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# Orthogonal scalar products on von Neumann algebras

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**Abstract.** Let $\langle\cdot,\cdot\rangle$ be a scalar product on a von Neumann algebra $A$, such that $\langle p, q\rangle = 0$ for any pair of mutually orthogonal projections $p, q \in A$. It is shown that, for a large class of von Neumann algebras, the general form of such a scalar product is given by the formula

$$\langle x, y\rangle = [\mu(y^* x + xy^*) + \varphi(y^* x - xy^*)]/2, \qquad x, y \in A,$$

where $\mu \in A_*^+$, $\varphi \in A_*^h$ and $-\mu \leqslant \varphi \leqslant \mu$. This result is applied to orthogonally scattered Gleason measures.

**Introduction.** In the paper, orthogonal scalar products on von Neumann algebras are considered. A scalar product $\langle\cdot,\cdot\rangle$ on a von Neumann algebra $A$ is called *orthogonal* if $\langle p, q\rangle = 0$ for any pair $p, q$ of mutually orthogonal projections from $A$. Scalar products of the above type were investigated because of their strong connection with orthogonally scattered Gleason measures (see [4] and Section 5). Namely, every orthogonally scattered Gleason measure gives rise to an orthogonal scalar product and the problem of finding the general form of the "correlation function" of such a measure is essentially equivalent to that of finding the general form of the scalar product. The theorem giving the general form of the correlation function of an orthogonally scattered Gleason measure on the full algebra $L(H)$, where $H$ is a separable Hilbert space of dimension $\geqslant 3$, was proved by Jajte and Paszkiewicz in [5]. In the language of scalar products it states that for any orthogonal scalar product $\langle\cdot,\cdot\rangle$ on $L(H)$ there exist two trace-class operators $m_1, m_2$ on $H$ such that, for each $x, y \in L(H)$,

$$\langle x, y\rangle = \operatorname{tr}(m_1 y^* x + m_2 xy^*).$$

The above formula may be rewritten in the following equivalent form:

$$(*) \qquad \langle x, y\rangle = [\mu(y^* x + xy^*) + \varphi(y^* x - xy^*)]/2,$$

where $\mu$ is a positive normal functional and $\varphi$ a hermitian normal functional such that $-\mu \leqslant \varphi \leqslant \mu$. The main result of the present paper is the following

THEOREM. *Let $A$ be a von Neumann algebra without a direct summand of type $\mathrm{II}_1$. If $A$ has separable predual or is a factor, then, for any orthogonal* *scalar product* $\langle\cdot,\cdot\rangle$ *on* $A$, *there exist* $\mu\in A_*^+$ *and* $\varphi\in A_*^h$, $-\mu\leqslant\varphi\leqslant\mu$, *such that formula* $(*)$ *holds.*

The conclusion of the theorem remains valid for an arbitrary approximately finite-dimensional von Neumann algebra (cf. Cor. 2.8) and some other cases (cf. Prop. 2.7, Cor. 3.11, Prop. 4.1, Cor. 4.2, Cor. 4.3).

Here are some comments on the content of each particular section: Section 1 contains all necessary definitions and some basic facts about orthogonal scalar products. The finite-dimensional and approximately finite-dimensional cases of the theorem are dealt with in Section 2. Also, the results in this section are formulated for the more general case of orthogonal sesquilinear forms. Since the original proof of the theorem for $A = L(H)$, with $H$ separable and of dimension $\geqslant 3$, presented in [5] was quite long and complicated, a new simpler proof, including also the case of factors of type $I_2$, is given here. Sections 3 and 4 are mainly devoted to investigating properly infinite von Neumann algebras. The following result of Paszkiewicz [6] is used to prove the respective part of the theorem: Every positive Gleason measure on a von Neumann algebra with separable predual (and not having a factor of type $I_2$ as a direct summand) or on a factor (not of type $I_2$) extends to a normal linear functional on the algebra (or the factor). Finally, in Section 5 the results obtained are applied to orthogonally scattered Gleason measures.

The notation used in the present paper is standard (see [7] and [9]). For the survey of results on Gleason measures the reader is referred to [3]. Other different approaches to the problem treated here can be found in [2] and [4].

**1. Definitions and fundamental properties of OSF-, OSHF- and OSP-algebras.**

DEFINITION 1.1. We call a sesquilinear form $\langle\cdot,\cdot\rangle$ on a von Neumann algebra $A$ *orthogonal* if

(a) $\langle p, q\rangle = 0$ for $p, q\in \mathrm{Proj}\, A$, $p\perp q$;

(b) the applications $x\mapsto\langle x, y\rangle$ and $x\mapsto\langle y, x\rangle$ are ultraweakly continuous on $A$ for every $y\in A$.

We say *orthogonal scalar product* instead of orthogonal *positive* sesquilinear form. We write OSF (resp. OSHF, OSP) in place of *orthogonal sesquilinear form* (resp. *orthogonal sesquilinear hermitian form*, *orthogonal scalar product*).

PROPOSITION 1.2. *Let* $\mu, \varphi\in A_*$. *The formula*

$$(*)\qquad \langle x, y\rangle = [\mu(y^*x+xy^*)+\varphi(y^*x-xy^*)]/2 \quad \text{for} \quad x, y\in A$$

*defines an* OSF *on* $A$. *If* $\mu, \varphi\in A_*^h$, *then the form is hermitian. If* $\mu\in A_*^+$ *and* $-\mu\leqslant\varphi\leqslant\mu$, *we obtain an* OSP *on* $A$.



Proof. It is obvious.

DEFINITION 1.3. If, for some $\mu, \varphi\in A^*$, an OSF $\langle\cdot,\cdot\rangle$ is given by $(*)$, the pair $(\mu, \varphi)$ is said to *determine* the form. OSF- (resp. OSHF-; OSP-) *algebra* is a von Neumann algebra $A$ such that an arbitrary OSF (resp. OSHF; OSP) on $A$ is determined by a pair $(\mu, \varphi)$ of functionals such that $\mu, \varphi\in A_*$ (resp. $\mu, \varphi\in A_*^h$; $\mu\in A_*^+$ and $-\mu\leqslant\varphi\leqslant\mu$).

PROPOSITION 1.4. *If both* $(\mu, \varphi)$ *and* $(\mu', \varphi')$ *determine an* OSF $\langle\cdot,\cdot\rangle$, *then* $\mu = \mu'\colon x\mapsto\langle x, 1\rangle$, *so that* $\mu\in A_*$. *Moreover, if* $\langle\cdot,\cdot\rangle$ *is* OSHF, *then* $\mu\in A_*^h$ *and if* $\langle\cdot,\cdot\rangle$ *is* OSP, *then* $\mu\in A_*^+$.

Proof. As $\mu(p) = \langle p, p\rangle = \langle p, 1\rangle$ for every $p\in\mathrm{Proj}\, A$, the proposition follows directly from the Spectral Theorem.

In the sequel, $\langle\cdot,\cdot\rangle$ is always an OSF, and $\mu$ is given by $\mu(x) = \langle x, 1\rangle$ for $x\in A$.

PROPOSITION 1.5. (a) *If* $x, y\in A$ *are normal and* $xy = yx$, *then*

$$(1)\qquad \langle x, y\rangle = \mu(y^*x).$$

(b) *If* $x, y\in A$ *and* $\mathrm{re}\, x\, \mathrm{re}\, y = \mathrm{re}\, y\, \mathrm{re}\, x$, $\mathrm{re}\, x\, \mathrm{im}\, y = \mathrm{im}\, y\, \mathrm{re}\, x$, $\mathrm{im}\, x\, \mathrm{re}\, y = \mathrm{re}\, y\, \mathrm{im}\, x$ *and* $\mathrm{im}\, x\, \mathrm{im}\, y = \mathrm{im}\, y\, \mathrm{im}\, x$, *then formula* (1) *holds.*

(c) *If* $x, y\in A$ *and* $l(x)\, l(y) = l(x)\, r(y) = r(x)\, l(y) = r(x)\, r(y) = 0$ (*where* $l(x)$ *and* $r(x)$ *are resp. left and right supports of* $x$), *then* $\langle x, y\rangle = 0$.

(d) *For any* $x\in A$,

$$(2)\qquad \langle x, x\rangle + \langle x^*, x^*\rangle = \mu(x^*x+xx^*).$$

Proof. Since (b) is a consequence of (a), and (c) is a consequence of (d), it is enough to show (a) and (d).

(a): If $p, q\in\mathrm{Proj}\, A$ and $pq = qp$, then $\langle p, q\rangle = \mu(pq)$ by the orthogonality of $\langle\cdot,\cdot\rangle$. Now, we obtain formula (1) in a standard way using the Spectral Theorem.

(d): Using (1), we get

$$\begin{aligned}\langle x, x\rangle + \langle x^*, x^*\rangle &= 2(\langle \mathrm{re}\, x, \mathrm{re}\, x\rangle + \langle \mathrm{im}\, x, \mathrm{im}\, x\rangle)\\ &= 2\mu\big((\mathrm{re}\, x)^2 + (\mathrm{im}\, x)^2\big) = \mu(x^*x+xx^*).\end{aligned}$$

PROPOSITION 1.6. *The form* $\langle\cdot,\cdot\rangle$ *is determined by a pair* $(\mu, \varphi)$ *iff any one of the following conditions holds*:

(a) $\langle x, x\rangle = [\mu(x^*x+xx^*)+\varphi(x^*x+xx^*)]/2$ *for* $x\in A$;

(b) $\langle x, x\rangle - \langle x^*, x^*\rangle = \varphi(x^*x-xx^*)$ *for* $x\in A$;

(c) $\langle x, y\rangle - \langle y, x\rangle = \varphi(yx-xy)$ *for* $x, y\in A_h$;

(d) $\langle p, q\rangle - \langle q, p\rangle = \varphi(qp-pq)$ *for* $p, q\in\mathrm{Proj}\, A$.

Proof. By easy calculations, using formula (2), the polarization formulae and standard limit procedures.

PROPOSITION 1.7. *If* $(\mu, \varphi)$ *determines* $\langle\cdot,\cdot\rangle$, *then* $(\mu, \varphi')$, *where* $\varphi' = \varphi+\tau_1-\tau_2+i\tau_3-i\tau_4$ *(the* $\tau$*'s being finite traces on* $A$*), also determines* $\langle\cdot,\cdot\rangle$. *Conversely, if both* $(\mu, \varphi)$ *and* $(\mu, \varphi')$ *determine* $\langle\cdot,\cdot\rangle$, *and*

(a) $A$ *is a factor of type* $I_n$ $(n<\infty)$, *then* $\varphi-\varphi' = \alpha\tau_n$ *for some* $\alpha\in C$ ($\tau_n$ *is here the normalized trace on* $A$); *or*

(b) $A$ *is a factor of type* $I_\infty$,

*then* $\varphi = \varphi'$.

Proof. The first part of the proposition is evident. Now, if both $(\mu, \varphi)$ and $(\mu, \varphi')$ determine $\langle\cdot,\cdot\rangle$, then

$$(\varphi-\varphi')(x^*x) = (\varphi-\varphi')(xx^*) \quad \text{for} \quad x\in A.$$

We may suppose that $\varphi-\varphi'\in A_*^h$. If $A$ is a factor of type $I_n$ $(n<\infty)$, then, for some $z\in L^1(A, \tau_n) = A$,

$$(\varphi-\varphi')(x) = \tau_n(zx) \quad \text{for} \quad x\in A.$$

Hence

$$|(\varphi-\varphi')(x)| \leqslant \|z\|\,\tau_n(x) \quad \text{for} \quad x\in A_+,$$

and the functional $\varphi-\varphi'+\|z\|\,\tau_n$ is positive, therefore it is a trace and the desired result follows. A similar reasoning shows that we must have $\varphi = \varphi'$ in the $I_\infty$ case.

PROPOSITION 1.8. (a) *If a von Neumann algebra* $A$ *is an* OSF- (*resp.* OSHF-, OSP-) *algebra, then every von Neumann algebra* $B$ *isomorphic to* $A$ *is also an* OSF- (*resp.* OSHF-, OSP-) *algebra.*

(b) *Every abelian von Neumann algebra is an* OSF-, OSHF- *and* OSP-*algebra.*

(c) *If von Neumann algebras* $A_i$, $i = 1,\ldots, n$, *are all* OSF- (*resp.* OSHF-) *algebras, then* $A = \bigoplus_{i=1}^{n} A_i$ *is also an* OSF- (*resp.* OSHF-) *algebra.*

(d) *If von Neumann algebras* $A_i$, $i\in J$ *(arbitrary), are all* OSP-*algebras, then* $\bigoplus_{i\in J} A_i$ *is also an* OSP-*algebra.*

Proof. (a): It is enough to observe that the orthogonality of projections is preserved under isomorphisms of von Neumann algebras.

(b): In an abelian von Neumann algebra, any $\varphi$ will do in the pair $(\mu, \varphi)$.

(c): This is a consequence of Prop. 1.5 (c).

(d): Let $(\mu_i, \varphi_i)$ be pairs determining $\langle\cdot,\cdot\rangle_i$ in $A_i$, where $\langle\cdot,\cdot\rangle_i = \langle\cdot,\cdot\rangle|A_i\times\times A_i$. It is easy to see that $\mu_i = \mu|A_i$ for every $i\in J$ ($\mu(x) \stackrel{\text{df}}{=} \langle x, 1\rangle$, $\mu_i(x) \stackrel{\text{df}}{=} \langle x, 1\rangle_i$). As $\|\varphi_i\| \leqslant \|\mu_i\|$ for $i\in J$, we have

$$\sum_{i\in J}\|\varphi_i\| \leqslant \sum_{i\in J}\|\mu_i\| = \sum_{i\in J}\mu_i(1_i) = \mu(1) < \infty,$$



and it is possible to define

$$\varphi\big((x_i)_{i\in J}\big) \stackrel{\text{df}}{=} \sum_{i\in J}\varphi_i(x_i).$$

Now, we easily check condition (b) of Prop. 1.6 for the pair $(\mu, \varphi)$, using Prop. 1.5 (c).

## 2. Finite-dimensional and approximately finite-dimensional cases.

THEOREM 2.1. $\boldsymbol{B}(\boldsymbol{C}^2)$ *is an* OSF-*algebra.*

Proof. Let $\langle\cdot,\cdot\rangle$ be an OSF on $\boldsymbol{B}(\boldsymbol{C}^2)$. If we treat the matrices in $\boldsymbol{B}(\boldsymbol{C}^2)$ as vectors from $\boldsymbol{C}^4$, the mapping $(x, y)\mapsto\langle x, y\rangle$ can be considered as a sesquilinear form on $\boldsymbol{C}^4$. Therefore there exists an operator $A\in\boldsymbol{B}(\boldsymbol{C}^4)$, $A = [a_{jk}]_{j,k=1,\ldots,4}$, such that

$$\langle x, y\rangle = (Ax, y) = \sum_{j,k=1}^{4} a_{jk}\,x_j\bar{y}_k$$

for

$$x = \begin{bmatrix} x_1 & x_2\\ x_3 & x_4\end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 & y_2\\ y_3 & y_4\end{bmatrix}.$$

From now on, we shall write elements of $\boldsymbol{B}(\boldsymbol{C}^2)$ as if they were vectors from $\boldsymbol{C}^4$, for example, the above $x$ would be written as $x = [x_1, x_2, x_3, x_4]$. Now, for any $c, s\in\boldsymbol{R}$ and $\theta\in\boldsymbol{C}$, $c^2+s^2 = 1$, $|\theta| = 1$, the operator $p\in\boldsymbol{B}(\boldsymbol{C}^2)$ of the form

$$p = [c^2, cs\theta, cs\bar{\theta}, s^2]$$

is a one-dimensional orthogonal projection, the projection $p^\perp = 1-p$ being equal to

$$p^\perp = [s^2, -cs\theta, -cs\bar{\theta}, c^2].$$

Then

$$\begin{aligned}\langle p, p^\perp\rangle = {} & c^2s^2(a_{11}-a_{22}-a_{33}+a_{44})+\\ & +\theta\,[c^3s(a_{24}-a_{13})+cs^3(a_{21}-a_{43})]+\\ & +\bar{\theta}\,[c^3s(a_{34}-a_{12})+cs^3(a_{31}-a_{42})]+\\ & +c^4a_{14}+s^4a_{41}-\theta^2c^2s^2a_{23}-\bar{\theta}^2c^2s^2a_{32}.\end{aligned}$$

The above expression is zero for any choice of the parameters $c$, $s$ and $\theta$. Hence (among others)

$$\begin{aligned}a_{24}-a_{13} = a_{21}-a_{43} = a_{34}-a_{12} = a_{31}-a_{42}\\ = a_{14} = a_{41} = a_{23} = a_{32} = 0.\end{aligned}$$

Using the above, we easily calculate

$$\langle x, x\rangle - \langle x^*, x^*\rangle = (a_{21}-a_{13})(\bar{x}_1 x_2+\bar{x}_3 x_4-x_1\bar{x}_3-x_2\bar{x}_4)+$$
$$+(a_{12}-a_{31})(x_1\bar{x}_2+x_3\bar{x}_4-\bar{x}_1 x_3-\bar{x}_2 x_4)+$$
$$+(a_{33}-a_{22})(|x_3|^2-|x_2|^2).$$

Since, for $x = [x_1, x_2, x_3, x_4]$, we have

$$x^*x-xx^* = [|x_3|^2-|x_2|^2,\ \bar{x}_1 x_2+\bar{x}_3 x_4-x_1\bar{x}_3-x_2\bar{x}_4,$$
$$x_1\bar{x}_2+x_3\bar{x}_4-\bar{x}_1 x_3-\bar{x}_2 x_4,\ |x_2|^2-|x_3|^2],$$

the desired functional $\varphi$ may be defined by

$$\varphi([x_1, x_2, x_3, x_4]) = a_{33}\, x_1+(a_{21}-a_{13})x_2+(a_{12}-a_{31})x_3+a_{22}\, x_4.$$

THEOREM 2.2. *$B(C^3)$ is an* OSF-*algebra.*

Proof. The proof can be carried out essentially in the same way as that of Th. 2.1, but this time the calculations are tedious and will be omitted.

THEOREM 2.3. *Factors of type* $I_n$ $(n<\infty)$ *are all* OSF-*algebras.*

Proof. We use induction on $n$. For $n = 1, 2, 3$, the theorem is true by Prop. 1.8 (b) and Th. 2.1, Th. 2.2. Assume that it is true for $k<n$ $(n\geq 4)$. We shall show that $B(C^n)$ is an OSF-algebra. Let $\langle\cdot,\cdot\rangle$ be an OSF on $B(C^n)$, and $\{e_1, \ldots, e_n\}$ an orthonormal basis in $C^n$. Put

$$A_{j_1,\ldots,j_m} \overset{\text{df}}{=} (\hat{e}_{j_1}+\ldots+\hat{e}_{j_m})B(C^n)(\hat{e}_{j_1}+\ldots+\hat{e}_{j_m}),$$

where $\hat{e}$ denotes the one-dimensional orthogonal projection in the direction of $e$. Now, define functionals $\varphi_1$ and $\psi$ so that the pair $(\mu|A_{1,\ldots,n-1}, \varphi_1)$ should determine $\langle\cdot,\cdot\rangle$ on $A_{1,\ldots,n-1}$ and $(\mu|A_{2,\ldots,n}, \psi)$ on $A_{2,\ldots,n}$. By Prop. 1.7, $\varphi_1-\psi = \alpha\tau$ on $A_{2,\ldots,n-1}$ for some $\alpha\in C$ ($\tau$ is the normalized trace on $B(C^n)$). Further, let us put $\varphi_n = \psi+\alpha\tau|A_{2,\ldots,n}$. Now $\varphi_1 = \varphi_n$ on $A_{2,\ldots,n-1}$. By Prop. 1.7, we may choose functionals $\varphi_k$ on $A_{1,k,n}$ $(k = 2, \ldots, n-1)$ so that the pair $(\mu|A_{1,k,n}, \varphi_k)$ should determine $\langle\cdot,\cdot\rangle$ on $A_{1,k,n}$, and that $\varphi_k(\hat{e}_k) = \varphi_1(\hat{e}_k) = \varphi_n(\hat{e}_k)$. Then $\varphi_k = \varphi_1$ on $A_{1,k}$ and $\varphi_k = \varphi_n$ on $A_{k,n}$ for $k = 2, \ldots, n-1$.

Now we define the desired linear functional $\varphi$ on $B(C^n)$ by giving its values on partial isometries $u_{ij}$, $i, j = 1, \ldots, n$ where $u_{ij}^* u_{ij} = \hat{e}_j$ and $u_{ij}u_{ij}^* = \hat{e}_i$:

$$\varphi(u_{ij}) \overset{\text{df}}{=} \begin{cases} \varphi_1(u_{ij}) & \text{for}\quad i, j = 1, \ldots, n-1;\\ \varphi_n(u_{ij}) & \text{for}\quad i, j = 2, \ldots, n;\\ \varphi_k(u_{ij}) & \text{for}\quad i = 1, j = n \text{ or } i = n, j = 1.\end{cases}$$

It is obvious that $\varphi|A_{1,\ldots,n-1} = \varphi_1$, $\varphi|A_{2,\ldots,n} = \varphi_n$ and $\varphi|A_{1,k,n} = \varphi_k$ for $k = 2, \ldots, n-1$. We shall show that the pair $(\mu, \varphi)$ determines $\langle\cdot,\cdot\rangle$. By Prop. 1.6 (b) and the additivity argument, it is enough to check that



$$(3)\qquad \langle u_{ij}, u_{lm}\rangle - \langle u_{lm}, u_{ij}\rangle = \varphi(u_{lm}^* u_{ij} - u_{ij}u_{lm}^*)$$

for any quadruple $(i, j, l, m)$, $i, j, l, m = 1, \ldots, n$. If $i\neq l$, $i\neq m$, $j\neq l$ and $j\neq m$, then, by Prop. 1.5 (c), the left-hand side of (3) is zero; of course, so is the right-hand side of (3). If any two of the indices coincide, $u_{ij}$ and $u_{lm}$ belong to one of the algebras $A_{1,\ldots,n-1}$, $A_{2,\ldots,n}$ or $A_{1,k,n}$ $(k = 2, \ldots, n-1)$. Thus, for some $k$ $(k = 1, \ldots, n)$,

$$\langle u_{ij}, u_{lm}\rangle - \langle u_{lm}, u_{ij}\rangle = \varphi_k(u_{lm}^* u_{ij} - u_{ij}u_{lm}^*)$$

and so (3) holds. This completes the proof of the theorem.

THEOREM 2.4. *Factors of type* $I_n$ $(n<\infty)$ *are all* OSHF-*algebras.*

Proof. Let $A$ be a factor of type $I_n$ and $\langle\cdot,\cdot\rangle$ an OSHF on $A$. Choose a one-dimensional $p\in \operatorname{Proj} A$ and a pair $(\mu, \varphi)$ determining $\langle\cdot,\cdot\rangle$, so that $\varphi(p) = 0$ (this is possible by virtue of Prop. 1.7 and Th. 2.3). Then, for any one-dimensional $q\in\operatorname{Proj} A$ we have

$$\varphi(q) = \varphi(q)-\varphi(p) = \langle u, u\rangle - \langle u^*, u^*\rangle \in R$$

(here $u\in A$ is such that $u^*u = q$, $uu^* = p$), and consequently, $\varphi\in A_*^h$.

THEOREM 2.5. *Factors of type* $I_n$ $(n<\infty)$ *are all* OSP-*algebras.*

Proof. Consider an $n$-dimensional Hilbert space $H$ and the algebra $A = B(H)$ with an OSP $\langle\cdot,\cdot\rangle$. Let $h\in A_+$ be the density of $\mu$ with respect to the normalized trace $\tau$ on $A$, i.e. $\mu(x) = \tau(hx)$ for $x\in A$. Denote by $\beta$ the least eigenvalue of the operator $h$. Then $\mu-\beta\tau\in A_*^+$ and $\mu-\beta\tau$ is not faithful. Let $p\in\operatorname{Proj} A$ be one-dimensional and such that $(\mu-\beta\tau)(p) = 0$. Select $\varphi\in A_*^h$ so that the pair $(\mu, \varphi)$ should determine $\langle\cdot,\cdot\rangle$ and that $\varphi(p) = 0$ (this is possible by virtue of Th. 2.4 and Prop. 1.7). Define $\langle\langle\cdot,\cdot\rangle\rangle$ to be the OSP on $A$ determined by $(\mu-\beta\tau, \varphi)$. Now, let $q$ be an arbitrary one-dimensional projection from $A$, and $u\in A$ the partial isometry with initial projection $p$ and final projection $q$. Then

$$|\varphi(q)-\varphi(p)| = |\langle\langle u, u\rangle\rangle - \langle\langle u^*, u^*\rangle\rangle|$$
$$\leq \langle\langle u, u\rangle\rangle + \langle\langle u^*, u^*\rangle\rangle = (\mu-\beta\tau)(p+q).$$

Hence $|\varphi(q)|\leq(\mu-\beta\tau)(q)$, which completes the proof.

THEOREM 2.6. *Finite-dimensional von Neumann algebras are all* OSF-, OSHF- *and* OSP-*algebras.*

Proof. Any such algebra is a finite direct sum of factors of type $I_n$ $(n<\infty)$, so it is enough to apply Ths. 2.3, 2.4, 2.5 and Prop. 1.8 (c).

PROPOSITION 2.7. *The ultraweak closure of an ascending sequence of* OSP-*algebras is an* OSP-*algebra.*

Proof. Let $A$ be the ultraweak closure of $\bigcup_{n=1}^{\infty} A_n$, where $(A_n)$ is an ascending sequence of OSP-algebras. Let $\langle\cdot,\cdot\rangle$ be an OSP on $A$, and put $\langle\cdot,\cdot\rangle_n = \langle\cdot,\cdot\rangle|A_n\times A_n$. Choose pairs $(\mu_n, \varphi_n)$ determining the OSP's $\langle\cdot,\cdot\rangle_n$ so that $-\mu_n \leqslant \varphi_n \leqslant \mu_n$. Clearly, $\mu_n = \mu|A_n$. As $0 \leqslant \varphi_n+\mu_n \leqslant 2\mu_n$, by the Radon–Nikodym-type theorem of Sakai (see [7], Prop. 1.24.4), there exists an $h_n \in A_n$, $0 \leqslant h_n \leqslant 1$ such that

$$(\varphi_n+\mu_n)(x) = \mu_n(h_n x + x h_n) \quad \text{for} \quad x \in A_n.$$

Put

$$\bar{\varphi}_n(x) \stackrel{\mathrm{df}}{=} \mu(h_n x + x h_n) - \mu(x) \quad \text{for} \quad x \in A.$$

Then, for $x \in A_+$,

$$|\bar{\varphi}_n(x)| \leqslant \mu(x) + 2\mu(h_n^2)^{1/2}\mu(x^2)^{1/2} \leqslant \mu(x) + 2\|\mu\|^{1/2}\mu(x^2)^{1/2}.$$

Therefore, if $p_k \in \operatorname{Proj} A$ for $k = 1, 2\ldots$ and $p_k \downarrow 0$ as $k \to \infty$, then $\bar{\varphi}_n(p_k) \to 0$ as $k \to \infty$ uniformly with respect to $n$. It is also clear from the inequality that $\|\bar{\varphi}_n\| \leqslant 3\|\mu\|$. Thus, the set $\{\bar{\varphi}_n\}_{n=1,2,\ldots}$ is relatively $\sigma(A_*, A)$-compact and consequently, relatively sequentially compact by virtue of Eberlein–Šmuljan theorem (see [9], Th. III. 5.4). Hence there exist a subsequence $(k_n)$ of positive integers and a functional $\varphi \in A_*$, such that $\bar{\varphi}_{k_n}(x) \to \varphi(x)$ for every $x \in A$. It is easy to check that

$$\langle x, y\rangle - \langle y, x\rangle = \varphi(xy - yx) \quad \text{for} \quad x, y \in \Big(\bigcup_{n=1}^{\infty} A_n\Big)_h.$$

By the continuity of $\varphi$ and $\langle\cdot,\cdot\rangle$, the above equality holds for $x, y \in A_h$, and by Prop. 1.6 (c), the pair $(\mu, \varphi)$ determines $\langle\cdot,\cdot\rangle$. It is clear that $-\mu \leqslant \varphi \leqslant \mu$.

Corollary 2.8. *Approximately finite-dimensional von Neumann algebras are all OSP-algebras.*

### 3. Semifinite properly infinite case.

Proposition 3.1. *Let $\langle\cdot,\cdot\rangle$ be an OSP on a von Neumann algebra $A$. Suppose there exists a functional $\varphi \in A_*$ such that, for any $p, q \in \operatorname{Proj} A$ with $p \sim q$ and $p \perp q$,*

$$\varphi(p) - \varphi(q) = \langle u, u\rangle - \langle u^*, u^*\rangle,$$

*where $u \in A$ is a partial isometry with initial projection $p$ and final projection $q$. Then the pair $(\mu, \varphi)$ determines $\langle\cdot,\cdot\rangle$.*

Proof. By Prop. 1.6 (d), it is enough to show that

$$\langle p, q\rangle - \langle q, p\rangle = \varphi(pq - qp) \quad \text{for} \quad p, q \in \operatorname{Proj} A. \tag{4}$$

Now, if $p, q \in \operatorname{Proj} A$, then there exist $p_1, p_2, q_1, q_2 \in \operatorname{Proj} A$ such that $p = p_1+p_2$, $q = q_1+q_2$, $p_2 q = q p_2$, $p q_2 = q_2 p$ and $p_1 \sim q_1$ (see [9], pp. 306–307).



Thus it suffices to establish (4) for pairs of *equivalent* projections from $A$. Now, if $p \sim q$, then there exist sequences $(p_n)$, $(q_n)$ of projections from $A$, such that $\|p_n - p\| \to 0$ and $\|q_n - q\| \to 0$ as $n \to \infty$, and that, for every $n$, the von Neumann algebra generated by $p_n$ and $q_n$ is a (finite) direct sum of *factors* of type $\mathrm{I}_2$ (see [6]). Suppose that, for every $n$,

$$\langle p_n, q_n\rangle - \langle q_n, p_n\rangle = \varphi(q_n p_n - p_n q_n).$$

It is clear that $\varphi(q_n p_n - p_n q_n) \to \varphi(qp - pq)$ as $n \to \infty$. On the other hand,

$$\begin{aligned} |\langle p, q\rangle - \langle p_n, q_n\rangle| &\leqslant |\langle p - p_n, q\rangle| + |\langle p_n, q - q_n\rangle| \\ &\leqslant [\mu((p-p_n)^2)]^{1/2}[\mu(q)]^{1/2} + \\ &\quad + [\mu(p_n)]^{1/2}[\mu((q-q_n)^2)]^{1/2} \to 0 \quad \text{as} \quad n \to \infty, \end{aligned} \tag{5}$$

since $(p-p_n)^2 \to 0$ ultraweakly as $n \to \infty$. Hence equality (4) holds for the pair $p, q$. Now it is clear from the above that it suffices to prove (4) for pairs $p, q$ of projections from $A$ such that the von Neumann algebra $E$ generated by $p$ and $q$ is a factor of type $\mathrm{I}_2$.

Let $\psi \in E_*$ be such that the pair $(\mu|E, \psi)$ determines $\langle\cdot,\cdot\rangle|E\times E$. If $r \in \operatorname{Proj} E$ is one-dimensional and $r^\perp = 1_E - r$, then, by (5),

$$\varphi(r) - \varphi(r^\perp) = \psi(r) - \psi(r^\perp).$$

Using the same method as in the proof of Th. 2.1, we get

$$\psi - \varphi|E = \alpha\tau \quad \text{for some} \quad \alpha \in \boldsymbol{C}$$

($\tau$ is the normalized trace on $E$) and equality (4) follows.

Theorem 3.2. *Let $A$ be a $\sigma$-finite semifinite properly infinite von Neumann algebra, and let $\langle\cdot,\cdot\rangle$ be an OSP on $A$. Then there exists a unique real Gleason measure $\varphi$ on $\operatorname{Proj} A$, such that, for any $p, q \in \operatorname{Proj} A$ with $p \sim q$ and $p \perp q$,*

$$\varphi(p) - \varphi(q) = \langle u, u\rangle - \langle u^*, u^*\rangle,$$

*where $u \in A$ is a partial isometry with initial projection $p$ and final projection $q$. Moreover, $|\varphi(p)| \leqslant \mu(p)$ for any $p \in \operatorname{Proj} A$.*

To prove the theorem, we need some technical lemmas. The assumptions on $A$ and $\langle\cdot,\cdot\rangle$ are those of Th. 3.2.

Lemma 3.3. *If $p, q \in \operatorname{Proj} A$, $p \perp q$ and $p = u^*u = v^*v$, $q = uu^* = vv^*$ for some $u, v \in A$, then $\langle u, u\rangle = \langle v, v\rangle$.*

Proof. As $uv = vu = 0$, the elements $u \pm v^*$ are normal. By Prop. 1.5 (a), we obtain

$$\langle u \pm v^*, u \pm v^*\rangle = \mu((u^* \pm v)(u \pm v^*)) = \mu(p+q).$$

Adding the above two equalities, we get

$$\langle u, u\rangle + \langle v^*, v^*\rangle = \mu(p+q),$$

and by Prop. 1.5 (d), $\langle u, u\rangle = \langle v, v\rangle$.

Lemma 3.3 assures the correctness of the following definition.

DEFINITION 3.4. For any $p, q \in \operatorname{Proj} A$ with $p \sim q$ and $p \perp q$,

$$\alpha_{pq} \overset{\text{df}}{=} \langle u, u\rangle - \langle u^*, u^*\rangle,$$

where $u \in A$ is such that $u^* u = p$ and $uu^* = q$.

LEMMA 3.5. *If $p, q, r \in \operatorname{Proj} A$, $p \sim q \sim r$, and $p, q, r$ are mutually orthogonal, then*

$$\alpha_{pq} + \alpha_{qr} = \alpha_{pr}.$$

Proof. Let $u, v \in A$ be such that $u^* u = p$, $uu^* = v^* v = q$ and $vv^* = r$. Then the elements $\varepsilon_1 u + \varepsilon_2 v + \varepsilon_3 (vu)^*$, where $\varepsilon_k = \pm 1$ for $k = 1, 2, 3$, are all normal. As in Lemma 3.3, we check that

$$\langle \varepsilon_1 u + \varepsilon_2 v + \varepsilon_3 (vu)^*, \varepsilon_1 u + \varepsilon_2 v + \varepsilon_3 (vu)^*\rangle = \mu(p+q+r).$$

Adding the above eight equalities, we obtain

$$\langle u, u\rangle + \langle v, v\rangle + \langle (vu)^*, (vu)^*\rangle = \mu(p+q+r).$$

By symmetry,

$$\langle u^*, u^*\rangle + \langle v^*, v^*\rangle + \langle vu, vu\rangle = \mu(p+q+r),$$

which gives the desired result.

LEMMA 3.6. *Let $p, q_n, r_n \in \operatorname{Proj} A$ $(n = 1, 2, \ldots)$ with $p = q_1 = r_1$ finite, $p \sim q_n \sim r_n$ for every $n$, $q_n$ mutually orthogonal and $r_n$ mutually orthogonal. Then*

$$|\alpha_{pq_n} - \alpha_{pr_n}| \to 0 \quad \text{as} \quad n \to \infty, \tag{6}$$

*and $(\alpha_{pq_n})$ is a Cauchy sequence.*

Proof. Observe that, by 1.5 (d), $|\alpha_{pq}| \leqslant \mu(p+q)$. To show that $(\alpha_{pq_n})$ is a Cauchy sequence, it now suffices to apply Lemma 3.5. As for (6), take an arbitrary $\varepsilon > 0$. Let $n$ be so large that $\mu(q_n) < \varepsilon$ and $\mu(r_n) < \varepsilon$. Choose an $s \in \operatorname{Proj} A$ so that $s \sim p$, $s \perp p \vee q_n \vee r_n$ and $\mu(s) < \varepsilon$. Then

$$\alpha_{ps} = \alpha_{pq_n} + \alpha_{q_n s} \quad \text{and} \quad \alpha_{ps} = \alpha_{pr_n} + \alpha_{r_n s}.$$

and thus

$$|\alpha_{pq_n} - \alpha_{pr_n}| \leqslant |\alpha_{q_n s}| + |\alpha_{r_n s}| < 4\varepsilon.$$

The correctness of the definition given below follows immediately from Lemma 3.6.



DEFINITION 3.7. For a finite $p \in \operatorname{Proj} A$,

$$\varphi(p) \overset{\text{df}}{=} \lim_{n\to\infty} \alpha_{pq_n},$$

where $(q_n)$ is any sequence of mutually orthogonal and equivalent projections from $A$, such that $q_1 = p$.

LEMMA 3.8. (a) *For any finite $p \in \operatorname{Proj} A$, $|\varphi(p)| \leqslant \mu(p)$.*

(b) *For finite and orthogonal $p, q \in \operatorname{Proj} A$, $\varphi(p+q) = \varphi(p) + \varphi(q)$.*

(c) *Let $p_n \in \operatorname{Proj} A$ $(n = 1, 2, \ldots)$ be finite and $p_n \uparrow \infty$. Then $(\varphi(p_n))$ is a Cauchy sequence.*

(d) *Let $r, p_n, q_n \in \operatorname{Proj} A$ with $p_n$ and $q_n$ finite $(n = 1, 2 \ldots)$ and such that $p_n \uparrow r$, $q_n \uparrow r$. Then*

$$|\varphi(p_n) - \varphi(q_n)| \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. (a): Let $(q_n)$ be as in Def. 3.7. Then

$$|\varphi(p)| = \lim_{n\to\infty} |\alpha_{pq_n}| \leqslant \limsup_{n\to\infty} \mu(p+q_n) = \mu(p).$$

(b): Let $(r_n)$ be a sequence of mutually orthogonal and equivalent projections from $A$, such that $r_1 = p+q$. It is now possible to build sequences $(p_n)$ and $(q_n)$ in such a way that $p_n + q_n = r_n$ for every $n$, $p_1 = p$, $q_1 = q$, $p_n$ are mutually orthogonal and equivalent and so are $q_n$. Using Prop. 1.5 (c), it is easy to demonstrate that

$$\alpha_{pp_n} + \alpha_{qq_n} = \alpha_{p+q, r_n}.$$

Letting $n \to \infty$ in the above equality yields the desired result.

(c): Follows from (a).

(d): Take an arbitrary $\varepsilon > 0$. Let $n$ be large enough so that $\mu(r - p_n) < \varepsilon$ and $\mu(r - q_n) < \varepsilon$. Put $r_n = p_n \vee q_n$. Then, by (a) and (b),

$$\begin{aligned}|\varphi(p_n) - \varphi(q_n)| &= |\varphi(r_n - p_n) - \varphi(r_n - q_n)| \\ &\leqslant \mu(r - p_n) + \mu(r - q_n) < 2\varepsilon.\end{aligned}$$

Parts (c) and (d) of Lemma 3.8 give the correctness of the following definition.

DEFINITION 3.9. For $p \in \operatorname{Proj} A$,

$$\varphi(p) \overset{\text{df}}{=} \lim_{n\to\infty} \varphi(p_n),$$

where $p_n \in \operatorname{Proj} A$ are finite and $p_n \uparrow p$.

Proof of Theorem 3.2. Let $\varphi\colon \operatorname{Proj} A \to \boldsymbol{R}$ be the functional of Def. 3.9. The verification of the additivity of $\varphi$ and the inequality $|\varphi(p)| \leqslant \mu(p)$ for $p \in \operatorname{Proj} A$ is routine. They both, taken together, imply the countable additivity of $\varphi$, so that $\varphi$ is a real Gleason measure.

Now we must show that, for $p, q \in \operatorname{Proj} A$ with $p \sim q$ and $p \perp q$, $\alpha_{pq} = \varphi(p) - \varphi(q)$. This is easily seen for finite $p, q$. If $p, q$ are arbitrary, we choose finite $p_n, q_n$ so that $p_n \uparrow p$, $q_n \uparrow q$ and $p_n \sim q_n$. Let $u, u_n \in A\ (n = 1, 2, \ldots)$ be partial isometries such that $u^* u = p$, $uu^* = q$, $u_n^* u_n = p_n$, $u_n u_n^* = q_n$. It suffices to establish the convergence

$$\langle u_n, u_n \rangle - \langle u_n^*, u_n^* \rangle \to \langle u, u \rangle - \langle u^*, u^* \rangle \quad \text{as} \quad n \to \infty.$$

But since $u_n \to u$ ultrastrongly as $n \to \infty$, the above can be shown in exactly the same way as (5)' in Prop. 3.1. The uniqueness of the Gleason measure is evident.

Remark. Theorem 3.2 remains valid if we assume that $A$ is an arbitrary, not necessarily $\sigma$-finite, factor of type $\mathrm{I}_\infty$ or $\mathrm{II}_\infty$. Then the above proof requires only slight modifications.

Denote by $\mathrm{GM}(A)$ the set of all positive Gleason measures on an algebra $A$ and by $\mathrm{EGM}(A)$ the set of those positive Gleason measures on $A$ which can be extended to a normal positive linear functional on $A$.

COROLLARY 3.10. *There is a one-to-one correspondence between the set of all* OSP's *on $A$ and a set of pairs $(\mu, \varphi)$ with $\mu \in A_*^+$ and $\varphi$ being a real Gleason measure such that $|\varphi(p)| \leqslant \mu(p)$ for $p \in \operatorname{Proj} A$ (if $\varphi + \mu|\operatorname{Proj} A \in \mathrm{EGM}(A)$, then $(\mu, \varphi)$ corresponds to some* OSP *on $A$). The above set of pairs $(\mu, \varphi)$ is unique.*

Proof. By virtue of Props. 1.2 and 1.5 (d), it is sufficient to show that, if two OSP's, $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, give the same real Gleason measure $\varphi$, then

$$\langle x, x \rangle_1 - \langle x^*, x^* \rangle_1 = \langle x, x \rangle_2 - \langle x^*, x^* \rangle_2 \quad \text{for} \quad x \in A$$

or, which is equivalent by Prop. 1.7, that

$$\langle p, q \rangle_1 - \langle q, p \rangle_1 = \langle p, q \rangle_2 - \langle q, p \rangle_2 \quad \text{for} \quad p, q \in \operatorname{Proj} A.$$

But the proof of this is essentially the same as that of Prop. 3.1.

COROLLARY 3.11. *Every semifinite properly infinite von Neumann algebra $A$ which is $\sigma$-finite or a factor and such that $\mathrm{EGM}(A) = \mathrm{GM}(A)$ is an* OSP-*algebra. In particular, semifinite properly infinite factors, and semifinite properly infinite algebras with separable preduals are all* OSP-*algebras.*

Proof. The first part of the corollary results at once from Prop. 3.1 and Th. 3.2 together with the remark. The second part follows from the first one by the theorem of Paszkiewicz quoted in the Introduction.

## 4. Algebras of type I, $\mathrm{II}_1$ and III.

PROPOSITION 4.1. *If $A$ is an* OSP-*algebra and $B$ a commutative von Neumann algebra, then $C = A \bar{\otimes} B$ is an* OSP-*algebra.*

Proof. Let $\langle \cdot, \cdot \rangle$ be an OSP on $C$. Put $\langle x, y \rangle_p = \langle x \otimes p, y \otimes p \rangle$ for $x, y \in A$ and $p \in \operatorname{Proj} B$. Then $\langle \cdot, \cdot \rangle_p$ is an OSP on $A$ so that there exists a functional $\psi_p \in A_*^h$ such that



$$\langle x, y \rangle_p - \langle y, x \rangle_p = \psi_p(yx - xy) \quad \text{for} \quad x, y \in A_h.$$

Let $D$ be the linear subspace of $C$, algebraically spanned by elements of the form $x \otimes p$, where $x \in A$ and $p \in \operatorname{Proj} B$. Put $\varphi(x \otimes p) \overset{\mathrm{df}}{=} \psi_p(x)$ and extend $\varphi$ in an obvious way to $D$. Using the commutativity of $B$, we easily check that $\varphi$ is well defined and linear on $D$. If $u = \sum_k x_k \otimes p_k$ with $x_k \in A_+$, then

$$|\varphi(u)| \leqslant \sum_k |\psi_{p_k}(x_k)| \leqslant \sum_k \mu_{p_k}(x_k) = \sum_k \langle x_k \otimes p_k, 1 \otimes p_k \rangle$$
$$= \sum_k \langle x_k \otimes p_k, 1 \rangle = \sum_k \mu(x_k \otimes p_k) = \mu(u),$$

so that $\varphi$ is ultraweakly continuous on $D$. Since $D$ is ultraweakly dense in $C$, $\varphi$ can be uniquely extended to a functional $\bar{\varphi} \in A_*^h$ such that $-\mu \leqslant \bar{\varphi} \leqslant \mu$. We shall now show that $(\mu, \bar{\varphi})$ determines $\langle \cdot, \cdot \rangle$. Let $u = \sum_k x_k \otimes p_k$, $v = \sum_j y_j \otimes q_j$ with $x_k, y_j \in A_h$, $p_k, q_j \in \operatorname{Proj} B$. By Prop. 1.5 (c),

$$\langle u, v \rangle - \langle v, u \rangle = \sum_{k,j} \left( \langle x_k \otimes p_k q_j, y_j \otimes p_k q_j \rangle - \langle y_j \otimes p_k q_j, x_k \otimes p_k q_j \rangle \right)$$
$$= \sum_{k,j} \psi_{p_k q_j}(y_j x_k - x_k y_j) = \varphi(uv - vu)$$

and the proof is done.

COROLLARY 4.2. *Every type* I *von Neumann algebra is an* OSP-*algebra.*

Proof. This is a direct consequence of Prop. 4.1. and Cor. 3.11.

COROLLARY 4.3. *Each subalgebra of the hyperfinite factor $R$ of Murray and von Neumann is an* OSP-*algebra* (see [2], p. 73).

THEOREM 4.4. *If $A$ is a type* III *von Neumann algebra with separable predual or a type* III *factor, then $A$ is an* OSP-*algebra.*

Proof. Let $p \in \operatorname{Proj} A$ be such that $c(p) = c(1-p) = 1$, where $c(p)$ denotes the central support of $p$. Then there exists a sequence $(p_n)$ of mutually orthogonal and equivalent projectors of $A$ with $p = p_1$. Let $(u_n)$ be a sequence of partial isometries such that $u_n^* u_n = p$ and $u_n u_n^* = p_n$. For any $q \in \operatorname{Proj}(pAp)$ we put $q_n = u_n^* q u_n$ and $\varphi_p(q) \overset{\mathrm{df}}{=} \lim_{n \to \infty} \alpha_{q q_n}$ (compare this with Def. 3.4). Note that Lemmas 3.3, 3.5 and the second part of Lemma 3.6 are still valid in the case considered, so that $\varphi_p(q)$ is well defined. The mapping $\varphi_p$ is a real Gleason measure on $pAp$ (cf. parts (a) and (b) of Lemma 3.8) such that

$$\varphi(q) - \varphi(r) = \langle u, u \rangle - \langle u^*, u^* \rangle$$

for $q, r \in \operatorname{Proj}(pAp)$ with $q \sim r$ and $q \perp r$, where $u^*u = q$, $uu^* = r$. By Prop. 3.1, the extension of $\varphi_p$ to a functional $\bar{\varphi}_p \in (pAp)_*^h$ is such that the pair $(\mu|pAp, \bar{\varphi}_p)$ determines $\langle \cdot, \cdot \rangle$ on $pAp$. Choose now $r_n \in \operatorname{Proj} A$ with $c(r_n) = c(1-r_n) = 1$ so that $r_n \uparrow 1$. It is clear that $\bigcup_{n=1}^{\infty} (r_n A r_n)$ is ultraweakly dense in $A$, and that, for every $n$, $-\mu \leqslant \bar{\varphi}_{r_n} \leqslant \mu$. To end the proof of the theorem, it suffices now to apply Prop. 2.7.

**5. Correlation function of orthogonally scattered Gleason measure.** Let $A$ be a factor von Neumann algebra acting in a separable Hilbert space, and $H$ a (complex) Hilbert space of dimension $\geqslant 3$. An orthogonally scattered Gleason measure (or OSG-measure for short) was defined in [5], [3] as an $H$-valued Gleason measure $\xi$: $\operatorname{Proj} A \to H$ such that $(\xi(p), \xi(q)) = 0$ for $p \perp q$. By the above-mentioned theorem of Paszkiewicz [6], $\xi$ extends to an ultraweakly continuous operator $\xi$: $A \to H$ ($H$ with the weak topology), so that we may define

$$\langle x, y \rangle \stackrel{\mathrm{df}}{=} (\bar{\xi}(x), \bar{\xi}(y)).$$

It can easily be seen that $\langle \cdot, \cdot \rangle$ is an OSP on $A$. The results of Sections 2, 3 and 4 now yield the general form of the "correlation function" $(p, q) \mapsto (\xi(p), \xi(q))$ of the OSG-measure $\xi$: there exists a pair $(\mu, \varphi)$ with $\mu \in A_*^+$, $\varphi \in A_*^h$ and $-\mu \leqslant \varphi \leqslant \mu$, such that

$$(\xi(p), \xi(q)) = [\mu(qp + pq) + \varphi(qp - pq)]/2. \tag{7}$$

The above formula is proved in the paper for any factor which is not of type $\mathrm{II}_1$, and for the hyperfinite factor $R$ of Murray and von Neumann. One is tempted to believe that it remains valid for arbitrary factors of type $\mathrm{II}_1$.

Formula (7) may be rewritten in the following way:

$$(\xi(p), \xi(q)) = \psi_1(qp) + \psi_2(pq), \tag{8}$$

where $\psi_1 = (\mu+\varphi)/2$ and $\psi_2 = (\mu-\varphi)/2$. If $A$ is of type $\mathrm{I}_n$ $(3 \leqslant n < \infty)$ or $\mathrm{I}_\infty$, then

$$\psi_1 = \operatorname{tr}(m_1 \cdot) \quad \text{and} \quad \psi_2 = \operatorname{tr}(m_2 \cdot),$$

where $m_1, m_2 \in A$ are positive trace-class operators, and we get the formula from [5]:

$$(\xi(p), \xi(q)) = \operatorname{tr}(m_1 qp) + \operatorname{tr}(m_2 pq). \tag{9}$$

If $A$ is of type $\mathrm{II}_\infty$, we can still write formula (8) in the form (9) with tr replaced by $\tau$, the normalized trace on $A$, but now the operators $m_1, m_2 \in L_+^1(A, \tau)$ may not be bounded (see [8], [9]).



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