converge also, to limits forming an l^p sequence. To verify that the atomic sum converges in the distribution sense, we notice that the terms corresponding to cubes larger than some $\delta > 0$ are finite in number, so their sum converges. We need thus only verify that the integrals of the remaining terms have a small sum when integrated against a test function φ. Now $\varphi \in C^\infty$, so φ is small in the dual space $\dot{\Lambda}_\alpha$ in a small dyadic cube, and this gives the necessary estimate. Hence, the atomic decomposition of $\tilde{f}_{\varepsilon_j}$ converges in the distribution sense to an atomic sum representing an H^p distribution which must be f. This gives the required decomposition of f, completing the proof.

References

[1] S. Campanato, *Proprietà di una famiglia di spazi funzionali*, Ann. Scuola Norm. Sup. Pisa 18 (1964), 137–160.

[2] R. R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[3] C. Fefferman, E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[4] O. Frostman, H. Wallin, *Riesz potentials on manifolds*, in: Proceedings of a Conference on Potential Theory (Copenhagen 1979), pp. 106–120. Lecture Notes in Math. 787, Springer, Berlin-Heidelberg-New York 1980.

[5] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J. 46(1979), 27–42.

[6] A. Jonsson, H. Wallin, *A Whitney extension theorem in L^p and Besov spaces*, Ann. Inst. Fourier (Grenoble) 28 (1978), 139–192.

[7] —, —, *The trace to closed sets of functions in R^n with second difference of order $O(h)$*, J. Approx. Theory 26 (1979), 159–184.

[8] —, —, *Local polynomial approximation and Lipschitz type conditions on general closed sets*, Department of Math., Univ. of Umeå, No 1, 1980.

[9] —, —, *Local polynomial approximation and Lipschitz functions on closed sets*, to appear in: Proceedings of a Conference on Constructive Function Theory, Varna, Bulgaria, 1981.

[10] R. H. Latter, *A decomposition of $H^p(R^n)$ in terms of atoms*, Studia Math. 62 (1977), 92–101.

[11] R. H. Latter, A. Uchiyama, *The atomic decomposition for parabolic H^p spaces*, Trans. Amer. Math. Soc. 253 (1979), 391–398.

[12] R. A. Macias, C. Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type*, Adv. in Math. 33 (1979), 271–309.

[13] P. Sjögren, *Lectures on atomic H^p space theory in R^n*, Department of Math., Univ. of Umeå, No 5, 1981.

[14] —, *A weak spectral synthesis property for Hardy and Lipschitz spaces*, in: Harmonic analysis, Proceedings 1981, Lecture Notes in Math. 908 Springer, 1982, 285–296.

[15] J. O. Strömberg, A. Torchinsky, *Weights, sharp maximal functions and Hardy spaces*, Bull. Amer. Math. Soc. 3 (1980), 1053-1056.

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Generalized convolutions III

by

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Abstract. This is a study of characteristic functions of generalized convolutions. In particular, we obtain some uniqueness and characterization theorems. Moreover, the concepts of representability and order of generalized convolutions are discussed. The paper is a continuation of [11] and [13].

1. Preliminaries and notation. We denote by $\mathscr{C}_b$ the space of bounded continuous real-valued functions on the positive half-line R^+ with the topology of uniform convergence on every compact subset of R^+. Further, by $\mathfrak{P}$ we shall denote the set of all probability measures defined on Borel subsets of R^+. The set $\mathfrak{P}$ is endowed with the topology of weak convergence. For $a \in R^+$ we define the mapping T_a: $R^+ \to R^+$ by $T_a x = ax$. For a function $f \in \mathscr{C}_b$, $T_a f$ denotes the function $(T_a f)(x) = f(ax)$ and for a measure $\mu \in \mathfrak{P}$, $T_a \mu$ denotes the measure defined by $(T_a \mu)(E) = \mu(a^{-1} E)$ if $a > 0$ and $T_0 \mu = \delta_0$, where $a^{-1}E = \{a^{-1}x: x \in E\}$ and δ_c is the probability measure concentrated at the point c. We say that two functions f and g from $\mathscr{C}_b$ are *similar*, in symbols $f \sim g$, if $f = T_a g$ for a certain positive number a. Further, two measures μ and ν from $\mathfrak{P}$ are said to be *similar*, in symbols $\mu \sim \nu$, if $\mu = T_a \nu$ for a certain positive number a.

A continuous commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ defined on $\mathfrak{P}$ is called a *generalized convolution* if the following conditions are fulfilled:

(i) the measure δ_0 is a unit element, i.e. $\mu \circ \delta_0 = \mu$ $(\mu \in \mathfrak{P})$,

(ii) $(c\mu + (1-c)\nu) \circ \lambda = c(\mu \circ \lambda) + (1-c)(\nu \circ \lambda)$ $(0 \leqslant c \leqslant 1;\ \mu, \nu, \lambda \in \mathfrak{P})$,

(iii) $(T_a \mu) \circ (T_a \nu) = T_a(\mu \circ \nu)$ $(a \in R^+,\ \mu, \nu \in \mathfrak{P})$,

(iv) there exists a sequence $c_1, c_2, \ldots$ of positive numbers such that the sequence $T_{c_n} \delta_1^{\circ n}$ converges to a measure different from δ_0. The power $\delta_1^{\circ n}$ is taken here in the sense of the operation $\circ$.

The set $\mathfrak{P}$ with the operation $\circ$ and the operations of convex linear combinations is called a *generalized convolution algebra* and denoted by $(\mathfrak{P}, \circ)$. For basic properties of generalized convolution algebras we refer to [2]–[7] and [10]–[14]. In particular, generalized convolution algebras admitting a non-trivial homomorphism into the algebra of real numbers with the operations of multiplication and convex linear combinations are called *regular*. We recall that a homomorphism h is trivial if either $h \equiv 0$ or $h \equiv 1$. All generalized convolution algebras under consideration in the sequel will tacitly be assumed to be regular.

Now we shall quote some simple examples of generalized convolutions. In all examples generalized convolutions $\mu \circ \nu$ will be defined by means of the functional $\int_0^\infty f(x)(\mu\circ\nu)(dx)$ $(f \in \mathscr{C}_b)$.

Example 1.1. The convolutions $*_\alpha$ $(0<\alpha<\infty)$

$$\int_0^\infty f(x)(\mu *_\alpha \nu)(dx) = \int_0^\infty\int_0^\infty f\big((x^\alpha+y^\alpha)^{1/\alpha}\big)\,\mu(dx)\,\nu(dy).$$

For $\alpha = 1$ we obtain the ordinary convolution.

Example 1.2. Kingman convolutions $*_{\alpha,\beta}$ $(0<\alpha<\infty,\ 1\leqslant\beta<\infty)$

$$\int_0^\infty f(x)(\mu *_{\alpha,1}\nu)(dx) = \tfrac{1}{2}\int_0^\infty\int_0^\infty \big[f\big((x^\alpha+y^\alpha)^{1/\alpha}\big)+f\big(|x^\alpha-y^\alpha|^{1/\alpha}\big)\big]\,\mu(dx)\,\nu(dy)$$

and for $\beta > 1$

$$\int_0^\infty f(x)(\mu *_{\alpha,\beta}\nu)(dx) = \frac{\Gamma(\beta/2)}{\sqrt{\pi}\,\Gamma((\beta-1)/2)}\int_0^\infty\int_0^\infty\int_{-1}^{1} f\big((x^{2\alpha}+y^{2\alpha}+2x^\alpha y^\alpha z)^{1/2\alpha}\big)\times$$
$$\times(1-z^2)^{(\beta-3)/2}\,dz\,\mu(dx)\,\nu(dy).$$

Example 1.3. The convolutions $\circ_{\alpha,n}$ $(0<\alpha<\infty,\ n=1,2,\ldots)$

$$\int_0^\infty f(x)(\mu\circ_{\alpha,n}\nu)(dx) = \int_0^\infty\int_0^\infty\Big[\big(1-\min(x^\alpha y^{-\alpha},\ y^\alpha x^{-\alpha})\big)^n f(\max(x,y))+$$
$$+\sum_{k=1}^{n}\alpha(n+1)\binom{n}{k}\binom{n}{k-1}x^{\alpha(n+1-k)}y^{\alpha k}\times$$
$$\times\int_{\max(x,y)}^{\infty} f(z)(z^\alpha-x^\alpha)^{n-1}(z^\alpha-y^\alpha)^{n-k}z^{-2\alpha n-1}\,dz\Big]\mu(dx)\,\nu(dy),$$

where $\min(0\cdot 0^{-1},\ 0\cdot 0^{-1})$ is assumed to be 0.

We say that an algebra $(\mathfrak{P}, \circ)$ *admits a characteristic function* if there exists a homeomorphic map from $\mathfrak{P}$ into $\mathscr{C}_b$: $\mu\to\hat{\mu}$ such that

$$\widehat{c\mu+(1-c)\nu} = c\hat{\mu}+(1-c)\hat{\nu}\qquad(0\leqslant c\leqslant 1),\tag{1.1}$$

$$\widehat{\mu\circ\nu} = \hat{\mu}\hat{\nu},\tag{1.2}$$

$$\widehat{T_a\mu} = T_a\hat{\mu}\qquad(a\in R^+)\tag{1.3}$$

for all $\mu,\nu\in\mathfrak{P}$. The characteristic function plays the same fundamental role in generalized convolution algebras as the Laplace transform in the ordinary

convolution algebra $(\mathfrak{P}, *_1)$. It has been proved in [11] (Theorem 6) that an algebra admits a characteristic function if and only if it is regular. Moreover, each characteristic function is an integral transform

$$\hat{\mu}(t) = \int_0^\infty \Omega(tx)\,\mu(dx)\tag{1.4}$$

with a continuous kernel Ω satisfying the conditions $|\Omega(t)|\leqslant 1$ $(t\in R^+)$ and

$$\Omega(t) = 1-t^\varkappa L(t),\tag{1.5}$$

where $\varkappa>0$ and the function L is slowly varying at the origin. Moreover, using Lemma 1 in [13] one can show that L is continuous at the origin. The constant $\varkappa$ in (1.5) does not depend upon the choice of a characteristic function and is called the *characteristic exponent* of the generalized convolution $\circ$, in symbols $\varkappa(\circ)=\varkappa$. For a description of kernels corresponding to characteristic functions of generalized convolutions we refer to [8].

A probability measure μ is said to be $\circ$-*stable* if the relations $\mu_1\sim\mu$ and $\mu_2\sim\mu$ yield $\mu_1\circ\mu_2\sim\mu$. It was shown in [11] (Chapter 6) that the set of all $\circ$-stable measures coincides with the set of all possible limit distributions of sequences $T_{c_n}\nu^{\circ n}$, where $c_n>0$ and $\nu\in\mathfrak{P}$. Moreover, the characteristic function of non-degenerate $\circ$-stable measures μ is of the form

$$\hat{\mu}(t) = \exp(-ct^p)$$

where $c>0$ and $0<p\leqslant\varkappa(\circ)$ ([13], Theorem 2). The constant p does not depend upon the choice of the characteristic function and is called the *exponent* of μ. By (1.3) all $\circ$-stable measures with the same exponent are similar. When in the sequel we are dealing with a fixed generalized convolution $\circ$, we always use the notation σ_p $(0<p\leqslant\varkappa(\circ))$ for the non-degenerate $\circ$-stable measure with exponent p and the characteristic function

$$\hat{\sigma}(t) = \exp(-t^p).$$

The measure $\sigma_\varkappa$ where $\varkappa=\varkappa(\circ)$ is called the *characteristic measure* of the generalized convolution in question.

Now we shall quote some examples of kernels of characteristic functions and characteristic exponents and measures of generalized convolutions.

Example 1.4. For the convolution $*_\alpha$ $(0<\alpha<\infty)$ we have $\Omega(t) = \exp(-t^\alpha)$, $\varkappa(*_\alpha)=\alpha$ and $\sigma_\varkappa=\delta_1$.

Example 1.5. For the convolution $*_{\alpha,\beta}$ $(0<\alpha<\infty,\ 1\leqslant\beta<\infty)$ we have $\Omega(t) = \Gamma\Big(\dfrac{\beta}{2}\Big)\Big(\dfrac{2}{t^\alpha}\Big)^{\beta/2-1}J_{\beta/2-1}(t^\alpha)$, where J_ν is the Bessel function, $\varkappa(*_{\alpha,\beta})=2\alpha$ and

$$\sigma_\varkappa(E) = \frac{\alpha}{2^{2\beta-2}\,\Gamma(\beta-\frac{1}{2})}\int_E x^{2\alpha\beta-\alpha-1}\exp\Big(-\frac{x^{2\alpha}}{4}\Big)dx.\tag{1.6}$$

EXAMPLE 1.6. For the convolution $\circ_{\alpha,n}$ $(0<\alpha<\infty,\ n=1,2,\ldots)$ we have $\Omega(t)=(1-t^{\alpha})^{n}$ if $0\leqslant t\leqslant 1$ and $\Omega(t)=0$ otherwise, $\varkappa(\circ_{\alpha,n})=\alpha$ and

$$\sigma_{\varkappa}(E)=\frac{\alpha}{n!}\int_{E}x^{-1-\alpha(n+1)}\exp(-x^{-\alpha})\,dx. \tag{1.7}$$

For any pair μ,ν from $\mathfrak{P}$, by $\mu\nu$ we shall denote the probability distribution of the product XY of two independent random variables X and Y with probability distributions μ and ν, respectively. The operation $\mu\nu$ is a commutative semigroup operation with the following properties:

$$(T_a\mu)\nu=T_a(\mu\nu)\quad (a\in R^{+}), \tag{1.8}$$

$$T_a\mu=\delta_a\mu\quad (a\in R^{+}), \tag{1.9}$$

$$(c\mu+(1-c)\nu)\lambda=c(\mu\lambda)+(1-c)(\nu\lambda)\quad (0\leqslant c\leqslant 1). \tag{1.10}$$

Moreover, we have the following propositions.

PROPOSITION 1.1. *If $\mu_n\to\mu$, then $\mu_n\nu\to\mu\nu$ for all $\nu\in\mathfrak{P}$.*

PROPOSITION 1.2. *If $\nu\neq\delta_0$ and $\mu_n\nu\to\lambda$, then the sequence μ_n is conditionally compact and each its limit point μ fulfils the equation $\mu\nu=\lambda$.*

This statement is an immediate consequence of the inequalities

$$(\mu_n\nu)([a,\infty))\geqslant\nu([b,\infty))\mu_n([ab^{-1},\infty))$$

for $a,b>0$, $n=1,2,\ldots$, and Proposition 1.1.

PROPOSITION 1.3. *Let $\hat{\mu}$ be a characteristic function of a generalized convolution. Then*

$$\widehat{\mu\nu}(t)=\int_{0}^{\infty}\hat{\mu}(tx)\,\nu(dx). \tag{1.11}$$

Proof. By Proposition 1.1 it is enough to show (1.11) for measures ν of the form $\nu=\sum_{k=1}^{n}c_k\delta_{a_k}$ $(c_k\geqslant 0,\ k=1,2,\ldots,n)$, because these measures form a dense subset of $\mathfrak{P}$. By (1.9) and (1.10) $\mu\nu=\sum_{k=1}^{n}c_kT_{a_k}\mu$. Consequently, by (1.1) and (1.3),

$$\widehat{\mu\nu}(t)=\sum_{k=1}^{n}c_k\hat{\mu}(a_jt)=\int_{0}^{\infty}\hat{\mu}(tx)\,\nu(dx)$$

which completes the proof.

PROPOSITION 1.4. *Let $\circ$ be an arbitrary generalized convolution with $\varkappa(\circ)=\varkappa$. Then each $\circ$-stable measure σ_p $(0<p\leqslant\varkappa(\circ))$ is of the form*

$$\sigma_p=\sigma_{\varkappa}\lambda_p,$$

where λ_p denotes the $*_{\varkappa}$-stable measure with exponent p.

Proof. Taking into account Example 1.4, we infer

$$\int_{0}^{\infty}\exp(-t^{\varkappa}x^{\varkappa})\lambda_p(dx)=\exp(-t^{p}),$$

$$\int_{0}^{\infty}\hat{\sigma}_{\varkappa}(tx)\lambda_p(dx)=\hat{\sigma}_p(t).$$

Now our assertion is a direct consequence of Proposition 1.3.

We apply Proposition 1.4 to the case $\circ=*_{\alpha,\beta}$. The Smirnov measure

$$\lambda_{\alpha}(E)=\alpha\sqrt{2/\pi}\int_{E}x^{-\alpha-1}\exp(-2^{-1}x^{-2\alpha})\,dx \tag{1.12}$$

is $*_{2\alpha}$-stable with exponent α. Using (1.6) we get by a simple computation the $*_{\alpha,\beta}$-stable measure σ_{α} with exponent α,

$$\sigma_{\alpha}(E)=\frac{2\Gamma(\beta)2^{\alpha-1/2}}{\sqrt{\pi}\,\Gamma(\beta-\frac{1}{2})}\int_{E}\frac{x^{2\alpha\beta-\alpha-1}}{(2+x^{2\alpha})^{\beta}}\,dx. \tag{1.13}$$

A measure μ from $\mathfrak{P}$ is said to be *cancellable* if the equation $\mu\nu=\mu\lambda$ yields $\nu=\lambda$. The following statement is obvious.

PROPOSITION 1.5. *All factors of a cancellable measure are cancellable too.*

Furthermore, Propositions 1.1 and 1.2 imply the following result.

PROPOSITION 1.6. *For cancellable measures ν the relation $\mu_n\nu\to\lambda$ yields the convergence of the sequence μ_n to a measure μ with the property $\mu\nu=\lambda$.*

PROPOSITION 1.7. *For every generalized convolution $\circ$ all $\circ$-stable measures σ_p $(0<p\leqslant\varkappa(\circ))$ are cancellable.*

Proof. Suppose that $\sigma_p\nu=\sigma_p\lambda$. Then, by (1.11),

$$\int_{0}^{\infty}\hat{\sigma}_p(tx)\nu(dx)=\int_{0}^{\infty}\hat{\sigma}_p(tx)\lambda(dx)$$

and, consequently,

$$\int_{0}^{\infty}\exp(-t^{p}x^{p})\nu(dx)=\int_{0}^{\infty}\exp(-t^{p}x^{p})\lambda(dx).$$

Now our assertion is a direct consequence of the Uniqueness Theorem for the Laplace transform.

2. Characteristic functions. The main aim of this section is to prove some uniqueness and characterization theorems for characteristic functions of generalized convolutions. We start by establishing some properties of linear operators induced by probability measures.

Let $\mathscr{B}_0$ be the space of all real-valued bounded and continuous at the origin Borel functions on R^+. For every $\mu \in \mathfrak{P}$ we define the linear operator L_μ on $\mathscr{B}_0$ by setting

$$L_\mu(f)(x) = \int_0^\infty f(xy)\mu(dy).$$

It is clear that the operators L_μ $(\mu \in \mathfrak{P})$ commute with one another and

$$L_\mu(f)(0) = f(0) \quad (\mu \in \mathfrak{P}). \tag{2.1}$$

Moreover, for any characteristic function $\hat{\mu}$ of a generalized convolution we have, by (1.11),

$$L_\mu(\hat{\nu}) = \widehat{\mu\nu} \quad (\mu, \nu \in \mathfrak{P}). \tag{2.2}$$

In particular,

$$L_\mu(\Omega) = \hat{\mu} \tag{2.3}$$

where Ω is the kernel of the characteristic function $\hat{\mu}$.

LEMMA 2.1. *Suppose that* $f, g \in \mathscr{B}_0$, $\mu, \nu \in \mathfrak{P}$,

$$f(0) = 0 \tag{2.4}$$

and

$$\int_0^\infty x^{-1}|L_\mu(g)(x)|\,dx < \infty, \quad \int_0^\infty x^{-1}|L_\nu(g)(x)|\,dx < \infty. \tag{2.5}$$

Then

$$\int_0^\infty x^{-1} L_\mu(g)(x) L_\nu(f)(x^{-1})\,dx = \int_0^\infty \int_0^\infty x^{-1} f(x^{-1}) L_\nu(g)(xy)\,dx\mu(dy). \tag{2.6}$$

Proof. By (2.5) we have

$$\int_0^\infty x^{-1} L_\mu(g)(x) L_\nu(f)(x^{-1})\,dx = \int_0^\infty \int_0^\infty z^{-1} L_\mu(g)(z) f(z^{-1}y)\nu(dy)\,dz.$$

Taking into account (2.4) and setting $z = xy$ $(y > 0)$, we get the formula

$$\int_0^\infty x^{-1} L_\mu(g)(x) L_\nu(f)(x^{-1})\,dx = \int_0^\infty \int_0^\infty x^{-1} L_\mu(g)(xy) f(x^{-1})\nu(dydx)$$
$$= \int_0^\infty x^{-1} L_\nu(L_\mu(g))(x) f(x^{-1})\,dx$$

which, by the commutability of L_ν and L_μ, yields the assertion of the lemma.

LEMMA 2.2. *Let* σ_p $(0 < p \leqslant \varkappa(\circ))$ *be a* $\circ$*-stable measure. If* $f \in \mathscr{B}_0$ *and* $L_{\sigma_p}(f) = 0$ *almost everywhere with respect to the Lebesgue measure on* R^+, *then* $f = 0$ *almost everywhere with respect to the Lebesgue measure on* R^+.

Proof. Put $\mu_n = \delta_1 \circ T_{1/n}\sigma_p$ $(n = 1, 2, \ldots)$ and

$$g_t(x) = \Omega(t^{1/p}x) - \Omega((t+1)^{1/p}x) \quad (t \in R^+)$$

where Ω is the kernel corresponding to the characteristic function of $\circ$. It is easy to verify the formulas

$$\begin{aligned} L_{\sigma_p}(g_t)(x) &= \exp(-tx^p) - \exp(-(t+1)x^p), \\ L_{\mu_n}(g_t)(x) &= \Omega(tx)\exp(-n^{-p}tx^p) - \Omega((t+1)x)\exp(-n^{-p}(t+1)x^p) \end{aligned} \tag{2.7}$$

which, by (1.5), yield the inequalities

$$x^{-1}|L_{\sigma_p}(g_t)(x)| \leqslant cx^{p-1}\exp(-tx^p),$$
$$x^{-1}|L_{\mu_n}(g_t)(x)| \leqslant c(x^{p-1} + x^{\varkappa-1})\exp(-n^{-p}tx^p),$$

where $\varkappa = \varkappa(\circ)$ and c is a positive constant. Moreover, by (2.1), $f(0) = L_{\sigma_p}(f)(0) = 0$. Thus, setting $\mu = \mu_n$, $\nu = \sigma_p$ and $g = g_t$, we infer that the conditions of Lemma 2.1 are fulfilled. By the assumption, the left-hand side of (2.6) is equal to 0. Consequently,

$$\int_0^\infty \int_0^\infty x^{-1} f(x^{-1}) L_{\sigma_p}(g_t)(xy)\,dx\mu_n(dy) = 0 \quad (n = 1, 2, \ldots, t \in R^+).$$

Since, by (2.7), the function

$$\int_0^\infty x^{-1} f(x^{-1}) L_{\sigma_p}(g_t)(xy)\,dx$$

is continuous and $\mu_n \to \delta_1$, the last equation yields, as $n \to \infty$,

$$\int_0^\infty x^{-1} f(x^{-1}) L_{\sigma_p}(g_t)(x)\,dx = 0 \quad (t \in R^+).$$

Thus, by (2.7),

$$\int_0^\infty x^{-1} f(x^{-1})(1 - e^{-x^p})e^{-tx^p}\,dx = 0 \quad (t \in R^+)$$

which, by the Uniqueness Theorem for the Laplace transform implies the assertion of the lemma.

We can now formulate a result on the uniqueness of the characteristic function, which plays a crucial role in our considerations.

THEOREM 2.1. *All kernels corresponding to characteristic functions of a generalized convolution are similar.*

Proof. Let Ω and Ω' be two kernels corresponding to characteristic functions of the generalized convolution in question. Passing if necessary to similar kernels we may assume without loss of generality that for the characteristic measure $\sigma_\varkappa$ the equation

$$\int_0^\infty \Omega(tx)\sigma_\varkappa(dx) = \int_0^\infty \Omega'(tx)\sigma_\varkappa(dx) = \exp(-t^\varkappa)$$

is true. In other words, we have the equation $L_{\sigma_\varkappa}(\Omega) = L_{\sigma_\varkappa}(\Omega')$ which by Lemma 2.2 and continuity of both kernels Ω and Ω' yields $\Omega = \Omega'$. The theorem is thus proved.

Theorem 2.1 enables us to associate with every generalized convolution $\circ$ the set

$$\mathscr{C}(\circ) = \{\hat{\mu}\colon \mu \in \mathfrak{P}\}$$

which does not depend upon the choice of a characteristic function. Taking into account Example 1.4, we obtain the inclusion

(2.8) $$\mathscr{C}(*_\alpha) \subset \mathscr{C}(\circ) \quad (0 < \alpha \leqslant \varkappa(\circ)).$$

THEOREM 2.2. *$\mathscr{C}(\circ) = \mathscr{C}(\circ')$ if and only if $\circ = \circ'$.*

Proof. Let Ω and Ω' be the kernels of characteristic functions of $\circ$ and $\circ'$, respectively. Suppose that $\mathscr{C}(\circ) = \mathscr{C}(\circ')$. Then $\Omega \in \mathscr{C}(\circ')$ and $\Omega' \in \mathscr{C}(\circ)$ or, equivalently,

$$\Omega(t) = \int_0^\infty \Omega'(tx)\mu(dx), \quad \Omega'(t) = \int_0^\infty \Omega(tx)\nu(dx)$$

for some measures $\mu, \nu \in \mathfrak{P}$. Hence it follows that

$$\Omega(t) = \int_0^\infty\int_0^\infty \Omega(txy)\mu(dx)\nu(dy)$$

or, in other words, $\delta_1 = \mu\nu$. The last equation yields $\mu = \delta_c$ for a certain positive number c. Thus $\Omega(t) = \Omega'(ct)$ which implies the equation $\circ = \circ'$.

From Theorem 2.1 it follows that the generalized convolution is completely described by its characteristic exponent and characteristic measure. More precisely, we have the following theorem.

THEOREM 2.3. *If $\varkappa(\circ) = \varkappa(\circ')$ and the characteristic measures of $\circ$ and $\circ'$ are similar, then $\circ = \circ'$.*

Proof. Suppose that $\varkappa = \varkappa(\circ) = \varkappa(\circ')$ and $\sigma_\varkappa$ is the characteristic measure of $\circ$ and $\circ'$ simultaneously. Then

$$\int_0^\infty \Omega(tx)\sigma_\varkappa(dx) = \int_0^\infty \Omega'(tx)\sigma_\varkappa(dx) = \exp(-t^\varkappa)$$

for suitably chosen kernels Ω and Ω' of characteristic functions of $\circ$ and $\circ'$, respectively. Consequently, $L_{\sigma_\varkappa}(\Omega) = L_{\sigma_\varkappa}(\Omega')$, which, by Theorem 2.1 and the continuity of Ω and Ω', yields the equation $\Omega = \Omega'$. Thus $\mathscr{C}(\circ) = \mathscr{C}(\circ')$, which, by Theorem 2.2, completes the proof.

We proceed now to a description of the set $\mathscr{C}(\circ)$ in terms of $\circ$-stable measures.

LEMMA 2.3. *Let σ_p $(0 < p \leqslant \varkappa(\circ))$ be a $\circ$-stable measure with exponent p. Let $f \in \mathscr{B}_0$ and $f(0) = 1$. If the function $L_{\sigma_p}(f)(t^{1/p})$ is completely monotone, then there exists a function $f_0 \in C(\circ)$ such that $f = f_0$ almost everywhere with respect to the Lebesgue measure on R^+.*

Proof. By the Bernstein Theorem $L_{\sigma_p}(f)(t^{1/p}) = \int_0^\infty \exp(-tx^p)\mu(dx)$ for a certain $\mu \in \mathfrak{P}$. Put $f_0 = \hat{\mu} = L_\mu(\Omega)$. Then, by the commutability of L_{σ_p} and L_μ we have $L_{\sigma_p}(f_0) = L_\mu(L_{\sigma_p}(\Omega)) = L_\mu(\hat{\sigma}_p) = \int_0^\infty e^{-t^p x^p}\mu(dx)$. Consequently, $L_{\sigma_p}(f) = L_{\sigma_p}(f_0)$ which, by Theorem 2.1, yields the assertion of the lemma.

LEMMA 2.4. *Let σ_p $(0 < p \leqslant \varkappa(\circ))$ be a $\circ$-stable measure with exponent p. If $f \in \mathscr{C}(\circ)$, then the function $L_{\sigma_p}(f)(t^{1/p})$ is completely monotone.*

Proof. By the assumption and (2.3) $f = \hat{\mu} = L_\mu(\Omega)$ for a certain $\mu \in \mathfrak{P}$. Thus

$$L_{\sigma_p}(f)(t) = L_{\sigma_p}(L_\mu(\Omega))(t) = L_\mu(L_{\sigma_p}(\Omega))(t) = \int_0^\infty \exp(-t^p x^p)\mu(dx)$$

which completes the proof.

As a consequence of Lemmas 2.3 and 2.4 we get the following characterization of the set $\mathscr{C}(\circ)$.

THEOREM 2.4. *Let σ_p $(0 < p < \varkappa(\circ))$ be a $\circ$-stable measure with exponent p. Let $f \in \mathscr{C}_b$ and $f(0) = 1$. Then $f \in \mathscr{C}(\circ)$ if and only if the function $\int_0^\infty f(t^{1/p}x)\sigma_p(dx)$ is completely monotone.*

We shall now illustrate the above theorem by some examples.

EXAMPLE 2.1. *The ordinary convolution $*_1$.* Taking the Smirnov measure $\lambda_{1/2}$ defined by (1.12), we infer that a continuous function f with $f(0) = 1$ is in $\mathscr{C}(*_1)$ or, in other words, is completely monotone if and only if the function $\int_0^\infty f(t^2 x)x^{-3/2}\exp(-(2x)^{-1})dx$ is completely monotone.

EXAMPLE 2.2. *The convolution $*_{1,\beta}$ $(\beta \geqslant 1)$.* Taking the characteristic measure σ_2 defined by (1.6), we conclude that a continuous function f with $f(0) = 1$ admits a representation

(2.9) $$f(t) = \Gamma\left(\frac{\beta}{2}\right)\left(\frac{2}{t}\right)^{\beta/2-1}\int_0^\infty J_{\beta/2-1}(tx)x^{1-\beta/2}\mu(dx)$$

where $\mu \in \mathfrak{P}$ if and only if the function $\int_0^\infty f(t^{1/2}x)x^{2\beta-2}\exp(x^2/4)dx$ is completely monotone. Further, taking the measure σ_1 defined by (1.13), we infer that (2.9) is equivalent to the complete monotonicity of the function $\int_0^\infty f(tx)x^{2\beta-2}(2+x^2)^{-\beta}dx$.

EXAMPLE 2.3. *The convolution* $\circ_{1,n}$ $(n = 1, 2, \ldots)$. Taking the characteristic measure σ_1 defined by (1.7), we conclude that a continuous function f with $f(0) = 1$ admits a representation

$$f(t) = \int_0^{1/t} (1-tx)^n \mu(dx)$$

if and only if the function $\int_0^\infty f(tx)x^{-n-2}\exp(-x^{-1})dx$ is completely monotone.

3. Representability of generalized convolutions. Let us consider two generalized convolutions $\circ$ and $\circ'$. The convolution $\circ$ is said to be *representable in* $\circ'$, in symbols $\circ \prec \circ'$, if there exists a continuous map h: $\mathfrak{P} \to \mathfrak{P}$ with the properties

$$h(\mu) \not\equiv \delta_0, \tag{3.1}$$

$$h(c\mu+(1-c)\nu) = ch(\mu)+(1-c)h(\nu) \quad (0 \leqslant c \leqslant 1), \tag{3.2}$$

$$h(\mu\circ\nu) = h(\mu)\circ' h(\nu), \tag{3.3}$$

$$h(T_a\mu) = T_a h(\mu) \tag{3.4}$$

for all $\mu, \nu \in \mathfrak{P}$. Some of the most important properties of the map h are summed up in the following lemma.

LEMMA 3.1. *Suppose that h realizes the relation $\circ \prec \circ'$. Then*

$$h(\mu) = h(\delta_1)\mu \quad (\mu \in \mathfrak{P}), \tag{3.5}$$

the measure $h(\delta_1)$ is cancellable, for any $\circ$-stable measure σ_p $(0 < p \leqslant \varkappa(\circ))$ the measure $h(\sigma_p)$ is $\circ'$-stable with the same exponent p and if $\mu \to \hat{\mu}'$ is a characteristic function of $\circ'$, then $\mu \to \widehat{h}(\mu)'$ is a characteristic function of $\circ$.

Proof. If a measure μ is of the form

$$\mu = \sum_{j=1}^n c_j \delta_{a_j}, \tag{3.6}$$

where $c_j, a_j \in R^+$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^n c_j = 1$, then, by (1.2), (1.3), (3.2) and (3.4) we have the formula

$$h(\mu) = \sum_{j=1}^n c_j T_{a_j} h(\delta_1) = \sum_{j=1}^n c_j h(\delta_1)\delta_{a_j} = h(\delta_1)\mu.$$

Since the measures (3.6) form a dense subset of $\mathfrak{P}$, we obtain, by (1.5), formula (3.5) for all $\mu \in \mathfrak{P}$. Since $h(\mu) \not\equiv \delta_0$, formula (3.5) yields

$$h(\delta_1) \neq \delta_0. \tag{3.7}$$

Let σ_p be a $\circ$-stable measure and $T_{a_n}\nu^{\circ n} \to \sigma_p$ for some constants a_n and $\nu \in \mathfrak{P}$. Then, by the continuity of h, (3.3) and (3.4), we have $T_{a_n} h(\nu)^{\circ' n} \to h(\sigma_p)$. Thus the measure $h(\sigma_p)$ is $\circ'$-stable and by (3.5), $h(\sigma_p) = h(\delta_1)\sigma_{p'}$, which by (3.7) yields $h(\sigma_p) \neq \delta_0$. Consequently, $h(\sigma_p)$ is non-degenerate. The measure $h(\delta_1)$, being a factor of $h(\sigma_p)$, is cancellable by Propositions 1.5 and 1.7. Thus, by (3.5), the map $\mu \to h(\mu)$ is one-to-one. Consequently the map $\mu \to \widehat{h}(\mu)'$ is also one-to-one. Evidently, it is continuous and, by (3.2), (3.3) and (3.4), fulfils conditions (1.1), (1.2) and (1.3) for characteristic functions. Moreover, by (3.5) and Proposition 1.6, the inverse map $\widehat{h}(\mu)' \to \mu$ is also continuous. In other words, the map $\mu \to \widehat{h}(\mu)'$ is a characteristic function of the convolution $\circ$. By the Uniqueness Theorem 2.1, $\widehat{h}(\sigma_p)' = e^{-ct^p}$. Since $h(\sigma_p) \neq \delta_0$, we conclude that $h(\sigma_p)$ has the exponent p. The lemma is thus proved.

THEOREM 3.1. *The following conditions are equivalent*:

$$\circ \prec \circ', \tag{3.8}$$

$$\varkappa(\circ) \leqslant \varkappa(\circ') \quad \textit{and} \quad \sigma_p' = \sigma_p \lambda \tag{3.9}$$

for some $\lambda \in \mathfrak{P}$ and $0 < p \leqslant \varkappa(\circ)$, where σ_p, σ_p' are $\circ$-stable and $\circ'$-stable measures with exponent p, respectively.

$$\mathscr{C}(\circ) \subset \mathscr{C}(\circ'). \tag{3.10}$$

Moreover, if (3.9) holds for a certain p, then it holds for all p $(0 < p \leqslant \varkappa(\circ))$.

Proof. Suppose that $\circ \prec \circ'$. Then, by Lemma 3.1, for every positive number $p \leqslant \varkappa(\circ)$ the measure $h(\sigma_p)$ is $\circ'$-stable with exponent p. Consequently, $\varkappa(\circ) \leqslant \varkappa(\circ')$. Moreover, by (3.5), $h(\sigma_p) = h(\delta_1)\sigma_p$ which yields (3.9) for all p $(0 < p \leqslant \varkappa(\circ))$.

Suppose now that (3.9) is fulfilled for a certain p $(0 < p \leqslant \varkappa(\circ))$. Passing to similar kernels if necessary, we may assume without loss of generality that $\hat{\sigma}_p = \hat{\sigma}_p'$ where $\mu \to \hat{\mu}$ and $\mu \to \hat{\mu}'$ are characteristic functions of $\circ$ and $\circ'$, respectively. Let Ω be the kernel of the characteristic function $\mu \to \hat{\mu}$. Then, by (2.3), $L_{\sigma_p}(\Omega) = \hat{\sigma}_p$ and $L_{\sigma_p}(\hat{\lambda}') = \widehat{\sigma_p\lambda}' = \hat{\sigma}_p'$ which implies the equation $L_{\sigma_p}(\Omega) = L_{\sigma_p}(\hat{\lambda}')$. Thus, by Lemma 2.2, $\Omega = \hat{\lambda}'$, which yields $\Omega \in \mathscr{C}(\circ')$ and, consequently, $\mathscr{C}(\circ) \subset \mathscr{C}(\circ')$.

Finally, let us assume that $\mathscr{C}(\circ) \subset \mathscr{C}(\circ')$. Let Ω and Ω' be the kernels of characteristic functions $\mu \to \hat{\mu}$ and $\mu \to \hat{\mu}'$ of $\circ$ and $\circ'$, respectively. Then

$$\Omega(t) = \int_0^\infty \Omega'(tx)\lambda(dx) \tag{3.11}$$

for a certain $\lambda \in \mathfrak{P}$. Of course, $\lambda \neq \delta_0$. Put $h(\mu) = \lambda\mu$ ($\mu \in \mathfrak{P}$). The map h fulfils condition (3.1). Moreover, by (1.8), (1.10) it fulfils conditions (3.2) and (3.4). The continuity of h is a consequence of Proposition 1.1. From (3.11) we get the formula $\hat{\mu} = \widehat{\lambda\mu'} = \hat{h}(\mu)'$ which for all $\mu, \nu \in \mathfrak{P}$ yields

$$\hat{h}(\mu \circ \nu)' = \widehat{\mu \circ \nu} = \hat{\mu}\hat{\nu} = \hat{h}(\mu)'\,\hat{h}(\nu)' = \hat{h}(\mu) \circ' \hat{h}(\nu)'.$$

Consequently, h fulfils condition (3.3). In other words, the map h defines the relation $\circ \prec \circ'$ which completes the proof of the theorem.

As a direct consequence of Theorems 2.2 and 3.1 we get the following result.

COROLLARY 3.1. *The relation $\prec$ is transitive. If $\circ \prec \circ'$ and $\circ' \prec \circ$, then $\circ = \circ'$.*

As an immediate consequence of inclusion (2.8) and Theorem 3.1 we get the following property.

PROPOSITION 3.1. *The convolution $*_\alpha$ is representable in a generalized convolution $\circ$ if and only if $\alpha \leqslant \varkappa(\circ)$.*

4. Examples of representability of $\circ_{\alpha,n}$ and $*_{\alpha,\beta}$. Theorem 3.1 gives useful criterions for the representability of generalized convolutions. We shall illustrate this by some examples of the representability of generalized convolutions $\circ_{\alpha,n}$ and $*_{\alpha,\beta}$.

LEMMA 4.1. *Let $\circ$ be an arbitrary generalized convolution with $\varkappa(\circ) \geqslant \alpha$. Suppose that the $\circ$-stable measure with exponent α has the density d_α. Then $\circ_{\alpha,n} \prec \circ$ if and only if the function $d_\alpha(t^{-1/\alpha})t^{-1/\alpha-n-1}$ is completely monotone for $t > 0$.*

Proof. Let $\sigma_{\alpha,n}$ be the characteristic measure of $\circ_{\alpha,n}$. Denoting by ν_α the $\circ$-stable measure with the density d_α, we conclude, by Theorem 3.1, that $\circ_{\alpha,n} \prec \circ$ if and only if $\nu_\alpha = \sigma_{\alpha,n}\lambda$ for a certain $\lambda \in \mathfrak{P}$. Taking into account (1.7) and writing the above criterion in terms of densities, we have $\circ_{\alpha,n} \prec \circ$ if and only if

$$d_\alpha(x) = \frac{\alpha}{n!}\int_0^\infty \frac{y^{\alpha(n+1)}}{x^{1+\alpha(n+1)}} e^{-y^\alpha/x^\alpha}\lambda(dy).$$

Substituting $x^{-\alpha} = t$, we get an equivalent criterion: $\circ_{\alpha,n} \prec \circ$ if and only if

$$d_\alpha(t^{-1/\alpha})t^{-1/\alpha-n-1} = \frac{\alpha}{n!}\int_0^\infty y^{\alpha(n+1)} e^{-ty^\alpha}\lambda(dy)$$

for $t > 0$, whence, by the Bernstein representation theorem, the assertion of the lemma follows.

Putting $\circ = \circ_{\alpha,n-1}$ ($n \geqslant 2$) into Lemma 4.1, we observe that $\varkappa(\circ_{\alpha,n}) = \varkappa(\circ_{\alpha,n-1}) = \alpha$ and in this case d_α is the density of the characteristic measure of $\circ_{\alpha,n-1}$. Consequently, by (1.7),

$$d_\alpha(t^{-1/\alpha})t^{-1/\alpha-n-1} = ct^{-1}e^{-t}$$

where c is a positive constant. Since $t^{-1}e^{-t}$ is completely monotone for $t > 0$, we have by Lemma 4.1 the following corollary.

COROLLARY 4.1. $\circ_{\alpha,n+1} \prec \circ_{\alpha,n}$ ($\alpha > 0$, $n = 1, 2, \ldots$).

It is convenient to introduce the operators U_γ ($\gamma > 0$) defined on $\mathscr{C}_b$ by

$$(U_\gamma f)(x) = \gamma x^\gamma \int_x^\infty f(y)y^{-\gamma-1}\,dy.$$

It is clear that $U_\gamma(\mathscr{C}_b) \subset \mathscr{C}_b$ and $(Uf)(0) = f(0)$ ($f \in \mathscr{C}_b$). Moreover, by integrating by parts it is easy to establish the following formula:

$$\int_0^\infty (U_{\alpha(n+1)}f)(t^{1/\alpha}x)\,\sigma_{\alpha,n+1}(dx) = \int_0^\infty f(t^{1/\alpha}x)\,\sigma_{\alpha,n}(dx) \qquad (\alpha > 0,\ n = 1, 2, \ldots)$$

for all $f \in \mathscr{C}_b$, where $\sigma_{\alpha,n}$ is the characteristic measure of $\circ_{\alpha,n}$. This proves, by Theorem 2.4, the following result.

LEMMA 4.2. *Let $f \in \mathscr{C}_b$ and $f(0) = 1$. Then $f \in \mathscr{C}(\circ_{\alpha,n})$ if and only if $U_{\alpha(n+1)}f \in \mathscr{C}(\circ_{\alpha,n+1})$ ($\alpha > 0$, $n = 1, 2, \ldots$).*

We can now formulate the following corollary.

COROLLARY 4.2. $\circ_{\alpha,n} \prec \circ_{\beta,n}$ ($\alpha \leqslant \beta$, $n = 1, 2, \ldots$).

Proof. We shall prove our corollary by induction with respect to n. Let $\Omega_{\alpha,n}$ denote the kernel of the characteristic function of $\circ_{\alpha,n}$ defined in Example 1.6. Setting for $\alpha \leqslant \beta$

$$\nu_{\alpha,\beta}(E) = \frac{\alpha}{\beta}\delta_1(E) + \frac{\alpha}{\beta}(\beta-\alpha)\int_{E\cap[1,\infty)} x^{-\alpha-1}\,dx,$$

we have the formula

$$\Omega_{\alpha,1}(t) = \int_0^\infty \Omega_{\beta,1}(tx)\,\nu_{\alpha,\beta}(dx)$$

which shows that $\mathscr{C}(\circ_{\alpha,1}) \subset \mathscr{C}(\circ_{\beta,1})$ and consequently, by Theorem 3.1, $\circ_{\alpha,1} \prec \circ_{\beta,1}$.

Suppose now that the relation $\circ_{\alpha,n} \prec \circ_{\beta,n}$ is valid. Put

$$g = \frac{\alpha}{\beta}\Omega_{\alpha,n} + \left(1 - \frac{\alpha}{\beta}\right)\Omega_{\alpha,n+1}.$$

We have, by Corollary 4.1 and Theorem 3.1, $g \in \mathscr{C}(\circ_{\alpha,n})$ which implies $g \in \mathscr{C}(\circ_{\beta,n})$. Now, applying Lemma 4.2, we get $U_{\beta(n+1)}g \in \mathscr{C}(\circ_{\beta,n+1})$. But, by a simple computation, $U_{\beta(n+1)} g = \Omega_{\alpha,n+1}$, which yields $\mathcal{C}(\circ_{\alpha,n+1}) \subset \mathcal{C}(\circ_{\beta,n+1})$ or in other words, by Theorem 3.1, $\circ_{\alpha,n+1} \prec \circ_{\beta,n+1}$. This completes the proof.

Further, as a direct consequence of the formula $\varkappa(\circ_{\alpha,n}) = \alpha$ and Theorem 3.1 we have the following result.

COROLLARY 4.3. *If* $\circ_{\alpha,n} \prec \circ_{\beta,m}$, *then* $\alpha \leqslant \beta$.

We next observe that the relation $\circ_{\alpha,n} \prec \circ_{\beta,m}$ yields the formula

$$\Omega_{\alpha,n}(t) = \int_0^\infty \Omega_{\beta,m}(tx)\,\lambda(dx)$$

for a certain $\lambda \in \mathfrak{P}$. This expression can be written in the form

$$\Omega_{\alpha,n}(t) = \int_0^{1/t} (1-t^\beta x^\beta)^m\,\lambda(dx)$$

which shows that $\lambda([0,1)) = 0$. Consequently, for $0 < t < 1$

$$(1-t^\alpha)^n = \int_0^{1/t} (1-t^\beta x^\beta)^m\,\lambda(dx) \leqslant (1-t^\beta)^m.$$

But this inequality is valid in the case $n \geqslant m$ only. Thus we obtain the following implication:

COROLLARY 4.4. *If* $\circ_{\alpha,n} \prec \circ_{\beta,m}$, *then* $n \geqslant m$.

Finally, by Corollaries 4.1–4.4, we obtain the following result.

PROPOSITION 4.1. *The generalized convolution* $\circ_{\alpha,n}$ *is representable in the generalized convolution* $\circ_{\beta,m}$ *if and only if* $\alpha \leqslant \beta$ *and* $n \geqslant m$.

We now proceed to the study of the generalized convolutions $*_{\alpha,\beta}$. We shall use the notation $\Omega_{(\alpha,\beta)}$ and $\sigma_{(\alpha,\beta)}$ for the kernel of the characteristic function and the characteristic measure of $*_{\alpha,\beta}$ defined in Example 1.5, respectively.

LEMMA 4.3. *Let* $\circ$ *be an arbitrary generalized convolution with* $\varkappa(\circ) \geqslant 2\alpha$. *Suppose that the* $\circ$*-stable measure with exponent* 2α *has the density* $d_{2\alpha}$. *Then* $*_{\alpha,\beta} \prec \circ$ *if and only if the function* $d_{2\alpha}(t^{1/2\alpha})\,t^{1/2+1/2\alpha-\beta}$ *is completely monotone for* $t > 0$.

Proof. Denoting by $\nu_{2\alpha}$ the $\circ$-stable measure with the density $d_{2\alpha}$, we conclude by Theorem 3.1 that $*_{\alpha,\beta} \prec \circ$ if and only if $\nu_{2\alpha} = \sigma_{(\alpha,\beta)}\lambda$ for a certain $\lambda \in \mathfrak{P}$. Taking into account (1.6) and writing the above criterion in terms of densities, we have $*_{\alpha,\beta} \prec \circ$ if and only if

$$d_{2\alpha}(x) = \frac{\alpha}{2^{2\beta-2}\Gamma(\beta-1/2)} \int_0^\infty \frac{x^{2\alpha\beta-\alpha-1}}{y^{2\alpha\beta-\alpha}} e^{-x^{2\alpha}/4y^{2\alpha}}\,\lambda(dy).$$

Substituting $x^{2\alpha} = t$, we get an equivalent criterion: $*_{\alpha,\beta} \prec \circ$ if and only if

$$d_{2\alpha}(t^{1/2\alpha})\,t^{1/2+1/2\alpha-\beta} = \frac{\alpha}{2^{2\beta-2}\Gamma(\beta-1/2)} \int_0^\infty y^{\alpha-2\alpha\beta} e^{-t/4y^{2\alpha}}\,\lambda(dy)$$

for $t > 0$, whence by the Bernstein Representation Theorem the assertion of the lemma follows.

Putting $\circ = *_{\alpha,\gamma}$ $(\gamma < \beta)$ into Lemma 4.3, we observe that $\varkappa(*_{\alpha,\gamma}) = \varkappa(*_{\alpha,\beta}) = 2\alpha$ and in this case $d_{2\alpha}$ is the density of $\sigma_{(\alpha,\gamma)}$. Consequently, by (1.6),

$$d_{2\alpha}(t^{1/2\alpha})\,t^{1/2+1/2\alpha-\beta} = c t^{\gamma-\beta} e^{-t/4}$$

where c is a positive constant. Since $t^{\gamma-\beta}e^{-t/4}$ is completely monotone for $\gamma < \beta$ and $t > 0$, we have by Lemma 4.3 the following result.

PROPOSITION 4.2. *If* $\gamma < \beta$, *then for every* α $*_{\alpha,\beta} \prec *_{\alpha,\gamma}$.

We note that from the Sonine integral ([1], 7.7 (5)) we get

$$\Omega_{(\alpha,\beta)}(t) = \frac{2\alpha\Gamma(\beta-1/2)}{\Gamma(\gamma-1/2)\Gamma(\beta-\gamma)} \int_0^1 \Omega_{(\alpha,\gamma)}(tx)\,x^{2\alpha\gamma-\alpha-1}(1-x^{2\alpha})^{\beta-\gamma-1}\,dx$$

for $\alpha > 0$ and $\beta > \gamma$. This shows that $\mathcal{C}(*_{\alpha,\beta}) \subset \mathcal{C}(*_{\alpha,\gamma})$ provided $\beta > \gamma$. Applying Theorem 3.1, we get an alternative proof of Proposition 4.2.

For the further discussion we need a lemma. For any $f \in \mathcal{C}_b$ we put

$$m(f) = \inf\{f(x):\ x \in R^+\}.$$

LEMMA 4.4. *Let* Ω *and* Ω' *be the kernels of characteristic functions of* $\circ$ *and* $\circ'$, *respectively. Suppose that* $\circ \prec \circ'$, $m(\Omega) = m(\Omega') = m$, *the set* $\{x:\ \Omega(x) = m\}$ *is non-void and the set* $\{x:\ \Omega'(x) = m\}$ *is bounded. Then* $\varkappa(\circ) = \varkappa(\circ')$.

Proof. Since $\circ \prec \circ'$, we have, by virtue of Theorem 3.1,

$$\Omega(t) = \int_0^\infty \Omega'(tx)\,\nu(dx) \tag{4.1}$$

for a certain $\nu \in \mathfrak{P}$. Evidently, $m < 1$, because the kernel Ω is not identically equal to 1. Therefore $\Omega(t_0) = m$ for a certain $t_0 > 0$. Then

$$\int_0^\infty (\Omega'(t_0 x) - m)\,\nu(dx) = 0,$$

which yields $\Omega'(t_0 x) = m$ ν-almost everywhere. In other words, by the continuity of Ω', the support of ν is contained in $t_0^{-1}\{x\colon \Omega'(x) = m\}$ and, consequently, is bounded. This, by (1.5) and (4.1), yields the formula

$$\lim_{t\to 0+} \frac{1-\Omega(tx)}{1-\Omega(t)} = x^{\varkappa(\circ')},$$

whence the equation $\varkappa(\circ) = \varkappa(\circ')$ follows.

Proposition 4.3. *Suppose that $\gamma > 1$. The relation $*_{\alpha,\gamma} \prec *_{\beta,\gamma}$ holds if and only if $\alpha = \beta$.*

Proof. Of course, it is enough to prove the necessity of the condition. It is clear that $m(\Omega_{(\alpha,\gamma)}) = m(\Omega_{(\beta,\gamma)}) = m < 0$ and $\lim_{t\to\infty} \Omega_{(\alpha,\gamma)}(t) = \lim_{t\to\infty} \Omega_{(\beta,\gamma)}(t) = 0$. Hence it follows that both sets $\{x\colon \Omega_{(\alpha,\gamma)}(x) = m\}$ and $\{x\colon \Omega_{(\beta,\gamma)}(x) = m\}$ are non-void and bounded. By Lemma 4.4, the relation $*_{\alpha,\gamma} \prec *_{\beta,\gamma}$ implies the equation $2\alpha = \varkappa(*_{\alpha,\gamma}) = \varkappa(*_{\beta,\gamma}) = 2\beta$ which completes the proof.

Proposition 4.4. *If $*_{\alpha,\gamma} \prec *_{\beta,\delta}$, then $\alpha \leqslant \beta$ and $\gamma \geqslant \delta$.*

Proof. The first inequality is a direct consequence of the formula $\varkappa(*_{\alpha,\beta}) = \alpha$ and Theorem 3.1. Suppose that

$$*_{\alpha,\gamma} \prec *_{\beta,\delta} \tag{4.2}$$

and

$$\gamma < \delta \tag{4.3}$$

Then, by Proposition 4.2, we have

$$*_{\alpha,\delta} \prec *_{\alpha,\gamma}, \tag{4.4}$$

which implies the relation $*_{\alpha,\delta} \prec *_{\beta,\delta}$. Taking into account Proposition 4.3, we infer that $\alpha = \beta$. Now, by Corollary 3.1, relations (4.2) and (4.4) imply $*_{\alpha,\gamma} = *_{\alpha,\delta}$. Consequently, $\gamma = \delta$, which contradicts (4.3). The proposition is thus proved.

We define the family of operators V_γ $(\gamma > 0)$ on $\mathscr{B}_0$ by setting

$$(V_\gamma f)(x) = \gamma x^{-\gamma} \int_0^x y^{\gamma-1} f(y)\, dy.$$

It is clear that $V(\mathscr{B}_0) \subset \mathscr{C}_b$ and $(V_\gamma f)(0) = f(0)$ $(f \in \mathscr{B}_0)$. Moreover, by integrating by parts it is easy to verify the formula

$$\int_0^\infty (V_{2\alpha\beta-\alpha} f)(t^{1/\alpha} x)\, \sigma_{(\alpha,\beta+1)}(dx) = \int_0^\infty f(t^{1/\alpha} x)\, \sigma_{(\alpha,\beta)}(dx) \qquad (\alpha > 0,\ \beta \geqslant 1)$$

for all $f \in \mathscr{B}_0$, where $\sigma_{(\alpha,\beta)}$ denotes the characteristic measure of $*_{\alpha,\beta}$ defined by (1.6). This proves, by Lemma 2.3 and Theorem 2.4, the following statement.

Lemma 4.5. *Let $f \in \mathscr{B}_0$ and $f(0) = 1$. Then $V_{2\alpha\beta-\alpha} f \in \mathscr{C}(*_{\alpha,\beta+1})$ if and only if there exists a function $f_0 \in \mathscr{C}(*_{\alpha,\beta})$ such that $f = f_0$ almost everywhere with respect to the Lebesgue measure on R^+.*

Proposition 4.5. *The generalized convolution $\circ_{\beta,n}$ is not representable in the generalized convolution $*_{\alpha,n+1}$ for any $n = 1, 2, \ldots$ and $\alpha, \beta > 0$.*

Proof. Let us suppose the contrary, i.e.

$$\circ_{\beta,n} \prec *_{\alpha,n+1} \tag{4.5}$$

for a triple α, β, n. Put

$$g(x) = \pi^{1/2} 2^{-n} \alpha^{-n} x^\alpha \Gamma(n+1/2)^{-1} \left(x^{1-2\alpha} \frac{d}{dx}\right)^n (x^{2n\alpha-\alpha}(1-x^\beta)^n)$$

in the interval $0 \leqslant x < 1$ and $g(x) = 0$ otherwise. It is easy to check by a simple computation that g is continuous in the interval $0 \leqslant x < 1$, $g(0) = 1$ and $\lim_{x\to 1-} g(x) = (-\beta)^n \alpha^{-n} 2^{-n} n!\, \pi^{1/2} \Gamma(n+1/2)^{-1}$. Thus $g \in B_0$ and g is not equal almost everywhere with respect to the Lebesgue measure on R^+ to a continuous function. Moreover, by a simple computation we get the formula

$$(V_{(2n-1)\alpha} V_{(2n-3)\alpha} \ldots V_\alpha g)(x) = (1-x^\beta)^n$$

in the interval $0 \leqslant x < 1$ and

$$(V_{(2n-1)\alpha} V_{(2n-3)\alpha} \ldots V_\alpha g)(x) = 0$$

otherwise. In other words,

$$V_{(2n-1)\alpha} V_{(2n-3)\alpha} \ldots V_\alpha g = \Omega_{\beta,n},$$

where $\Omega_{\beta,n}$ is the kernel of the characteristic function of $\circ_{\beta,n}$. Consequently, by (4.5) and Theorem 3.1,

$$V_{(2n-1)\alpha} V_{(2n-3)\alpha} \ldots V_\alpha g \in \mathscr{C}(*_{\alpha,n+1}).$$

Now applying Lemma 4.5 n times, we conclude that there exists a function $g_0 \in \mathscr{C}(*_{\alpha,1})$ such that $g = g_0$ almost everywhere with respect to the Lebesgue measure on R^+. But $\mathscr{C}(*_{\alpha,1}) \subset \mathscr{C}_b$ which yields the contradiction. The proposition is thus proved.

Put $\circ = *_{\alpha,\beta}$ in Lemma 4.1. We know that $\varkappa(*_{\alpha,\beta}) = 2\alpha$ and d_α is the density of the measure (1.13). Consequently,

$$d_\alpha(u^{-1/\alpha}) u^{-1/\alpha-n-1} = cu^{-n}(1+2u^2)^{-\beta}$$

where c is a positive constant. Substituting $u = 2^{-1/2} t$ into the right-hand side of the last formula and applying Lemma 4.1, we get the following result.

Corollary 4.5. *$\circ_{\alpha,n} \prec *_{\alpha,\beta}$ if and only if the function $t^{-n}(1+t^2)^{-\beta}$ is completely monotone.*

Put $\circ = *_{2\alpha,\gamma}$ in Lemma 4.3. We know that $\varkappa(*_{2\alpha,\gamma}) = 4\alpha$ and, by (1.13),

$$d_{2\alpha}(x) = c\frac{x^{4\alpha\gamma-2\alpha-1}}{(2+x^{4\alpha})^{\gamma}}$$

where c is a positive constant. Consequently,

$$d_{2\alpha}(u^{1/2\alpha})\,u^{1/2+1/2\alpha-\beta} = c\frac{u^{2\gamma-\beta-1/2}}{(2-u^2)^{\gamma}}.$$

Substituting $u = \sqrt{2}t$ into the right-hand side of the last formula and applying Lemma 4.3, we get the following criterion.

COROLLARY 4.6. *$*_{\alpha,\beta} \prec *_{2\alpha,\gamma}$ if and only if the function $t^{2\gamma-\beta-1/2}(1+t^2)^{-\gamma}$ is completely monotone.*

LEMMA 4.6. *Suppose that real numbers a and b fulfil the conditions*

(4.6) $$2a+4b \geqslant 3, \quad b \geqslant 1, \quad a < b.$$

Then the function $t^{-a}(1+t^2)^{-b}$ is not completely monotone.

Proof. Contrary to this let us assume that the function $t^{-a}(1+t^2)^{-b}$ is completely monotone. Setting $\beta = a+2b-1/2$, $\gamma = b$, we have, by (4.6), $\beta \geqslant 1$ and $\gamma \geqslant 1$. Consider the generalized convolutions $*_{1,\beta}$ and $*_{2,\gamma}$. Since $t^{2\gamma-\beta-1/2}(1-t^2)^{-\gamma} = t^{-a}(1-t^2)^{-b}$, we have, by Corollary 4.6,

(4.7) $$*_{1,\beta} \prec *_{2,\gamma}.$$

Taking into account the inequalities $b \geqslant 1$ and $a < b$, we may select a rational number b_0 satisfying the conditions $b_0 \geqslant 1$ and $a+2(b-b_0) < b_0 \leqslant b$. Let a_0 be an arbitrary positive rational number from the interval

(4.8) $$a+2(b-b_0) < a_0 < b_0.$$

Put $\beta_0 = a_0+2b_0-1/2$ and $\gamma_0 = b_0$. From the definition of a_0 and b_0 we get the inequalities $\beta < \beta_0$ and $1 \leqslant \gamma_0 \leqslant \gamma$. From these inequalities, by virtue of Proposition 4.2, we get the relations $*_{1,\beta_0} \prec *_{1,\beta}$ and $*_{2,\gamma} \prec *_{2,\gamma_0}$, which, by (4.7), yield $*_{1,\beta_0} \prec *_{2,\gamma_0}$. Consequently, by Corollary 4.6, we conclude that the function $t^{-a_0}(1-t^2)^{-b_0}$ is completely monotone. Writing the positive rational numbers a_0, b_0 in the form $a_0 = n/r$, $b_0 = m/r$, where n, m and r are positive integers, we infer by (4.8) that

(4.9) $$n < m.$$

Since the r-th power of a completely monotone function is completely monotone too, the function $t^{-n}(1+t^2)^m$ is completely monotone. By Corollary 4.5 this proves the relation $\circ_{1,n} \prec *_{1,m}$, which, by (4.9) and Proposition 4.2, yields $\circ_{1,n} \prec *_{1,n+1}$. But this contradicts Proposition 4.5. The lemma is thus proved.

PROPOSITION 4.6. *$\circ_{\alpha,n} \prec *_{\alpha,\beta}$ if and only if $\beta \leqslant n$.*

Proof. The necessity of the condition $\beta \leqslant n$ follows at once from Corollary 4.5 and Lemma 4.6. To prove the sufficiency, by Proposition 4.2 it is enough to show the relation

(4.10) $$\circ_{\alpha,n} \prec *_{\alpha,n}.$$

We note that $t^{-1}(1+t^2)^{-1}$ as the Laplace transform of $1-\cos x$ is completely monotone. Consequently, $t^{-n}(1+t^2)^{-n}$ is also completely monotone which, by Corollary 4.5, implies (4.10). The proposition is thus proved.

Arguing as before we obtain by Corollary 4.6 and Lemma 4.6 the following result.

PROPOSITION 4.7. *$*_{\alpha,\beta} \prec *_{2\alpha,n}$ if and only if $\beta \geqslant 3n-1/2$.*

5. The order of generalized convolutions. The main topic of this section is a description of generalized convolutions in terms of the asymptotic behaviour of their characteristic functions. By $\mathscr{C}_b^+$ we shall denote the subset of $\mathscr{C}_b$ containing all positive functions. Further, we put $\mathscr{C}^+(\circ) = \mathscr{C}(\circ) \cap \mathscr{C}_b^+$ for every generalized convolution $\circ$. It is clear that $\mathscr{C}^+(*_\alpha) = \mathscr{C}(*_\alpha)$ and, by (2.8),

$$\mathscr{C}(*_\alpha) \subset \mathscr{C}^+(\circ) \quad (0 < \alpha \leqslant \varkappa(\circ))$$

which shows that the set $\mathscr{C}^+(\circ)$ is always non-void. The order $\varrho(f)$ of a function f from $\mathscr{C}_b^+$ is defined by the formula

$$\varrho(f) = \overline{\lim_{t\to\infty}} \frac{\log^+(-\log f(t))}{\log t}$$

where $\log^+ x = \log x$ if $x \geqslant 1$ and $\log^+ x = 0$ otherwise.

We start by giving some elementary properties of the order ϱ.

PROPOSITION 5.1. *If $f(t) \geqslant g(t)$ for t large enough, then $\varrho(f) \leqslant \varrho(g)$.*

PROPOSITION 5.2. *If $a > 0$ and $f(t) = g(at)$ for t large enough, then $\varrho(f) = \varrho(g)$.*

PROPOSITION 5.3. *If $a > 0$ and $f(t) = ag(t)$ for t large enough, then $\varrho(f) = \varrho(g)$.*

PROPOSITION 5.4. *Setting for any $f \in \mathscr{C}_b^+$ $f_{\min}(t) = \min\{f(u)\colon 0 \leqslant u \leqslant t\}$, we have $\varrho(f) = \varrho(f_{\min})$.*

PROPOSITION 5.5. *If $g(t) = \int_0^\infty f(tx)\mu(dx)$ for a certain $\mu \in \mathfrak{P}$, then $\varrho(f) \geqslant \varrho(g)$.*

We define the order $\varrho(\circ)$ of a generalized convolution $\circ$ by assuming

$$\varrho(\circ) = \sup\{\varrho(f)\colon f \in \mathscr{C}^+(\circ)\}.$$

As an immediate consequence of Theorem 3.1 we get the following result.

THEOREM 5.1. *The relation* $\circ \prec \circ'$ *yields the inequality* $\varrho(\circ) \leqslant \varrho(\circ')$.

By Proposition 5.5 we have the following property.

PROPOSITION 5.6. *If the kernel* Ω *of the characteristic function of* $\circ$ *is positive, then* $\varrho(\circ) = \varrho(\Omega)$.

Applying this proposition to the generalized convolution $*_\alpha$ with the kernel defined in Example 1.4, we obtain the formula

$$\varrho(*_\alpha) = \alpha. \tag{5.1}$$

As an application of Proposition 3.1, Theorem 5.1 and formula (5.1) we get a relation between the characteristic exponent and the order of generalized convolutions.

THEOREM 5.2. *For every generalized convolutions* $\circ$ *the inequality* $\varkappa(\circ) \leqslant \varrho(\circ)$ *is fulfilled.*

We shall see that in the extremal case $\varkappa(\circ) = \varrho(\circ)$ the generalized convolution is uniquely determined. For the proof we will need some lemmas.

LEMMA 5.1. *If* $\varrho(\circ) < \infty$, *then the kernel of the characteristic function of* $\circ$ *is positive.*

Proof. Contrary to our assertion, let us suppose that the kernel Ω of the characteristic function of $\circ$ is non-positive at a certain point and $\varrho(\circ) < \infty$. By the continuity of Ω and the formula $\Omega(0) = 1$, we conclude that $\Omega(t_0) = 0$ for a certain $t_0 > 0$. Put $h(t) = \Omega^2(2^{-1} t_0 t)$. Of course, $h \in \mathscr{C}(\circ)$,

$$h(2) = 0 \tag{5.2}$$

and, setting $\varkappa(\circ) = \varkappa$, we have, by (1.5) and the continuity of L at the origin,

$$\lim_{t \to 0+} \frac{1 - h(t)}{t^\varkappa} < \infty.$$

Hence it follows that the infinite product

$$g(t) = \prod_{j=0}^{\infty} h(2^{-j} t)$$

is uniformly convergent on every compact subset of R^+. Consequently, $g \in \mathscr{C}(\circ)$ and, by (5.2),

$$g(2^m) = 0 \quad (m = 1, 2, \ldots). \tag{5.3}$$

For every positive integer n we put

$$f_n(t) = \sum_{k=0}^{\infty} c_{k,n}\, g(2^{-k} t)$$

where

$$c_{k,n} = \exp(-2^{kn}) - \exp(-2^{(k+1)n}) \quad (k = 1, 2, \ldots)$$

and

$$c_{0,n} = 1 - \exp(-2^n).$$

Since $\sum_{k=0}^{\infty} c_{kn} = 1$ and

$$c_{k,n} > 0 \quad (k = 0, 1, \ldots, n = 1, 2, \ldots), \tag{5.4}$$

we infer that $f_n \in \mathscr{C}(\circ)$ $(n = 1, 2, \ldots)$. The function g is non-negative on R^+ and positive in a neighbourhood of the origin because of the formula $g(0) = 1$ and the continuity. Hence by (5.4) it follows that the functions f_n belong to $\mathscr{C}^+(\circ)$. Further, by (5.3),

$$f_n(2^m) = \sum_{k=m}^{\infty} c_{k,n}\, g(2^{m-k}) \leqslant \sum_{k=m}^{\infty} c_{k,n} = \exp(-2^{mn}),$$

which yields $\varrho(f_n) \geqslant n$ $(n = 1, 2, \ldots)$. Consequently, $\varrho(\circ) = \infty$ which gives the contradiction. The lemma is thus proved.

As a direct consequence of Lemma 5.1 we get the following formulas

$$\varrho(*_{\alpha,\beta}) = \varrho(\circ_{\alpha,n}) = \infty.$$

LEMMA 5.2. *If* $\varrho(\circ) < \infty$, *then*

$$(\delta_1 \circ \delta_1)([0, 2^{1/\varrho(\circ)})) = 0.$$

Proof. Let Ω be the kernel of the characteristic function of $\circ$. By Lemma 4.1 the kernel Ω is positive on R^+. Consequently, by Proposition 5.6 and Theorem 5.2, we have $\varrho(\circ) = \varrho(\Omega)$ and $0 < \varrho(\Omega) < \infty$. Theorem 1.6 in [9] (p. 52) asserts that there exists then a differentiable function r on R^+ fulfilling the conditions

$$\lim_{t \to \infty} r(t) = \varrho(\circ), \tag{5.5}$$

$$\Omega(t) \geqslant \exp(-t^{r(t)}) \quad (t \in R^+), \tag{5.6}$$

$$\Omega(t_n) = \exp(-t_n^{r(t_n)}) \tag{5.7}$$

for a certain sequence $t_1 < t_2 < \ldots$ tending to ∞ and

$$\lim_{t \to \infty} t \log t \frac{d}{dt} r(t) = 0.$$

Applying the Mean Value Theorem, we get

$$\lim_{t \to \infty} \big(r(ct) - r(t)\big) \log t = 0$$

and, consequently, by (5.5),

$$\lim_{t\to\infty} t^{r(ct)-r(t)} c^{r(ct)} = c^{\varrho(\circ)} \tag{5.8}$$

for every $c > 0$. Setting $\nu = \delta_1 \circ \delta_1$, we have the formula

$$\Omega^2(t) = \int_0^\infty \Omega(tx)\,\nu(dx)$$

which, by (5.6), yields the inequality

$$\Omega^2(t) \geqslant \nu(\{0\}) + \int_a^b \exp\left(-(tx)^{r(tx)}\right)\nu(dx)$$

for all $b > a > 0$. Using (5.7), we have the inequality

$$1 \geqslant \nu(\{0\})\exp 2t_n^{r(t_n)} + \int_a^b \exp\left(2t_n^{r(t_n)} - (t_n x)^{r(t_n x)}\right)\nu(dx)$$

for $n = 1, 2, \ldots$ which, by (5.5), (5.8) and the Fatou Lemma, yields $\nu(\{0\}) + \nu([a, b]) = 0$ provided $b < 2^{1/\varrho(\circ)}$. Thus $\nu([0, 2^{1/\varrho(\circ)})) = 0$ which completes the proof.

We are now in a position to prove

THEOREM 5.3. *$\varrho(\circ) = \varkappa(\circ)$ if and only if $\circ = *_{\varkappa}$, where $\varkappa = \varkappa(\circ)$.*

Proof. Let Ω be the kernel of the characteristic function of $\circ$ and $\nu = \delta_1 \circ \delta_1$. Suppose that $\varrho(\circ) = \varkappa(\circ)$. Then, by Lemma 4.2, $\nu([0, 2^{1/\varkappa})) = 0$ and, consequently,

$$\Omega^2(t) = \int_{2^{1/\varkappa}}^\infty \Omega(tx)\,\nu(dx) \tag{5.9}$$

which yields the formula

$$1 + \Omega(t) = \int_{2^{1/\varkappa}}^\infty \frac{1-\Omega(tx)}{1-\Omega(t)}\,\nu(dx).$$

Using the Fatou Lemma as $t \to 0^+$, we get by (1.5)

$$2 \geqslant \int_{2^{1/\varkappa}}^\infty x^{\varkappa}\,\nu(dx).$$

It is clear that the last inequality is fulfilled if and only if the measure ν is concentrated at the point $2^{1/\varkappa}$. Thus, by (4.10), $\Omega^2(t) = \Omega(2^{1/\varkappa} t)$ $(t \in R^+)$. Since $\Omega(0) = 1$, we infer that Ω is positive on R^+. Applying Lemma 3.2 in [12] we conclude that $\circ = *_{\varkappa}$. The converse implication follows from (5.1) which completes the proof.

References

[1] H. Bateman and A. Erdélyi, *Higher transcendental functions*, Vol. 2, New York, Toronto, London 1953.

[2] N. H. Bingham, *Factorization theory and domains of attraction for generalized convolution algebras*, Proc. London Math. Soc. 23 (1971), 16–30.

[3] R. Jajte, *Quasi-stable measures in generalized convolution algebras*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), 505–511.

[4] —, *Quasi-stable measures in generalized convolution algebras II*, ibidem 25 (1977), 67–72.

[5] Z. J. Jurek, *Some characterizations of the class L in generalized convolution algebras*, ibidem 29 (1981), 409–415.

[6] —, *Limit distributions in generalized convolution algebras*, Probab. Math. Statistics (to appear).

[7] M. Kłosowska, *On the domain of attraction for generalized convolution algebras*, Rev. Roum. Math. Pures et Appl. 22 (1977), 669–677.

[8] J. Kucharczak and K. Urbanik, *Quasi-stable functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), 263–268.

[9] B. Ya. Levin, *The distribution of zeros of entire functions*, Moscow 1956 (in Russian).

[10] Nguyen van Thu, *Multiply self-decomposable measures in generalized convolution algebras*, Studia Math. 66 (1979), 177–184.

[11] K. Urbanik, *Generalized convolutions*, ibidem 23 (1964), 217–245.

[12] —, *A characterization of a class of convolutions*, Colloq. Math. 18 (1967), 239–249.

[13] —, *Generalized convolutions II*, Studia Math. 45 (1973), 57–70.

[14] V. E. Volkovich, *On an analytical description of Urbanik algebras*, Izv. Akad. Nauk UzSSR, Ser. Fiz.-Mat. Nauk 5 (1979), 12–17.

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