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Inverse elements in extensions of Banach algebras

by

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Abstract. We construct an example of a separable commutative Banach algebra A with unit and elements $u, v \in A$ such that the removable parts of $\sigma(u)$ and $\sigma(v)$ cannot be removed simultaneously. This solves also the problem of W. Żelazko concerning removability of compact sets of removable maximal ideals.

Let A be a commutative Banach algebra with unit and $x \in A$. We say that x is *permanently singular* if there is no superalgebra B (i.e. B is a commutative Banach algebra with unit and there is an isometric unit preserving isomorphism from A into B) in which x is invertible. This notion was introduced by R. Arens in [1] where it was proved that an element x is permanently singular if and only if it is a topological divisor of zero (i.e. $\inf_{\|y\|=1} \|xy\| = 0$). In [1] also the following problem was raised: Let M be a subset of A such that no $m \in M$ is a topological divisor of zero. Does there exist a superalgebra $B \supset A$ in which all elements of M are invertible?

The answer to this question is affirmative for M countable and negative in general as was shown by B. Bollobás [2].

In the present paper we exhibit an example of a commutative Banach algebra A and $u, v \in A$ such that we cannot adjoin inverses to all elements

$$\{u-\lambda,\ \lambda \in \sigma(u)-\tau(u)\} \cup \{v-\mu,\ \mu \in \sigma(v)-\tau(v)\}$$

where $\tau(x)$ is the approximate point spectrum of x, $\tau(x) = \{\lambda \in C,\ \lambda - x \text{ is a topological divisor of zero}\}$. Our example is simpler than the above-mentioned example of B. Bollobás; moreover, the constructed algebra A is separable. Let us point out that our construction is a modification of another example of Bollobás [2], Theorem 2.1.

The Banach algebra A gives also a negative answer to a question of W. Żelazko ([6], Problem 4) concerning removability of compact sets of removable ideals.

We construct a commutative Banach algebra A with generators $u, v, b_0^{(k)}, b_1^{(k)}, b_2^{(k)}$ $(k = 1, 2, \ldots)$. Fix $N \geq 3$. Write $T = \{u^i v^j,\ b_1^{(k)} u^i v^j,\ b_2^{(k)} u^i v^j;\ i, j = 0, 1, 2, \ldots,\ k = 1, 2, \ldots\} \cup \{b_0^{(k)} u^i v^j,\ k = 1, 2, \ldots, \min\{i, j\} < k\}$. Put

$$\|u^i v^j\| = \|b_1^{(k)} u^i v^j\| = \|b_2^{(k)} u^i v^j\| = N^{i+j}$$

(in particular, $\|b_1^{(k)}\| = \|b_2^{(k)}\| = 1$) and

$$\|b_0^{(k)} u^i v^j\| = (N-1) N^{m-1} \quad \text{where} \quad m = \max\{i, j, k\}.$$

The algebra A will be formed by all formal series $y = \sum_{t \in T} \alpha_t t$ with complex coefficients α_t for which $\|y\| = \sum_{t \in T} |\alpha_t| \|t\| < \infty$.

Clearly, A with natural addition and multiplication by scalars is a Banach space.

We define the multiplication in A by

$$b_i^{(k)} b_j^{(l)} = 0 \quad (i, j \in \{0, 1, 2\},\ k, l \in \{1, 2, \ldots\}),$$

$$b_0^{(k)} u^k v^k = b_1^{(k)} u^k + b_2^{(k)} v^k.$$

We have $\|ts\| \leqslant \|t\| \|s\|$ for $t, s \in T$ and therefore $\|yy'\| \leqslant \|y\| \|y'\|$ for all $y, y' \in A$. This means that A is a commutative Banach algebra with unit.

(1) $\|xu\| \geqslant \|x\|$, $\|xv\| \geqslant \|x\|$ *for every* $x \in A$.

Proof. We shall prove only the first inequality as the situation is symmetric with respect to u and v. Let

$$x \in A, \quad x = x_0 + \sum_{k=1}^{\infty} x_k, \quad x_0 = \sum_{i,j=0}^{\infty} \alpha_{ij} u^i v^j,$$

$$x_k = \sum_{i,j=0}^{\infty} \beta_{ij}^{(k)} b_1^{(k)} u^i v^j + \sum_{i,j=0}^{\infty} \gamma_{ij}^{(k)} b_2^{(k)} u^i v^j + \sum_{\min\{i,j\} < k} \delta_{ij}^{(k)} b_0^{(k)} u^i v^j$$

where $\alpha_{ij}, \beta_{ij}^{(k)}, \gamma_{ij}^{(k)}, \delta_{ij}^{(k)} \in C$. We have

$$\|x\| = \|x_0\| + \sum_{k=1}^{\infty} \|x_k\|, \quad \|x_0\| = \sum_{i,j=0}^{\infty} |\alpha_{ij}| N^{i+j},$$

$$\|xu\| = \|x_0 u\| + \sum_{k=1}^{\infty} \|x_k u\|, \quad \|x_0 u\| = N \|x_0\| > \|x_0\|.$$

It is sufficient to prove $\|x_k u\| \geqslant \|x_k\|$ $(k = 1, 2, \ldots)$.

For fixed k put

$$x_k = y_0 + y_1 + y_2 + \sum_{j=0}^{\infty} z_j$$

where

$$y_0 = \sum_{\substack{\text{either } i \neq k-1 \\ \text{or } j < k}} \delta_{ij}^{(k)} b_0^{(k)} u^i v^j,$$

$$y_1 = \sum_{i \neq k-1} \beta_{ij}^{(k)} b_1^{(k)} u^i v^j,$$

$$y_2 := \sum_{i,j=0}^{\infty} \gamma_{ij}^{(k)} b_2^{(k)} u^i v^j,$$

$$z_j = \delta_{k-1,k+j}^{(k)} b_0^{(k)} u^{k-1} v^{k+j} + \beta_{k-1,j}^{(k)} b_1^{(k)} u^{k-1} v^j \quad (j = 0, 1, 2, \ldots).$$

From the definition we get

$$\|x_k\| = \|y_0\| + \|y_1\| + \|y_2\| + \sum_{j=0}^{\infty} \|z_j\|,$$

$$\|x_k u\| = \|y_0 u\| + \|y_1 u\| + \|y_2 u\| + \sum_{j=0}^{\infty} \|z_j u\|,$$

$$\|y_0 u\| \geqslant \|y_0\|, \quad \|y_1 u\| = N \cdot \|y_1\| > \|y_1\|,$$

$$\|y_2 u\| = N \cdot \|y_2\| > \|y_2\|,$$

$$\|z_j\| = |\delta_{k-1,k+j}^{(k)}| (N-1) N^{k+j-1} + |\beta_{k-1,j}^{(k)}| N^{k+j-1},$$

$$\|z_j u\| = |\beta_{k-1,j}^{(k)} + \delta_{k-1,k+j}^{(k)}| N^{k+j} + |\delta_{k-1,k+j}^{(k)}| N^{k+j} \geqslant \|z_j\|.$$

This proves (1).

(2) *Let* $B \supset A$ *be a superalgebra of* A *such that* u^{-1} *and* v^{-1} *exist in* B. *Then* $\max\{r_B(u^{-1}), r_B(v^{-1})\} \geqslant N$ *(where* $r_B(x)$ *denotes the spectral radius of* x *in* B*).*

Proof. Let $B \supset A$ and $k \in \{1, 2, \ldots\}$. Then $b_0^{(k)} = b_1^{(k)} v^{-k} + b_2^{(k)} u^{-k}$ and

$$(N-1) N^{k-1} = \|b_0^{(k)}\| \leqslant \|v^{-k}\| + \|u^{-k}\|,$$

$$\max\{\|u^{-k}\|, \|v^{-k}\|\} \geqslant \frac{N-1}{2} N^{k-1}.$$

Using the spectral radius formula, we conclude that $\max\{r_B(u^{-1}), r_B(v^{-1})\} \geqslant N$. So we have proved:

Theorem. *There exists a separable commutative Banach algebra* A *with unit and elements* $u, v \in A$ *such that*

$$\|u\| = \|v\| = N \quad (N \geqslant 3), \quad \|ux\| \geqslant \|x\|, \quad \|vx\| \geqslant \|x\|$$

for every $x \in A$ *and* $\max\{r_B(u^{-1}), r_B(v^{-1})\} \geqslant N$ *in any superalgebra* $B \supset A$ *in which* u *and* v *are invertible.*

Remark 1. The set $M = \{u - \lambda, |\lambda| \leqslant 1/N\} \cup \{v - \mu, |\mu| \leqslant 1/N\}$ contains no topological divisor of zero. Let B be any superalgebra of A. Then either $\sigma_B(u) \cap \{\lambda \in C, |\lambda| \leqslant 1/N\} \neq \emptyset$ or $\sigma_B(v) \cap \{\mu \in C, |\mu| \leqslant 1/N\} \neq \emptyset$ which means that we cannot adjoin inverses to all elements of M simultaneously (see [2], Theorem 4.1).

Remark 2. In [3] B. Bollobás raised the following problem: Let x be an element of a commutative Banach algebra C. Does there exist a

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superalgebra $D \supset C$ such that $\sigma_D(x) = \tau_C(x)$ ($= \bigcap_{B \supset C} \sigma_B(x)$ according to the theorem of R. Arens [1])?

Let $x_1, \ldots; x_n \in C$. Denote by $\sigma_C(x_1, \ldots, x_n)$ the joint spectrum of the n-tuple $x_1, \ldots, x_n$ and by $\tau_C(x_1, \ldots, x_n)$ the joint approximate point spectrum of $x_1, \ldots, x_n$,

$$\tau_C(x_1, \ldots, x_n) = \{(\lambda_1, \ldots, \lambda_n) \in C^n, \inf_{\substack{y \in C \\ \|y\|=1}} \sum_{i=1}^{n} \|(x_i - \lambda_i) y\| = 0\}.$$

The constructed algebra A and elements $u, v \in A$ give a negative answer to the question analogical to the problem of Bollobás for the joint spectrum: in any $B \supset A$ either $\sigma_B(u) \neq \tau_A(u)$ or $\sigma_B(v) \neq \tau_A(v)$. As the joint approximate point spectrum possesses the projection property, $\sigma_B(u, v) \neq \tau_A(u, v)$ in any superalgebra $B \supset A$ ($\tau_A(u, v) = \bigcap_{B \supset A} \sigma_B(u, v)$ is still true according to [4]).

Remark 3. Problems concerning the possibility of adjoining inverses have an analogue for ideals (see e.g. [6]). We say that an ideal $I \subset A$ is *removable* if there exists $B \supset A$ such that I is contained in no proper ideal of B. We say that a family $\{I_j\}_{j \in J}$ is removable if there exists $B \supset A$ such that I_j is contained in no proper ideal of B for all $j \in J$. A countable family of removable ideals is removable (see [4], [5]). Let S be a compact subset of the maximal ideal space $\mathcal{M}(A)$ (in the Gelfand topology) which consists of removable ideals (i.e. $S \cap \operatorname{Cor} A = \emptyset$ where $\operatorname{Cor} A$ is the set of all maximal non-removable ideals). In [6], problem 4 there is a question whether such a set S is removable.

The constructed algebra A gives a negative solution of this problem. As $\mathcal{M}(A)$ is homeomorphic to $\sigma_A(u, v)$ ($b_i^{(k)} \in \operatorname{Rad} A$, $i = 0, 1, 2, k = 1, 2, \ldots$) and $\operatorname{Cor} A$ is homeomorphic to $\tau_A(u, v)$ (see [4]), the set

$$\{(\lambda, \mu) \in \sigma_A(u, v), \min\{|\lambda|, |\mu|\} \leqslant 1/N\}$$

is a counterexample to this question.

From this it also follows that in general the union of two removable families of ideals is not removable.

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