F_i = \{\sigma^i \mathring{\boldsymbol{a}}, \sigma^i \boldsymbol{b}\colon i \geq 0\}$ which is a subset of Ω_∞ (cf. Lemma 11). Hence $U_t \cap \bigcup_{i=1}^{\infty} \operatorname{bd} F_i = \emptyset$ for some t implying (5.3), because otherwise $x \in L_\infty$ by (5.1) contradicting the assumption. In all other cases $\bigcup_{i=0}^{n} \operatorname{bd} F_i$ is finite (cf. Proposition 3) and hence $U_t \cap \bigcup_{i=0}^{n} \operatorname{bd} F_i = \emptyset$ for some t.

We show $U_t \cap \Omega_k = \emptyset$. Then (5.3) implies (5.2). In cases (b) and (d) we can also choose t so that $U_t \cap Z = \emptyset$ since Z is finite. If $U_t \cap \Omega_k \neq \emptyset$ for all t, then $x \in L_k$ by (5.1), a contradiction to our assumptions. This shows (5.2).

Next we prove

$$\forall j \geq t\ \exists m \geq 1 \text{ with } \sigma^m U_j \cap U_j \neq \emptyset \Rightarrow x \in P. \tag{5.4}$$

Let k be as in (5.2). As $U_j \subset W_k$ for $j \geq t$, it follows from (4.1) that $U_j = \bigcup_{i=1}^{\infty} X_i$ where X_i are intervals with one endpoint $\varphi(x)$, $X_1 = U_j$, $X_{i+1} \subset X_i$ such that $\sigma^i|X_i$ is monotone, the interval $\sigma^i(X_i)$ is a subset of $F_k \setminus F_{k+1}$ and $\sigma^i(U_j \setminus X_i) \subset F_{k+1}$. If we have case (b) or (d) and $k = n$, then X_i is empty or a single point in $\sigma^{-i+1}\{\boldsymbol{a}\}$ (cf. Lemma 23) and we set F_{n+1}

$=\{\sigma^l a\colon l\geqslant 1\}$. As $U_j\subset F_k\setminus F_{k+1}$ by (5.2), $\sigma^m(U_j\setminus X_m)\subset F_{k+1}$ and $\sigma F_{k+1}\subset F_{k+1}$, we get that $\sigma^m U_j\cap U_j\neq\emptyset$ implies $\sigma^m(X_m)\cap U_j\neq\emptyset$. Since $x\in\Omega$, we have $x\notin\sigma^m(X_m)\subset W_k$. As $\sigma^m|X_m$ is monotone, we then find an $l>j$ with $U_l\subset X_m$ and $\sigma^m(U_l)\subset U_j$. If $\sigma^m(U_l)\subset U_j\setminus X_i$ for some i, then $\sigma^{m+i}(U_l)\subset F_{k+1}$ and $\sigma^p(U_l)\cap U_l=\emptyset$ for $p\geqslant 1$ since $\sigma F_{k+1}\subset F_{k+1}$ and $U_l\subset F_k\setminus F_{k+1}$. Hence $\sigma^m(U_l)$ and U_k have the same endpoint $\varphi(x)$, which implies $x\in P$ proving (5.4).

It follows from (5.2), $x=\varphi(I)\in\Omega$ and $\sigma(\Omega)\subset\Omega$ that

(5.5) $$\sigma^m x\notin U_t \quad \text{for} \quad m\geqslant 0.$$

We need another such assertion:

(5.6) $$\forall j\geqslant t\ \exists m\geqslant 1 \text{ with } x\in\sigma^m U_j\Rightarrow x\in P\cup Y.$$

Using the X_i defined above, we have $\sigma^m(X_m)\subset W_k$ and $\sigma^m(U_j\setminus X_m)\subset F_{k+1}$. As $x\in\Omega$, $x\in\sigma^m U_j$ implies $x\in F_{k+1}$. If $k<n$, this gives

$$x=\varphi(I)\in \operatorname{bd} F_{k+1}\subset\bigcap_{i=1}^{n} F_i \quad\text{and}\quad x\in\operatorname{bd} K_{k+1},$$

i.e., $x\in Y$ since x is an endpoint of $U_j\subset F_k\setminus F_{k+1}$ (cf. (5.2)). If we have $k=n$ and case (b) or (d), then x is the endpoint of the intervals U_j by (5.2) and Lemma 23. In case (b) it follows that $x\notin\sigma^m U_t\subset\bigcup_{i=-\infty}^{\infty}\sigma^i\{a\}$ for all m, because otherwise $x=\sigma^i a$ for $i\geqslant 1$, as $x\in\Omega$, and $U_t\not\subset\bigcup_{i=-\infty}^{0}\sigma^i\{a\}$ by (i) of Lemma 24, contradicting (5.2). In case (d) it follows from (ii) of Lemma 24 that $x\in\Omega_n$, i.e., x is periodic by (4.2). As $x\notin C$, because it is the initial point of $\varphi^{-1}\{x\}=\varphi^{-1}\{\sigma^l a\}$ and a is not periodic, we get from Lemma 17 that $x\in P$. This completes the proof of (5.6).

Now we can show Lemma 25. As $x\notin P$, we find by (5.4) a j with $T^m V_j\cap V_j=\emptyset$ for $m\geqslant 1$. By (5.5), $T^m I\cap V_j=\emptyset$ for $m\geqslant 0$. Furthermore, we find a subinterval I' of I with endpoint x such that $T^m I'\cap I'=\emptyset$ for $m\geqslant 1$. If x is not periodic, we can take I for I'. If $\sigma^p x=x$, then $T^p x\neq x$ since $x\notin P$ and I' is the interval with endpoints x and $T^p x$ (cf. the proof of Lemma 22). Therefore, as $x\in L$, for every j there must be an m with $T^m V_j\cap I\neq\emptyset$. Then (5.6) implies $x\in Y$, as $x\notin P$. This proves the lemma.

Remark. By Lemma 24, the nonwandering points of $\varphi^{-1}(Z)$ are contained in Y.

Lemma 26. $Y\subset L$.

Proof. Let $x\in Y$, i.e., x is the endpoint of an $I=\operatorname{int}\varphi^{-1}\{x\}$ and $x\in\operatorname{bd} K_i$ for some i. Then $x=\sigma^k a$ or $x=\sigma^k b$ for some k since $x\in\operatorname{bd} F_i$ for some i. As $x\in\bar{C}$, there are $c_j\notin\bigcap_{i=1}^{n}K_i$, $c_j\in C$, with c_j converging to x from the side not belonging to I. As in the proof of Lemma 25, let V_j be the open intervals with endpoints x and c_j and set $U_j=\varphi(V_j)$. As U_j is an open interval in Σ_T^+, we find a cylinder set $[y_0\ldots y_m]\subset U_j$. We represent $y_0\ldots y_m$ as a finite path in the Markov diagram. In all six cases we can continue this path so that it ends with

$$B_{s_{q-1}}\to\ldots\to B_{s_q}\to A_{s_q}\to A_{s_q+1}\to\ldots \quad \text{for some } q.$$

This path represents a $z\in[y_0\ldots y_m]$ with $\sigma^l z=a$ for some l. In cases (b) and (d) we then have $\sigma^{l+R_p+1}z=b$. In all other cases we can also continue the path so that it ends with

$$A_{R_{q-1}}\to\ldots\to A_{R_q}\to B_{r_q}\to B_{r_q+1}\to\ldots$$

representing a $z\in[y_0\ldots y_m]$ with $\sigma^l z=b$. Hence in any case we have a $z\in[y_0\ldots y_m]\subset U_j$ with $\sigma^i z=x$ for some i. Hence $x\in\sigma^i U_j$ or $I\subset T^i V_j$, which shows that $x\in L$.

Now we can show the main result.

Theorem 5. *We have* $L=\bigcup_{0\leqslant i\leqslant n} L_i\cup P\cup Y$.

Proof. It follows from Proposition 7 and Lemmas 21, 22 and 26 that $\bigcup L_i\cup P\cup Y\subset L$. For $x\in[0,1)$, we shall show that either $x\notin L$ or $x\in\bigcup L_i\cup P\cup Y$.

Suppose first that there is an i with $x\notin\bar{K}_i$. If $x\in C$, then $x\notin L$ by Proposition 5. If $x\in\varphi^{-1}W_j\setminus C$, $j<n$, then $x\notin L$ by Proposition 6. If $x\in\varphi^{-1}(\Omega_j)$, $j<n$, then $x\in L_j$ or $x\in P$ or $x\notin L$ by Proposition 7 and Lemmas 21, 22 and 25.

Now suppose that $x\in\bigcap_{i=1}^{n}\bar{K}_i$. In cases (a), (c), (e) and (f) it follows from Proposition 7 and Lemmas 21, 22 and 25 that $x\in L_n$, $x\in P\cup Y$ or $x\notin L$ since $\Omega_n=\bigcap_{i=1}^{n}F_i$. In cases (b) and (d) an $x\in C$ is between an $I\in\mathfrak{J}$ and an $J\in\mathfrak{J}$, as $\varphi^{-1}\{a\}$ and $\varphi^{-1}\{b\}$ are intervals (Lemma 20), so that Lemma 22 implies $x\notin L$ or $x\in P$. If

$$x\in\varphi^{-1}\Big(\bigcup_{k=-\infty}^{0}\sigma^k\{a\}\Big)\setminus C,$$

then $x\notin L$ by Proposition 6 and Lemma 15. If $x\in\varphi^{-1}(\Omega_n)$, then $x\in L_n$, $x\in P\cup Y$ or $x\notin L$ by Proposition 7 and Lemmas 21, 22 and 25.

Remark. For $i<n$, one easily shows that $\varphi^{-1}(\Omega_i)\cap C=\emptyset$ (cf. (ii) of Lemma 20). Hence φ is continuous on $\varphi^{-1}(\Omega_i)$ by Lemma 19. If $\varphi^{-1}(\Omega_i)\subset\bar{C}$, then $L_i=\varphi^{-1}(\Omega_i)$ is topologically isomorphic via φ to Ω_i, which is a finite type subshift. Furthermore, the L_i's are periodic orbits, Cantor sets, or finite unions of intervals.

This follows from results of [6] (cf. the remark at the end of § 4) and because φ is order-preserving.

It follows from the definitions of L_i, P and Y, that L can be determined if one knows the Ω_i, which are determined by the Markov diagram, and if one knows $\bar{C}$ or $\mathfrak{J}$. In [1] $\mathfrak{J}$ is determined for certain transformations. One can find examples for all possible cases described in the paper. For example, T defined by

$$T(x) = \begin{cases} \frac{1}{8}x+\frac{21}{80} & \text{for} \quad x\in[0, \frac{27}{260}], \\ \frac{27}{8}x-\frac{3}{40} & \text{for} \quad x\in[\frac{27}{260}, \frac{1}{5}], \\ 2x+\frac{1}{5} & \text{for} \quad x\in[\frac{1}{5}, \frac{2}{5}), \\ x-\frac{2}{5} & \text{for} \quad x\in[\frac{2}{5}, 1) \end{cases}$$

belongs to case (a), but $Y = \{\frac{1}{4}, \frac{7}{10}\}$.

By the methods of [3] one can transmit $x \to ax(1-x)$, where $2 \leqslant a \leqslant 4$, into a piecewise monotonic transformation. If one blows up each of the points in $\bigcup_{j=0}^{\infty} T^{-j}\{T^i(0), i \geqslant 0\}$ to an interval, one gets again a nonempty Y which has infinitely many elements for certain values of a.

References

[1] J. Guckenheimer, *Sensitive dependence to initial conditions for one dimensional maps*, Comm. Math. Phys. 70 (1979), 133–160.

[2] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math. 34 (1979), 213–237.

[3] — *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy* II, Israel J. Math. 38 (1981), 107–115.

[4] — *Maximal measures for simple piecewise monotonic transformations*, Z. Wahrsch. Verw. Gebiete 52 (1980), 289–300.

[5] — *The maximal measure for linear mod one transformations*, J. London Math. Soc. 23 (1981), 92–112.

[6] — *The structure of piecewise monotonic transformations*, Erg. Th. &. Dyn. Syst. 1 (1981), 159–178.

[7] — *Kneading invariants and Markov diagrams*, in: *Ergodic theory and related topics*, Proceedings, Akademie-Verlag, Berlin 1982.

[8] L. Jonker, D. Rand, *Bifurcations in one dimension*, Invent. Math. 62 (1981), 347–365.

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On the Wiener–Eberlein theorem

by

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Abstract. A counterexample is presented to the main theorem of a paper by J.-M. Belley and P. Morales that appeared in Studia Mathematica 72 (1982), pp. 27–36.

Given a locally compact Abelian group G, let μ be a bounded complex-valued countably additive measure defined on the Borel sets of the character group G^*. Then the Fourier transform $\hat{\mu}$,

$$\hat{\mu}(x) = \int_{G^*} (x, -y)\, d\mu(y) \qquad (x\in G)$$

is a weakly almost periodic (w.a.p.) function on G [3]. The following result is due to Norbert Wiener ([6], Vol. 2, pp. 259–261, and Vol. 1, p. 108; [5]) in the special cases $G = R$ and $G = Z$ and to the author in the general case [4].

THEOREM. $M[|\hat{\mu}|^2] = \sum_{y\in G^*}' |\mu\{y\}|^2$.

Here the mean value $M(f)$ of a w.a.p. function f may be defined as the (necessarily unique) constant that is the uniform limit of convex combinations of translates of f. When $G = R$, the additive group of the reals, M has the representation

$$M(f) = \lim_{L\to\infty} (2L)^{-1} \int_{-L}^{L} f(x)\, dx.$$

In a recent paper in this journal, Belley and Morales [1] purport to generalize this theorem to the case of *finitely* additive μ. Here is a counterexample to the extended theorem when G^* is non-compact (= G non-discrete) – say $G = R = G^*$: Pick any point y_0 in $\beta G^* - G^*$, where βG^* is the Čech–Stone compactification of G^*, and let ν be a unit measure concentrated at y_0. If f is any bounded continuous function on G^*, denote its extension to βG^* by F. Then ν induces a finitely additive bounded regular measure μ on the Borel subsets of G^* such that $\int_{G^*} f d\mu = \int_{\beta G^*} F d\nu$ ([2], p. 262). Clearly, $\mu\{y\} = 0$ for any y in G^*. But when $f(y) = (x, -y)$, $|f| = 1$ on G^* and $|F| = 1$ on βG^*, whence $|\hat{\mu}(x)| = |F(y_0)| = 1$. Hence $M[|\hat{\mu}|^2] = M(1) = 1 \neq 0 = \sum_{y\in G^*} |\mu\{y\}|^2$.