A Note on Unconditional Convergence

by

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A series

(1) $$\sum_{i=1}^{\infty} x_i$$

of elements x_i of a space E of type (B) is said to converge unconditionally if it converges no matter in what order the terms are arranged. Concerning such series W. ORLICZ has proved the following result[2]):

If the space E is weakly complete a necessary and sufficient condition for the unconditional convergence of the series (1) *is that the series*

(2) $$\sum_{i=1}^{\infty} |f(x_i)|$$

be convergent for each linear functional f in the conjugate space $\bar{E}$.

In this note we shall prove a similar theorem about series of elements of the conjugate space $\bar{E}$. We shall assume that the unit sphere: $\|x\| \leqslant 1$ of the space E is *weakly compact* — that is, given an infinite set $\{x_n\}$ with $\|x_n\| \leqslant 1$, it is possible to choose a subsequence $\{x_{n_i}\}$ and an element x_0 of E such that

$$\lim_{i \to \infty} f(x_{n_i}) = f(x_0)$$

for all f in $\bar{E}$. (It then follows that all bounded sets in E are weakly compact). The spaces L^p, l^p $(p > 1)$ have this property. The theorem to be proved is the following.

[1]) National Research Fellow at Princeton University (1937—38).

[2]) W. Orlicz, Beiträge zur Theorie der Orthogonalentwicklungen, II, Studia Math. 1 (1929) p. 1—39, Theorem 2.

Theorem 1. *Let E be a space of type (B) in which the unit sphere is weakly compact, and let*

(3) $$\sum_{i=1}^{\infty} f_i$$

be a series of elements of the conjugate space $\bar{E}$ (also of type (B)). Then a necessary and sufficient condition that the series (3) be unconditionally convergent is that the series

(4) $$\sum_{i=1}^{\infty} |f_i(x)|$$

converge for each element x of E.

Remarks: It is easily proved that if the unit sphere of E is weakly compact E is weakly complete. The converse is not true, however; for example, L is weakly complete but its unit sphere is not weakly compact. It is also known that if the unit sphere of E is weakly compact then the space $\bar{E}$ has this same property[3]). Consequently we required more of the space $\bar{E}$ than would be required by the theorem of ORLICZ, if applied to the series (3). The condition imposed by our theorem on series (4) is less stringent than that given by ORLICZ, however, for in general not every element of the conjugate space $\bar{\bar{E}}$ of $\bar{E}$ can be expressed in the form $f(x)$, where x is some element of E[4]).

The proof of the theorem rests on the following proposition:

Lemma. *Let E satisfy the condition of Theorem 1. Let the series (4) converge for each element x of E. Then the series converges uniformly for all elements x such that $\|x\| \leqslant 1$, and there exists a constant C, depending only on the $\{f_i\}$, such that*

(5) $$\sum_{i=1}^{\infty} |f_i(x)| \leqslant C \|x\|.$$

[3]) V. Gantmakher and V. Šmulian, Sur les espaces linéaires dont la sphère unitaire est faiblement compacte, Comptes Rendus (Doklady) de l'Académie des Sciences de l'U. R. S. S. 17 (1937) p. 91—94, Theorem 3.

[4]) If E is separable, however, our theorem is the same as that of Orlicz, for in this case every linear functional on $\bar{E}$ can be expressed in the form $f(x)$, where x is a suitably chosen element of E. (S. Banach, Opérations linéaires, p. 189—191).

Proof: Suppose the first assertion of the theorem were false. Then for some positive ε_0 there will exist a sequence of elements $\{x_\nu\}$, $\|x_\nu\| \leqq 1$, and a sequence of integers $\{n_\nu\}$, $\lim_{\nu\to\infty} n_\nu = \infty$, such that

$$\sum_{i=n_\nu}^{\infty} |f_i(x_\nu)| \geqq \varepsilon_0 \qquad (\nu = 1, 2, \ldots).$$

Since the unit sphere in E is weakly compact we may select a subsequence of $\{x_\nu\}$, say $\{y_\nu\}$, and an element y_0 of E such that $\lim_{\nu\to\infty} f(y_\nu) = f(y_0)$ for each f in $\bar{E}$. Therefore we may as well assume that

$$\sum_{i=n_\nu}^{\infty} |f_i(y_\nu)| \geqq \varepsilon_0 \qquad (\nu = 1, 2, \ldots). \tag{6}$$

Now let $\{\xi_i\}$ be a sequence of numbers forming an element of the space (m) so that $|\xi_i| \leqq M$ for some constant M. The series

$$f_\xi(x) = \sum_{i=1}^{\infty} \xi_i f_i(x)$$

is then evidently convergent, and it is clear, by a well known theorem[5]), that $f_\xi(x)$ is a linear functional defined on E. Hence $\lim_{\nu\to\infty} f_\xi(y_\nu) = f(y_0)$.

Consider next the matrix A of elements

$$A_{\nu i} = f_i(y_\nu).$$

We have

$$\sum_{i=1}^{\infty} A_{\nu i}\, \xi_i = f_\xi(y_\nu) \qquad (\nu = 1, 2, \ldots).$$

Thus A orders to the element ξ of (m) a convergent sequence $\{f_\xi(y_\nu)\}$ — that is, an element of the space (c). By a theorem of I. Schur[6]) it follows that we must have, given any positive ε, an N, depending on ε, such that

$$\sum_{i=N}^{\infty} |A_{\nu i}| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots).$$

[5]) Banach, loc. cit. p. 80, Theorem 5.

[6]) I. Schur, Über lineare Transformationen in der Theorie der unendlichen Reihen, Journ. f. Math. 151 (1921) p. 82 and 88—90.

In other words,

$$\sum_{i=N}^{\infty} |f_i(y_\nu)| \leqq \varepsilon \qquad (\nu = 1, 2, \ldots).$$

But this contradicts (6). The validity of the inequality (5) is easily deduced from what has just been established.

It is now easy to prove Theorem 1. As for the sufficiency of the condition, we define a functional f by the series

$$f(x) = \sum_{i=1}^{\infty} f_i(x).$$

This is the f_ξ used in the proof of the lemma, for the special choice $\xi_i = 1$, and it is an element of $\bar{E}$. We assert that

$$f = \sum_{i=1}^{\infty} f_i,$$

the convergence being according to the norm in $\bar{E}$. For

$$\|f - \sum_{i=1}^{n} f_i\| = \sup_{\|x\|\leqq 1} |f(x) - \sum_{i=1}^{n} f_i(x)|$$

$$= \sup_{\|x\|\leqq 1} |\sum_{i=n+1}^{\infty} f_i(x)| \leqq \sup_{\|x\|\leqq 1} \Big(\sum_{i=n+1}^{\infty} |f_i(x)|\Big)$$

and the expression on the right approaches zero as $n \to \infty$, by the lemma. Since the condition on series (4) is independent of the ordering of the f_i we conclude that (3) is unconditionally convergent.

The necessity of the condition is trivial in view of the equivalence of the concepts of absolute and unconditional convergence for numerical series.

As an application of the above results we shall prove a theorem concerning the nature of a linear transformation on E to (l).

Theorem 2. *If E is a space of type (B) in which the unit sphere is weakly compact the most general linear transformation on E to (l) has the form $T(x) = \xi = (\xi_1, \xi_2, \ldots)$, where $\xi_i = f_i(x)$, and $\sum_{i=1}^{\infty} f_i$ is an unconditionally convergent series of elements of $\bar{E}$. Conversely, every such series defines a linear transformation T. T is completely continuous.*

Proof. If $T(x)=\xi=(\xi_1,\xi_2,\ldots)$, and we define f_i by $\xi_i=f_i(x)$, it is clear that if T is linear, then since

$$\|\xi\|=\sum_{i=1}^{\infty}|f_i(x)|\leqslant\|T\|\|x\|,$$

the functionals f_i are also linear. By Theorem 1 they form an unconditionally convergent series.

Conversely, if, starting from the unconditionally convergent series (3), we define

$$T(x)=(f_1(x),f_2(x),\ldots)$$

then all that we need to do to prove that T is linear is to establish an inequality of the form

$$|\sum_{i=1}^{\infty}f_i(x)|\leqslant C\|x\| \tag{7}$$

where C is some positive constant. That such an inequality must hold is a trivial consequence of Theorem 1 and the lemma.

Theorem 2 is a sharpening of a theorem of B. J. Pettis [7]). That T is completely continuous may be proved directly with the aid of the lemma, or by an argument involving the fact that if a sequence in (l) converges weakly it also converges according to the norm. Since both proofs are essentially contained in the paper of Pettis (who makes use of the theorem of Orlicz), we shall not give any details.

In conclusion we shall show that for the validity of the criterion in Theorem 1 it is not sufficient to assume that E is weakly complete. For let E be the space L of functions defined and integrable (Lebesgue) on (0,1). L is weakly complete, as is well known. As the sequence $\{f_i\}$ we choose the functionals

$$f_i(x)=\int_0^1 x(t)\varphi_i(t)\,dt \qquad (i=1,2,\ldots),$$

where $x(t)\in L$ and

$$\varphi_i(t)=\begin{cases}1 \text{ for } & 1/2^{i+1}<t<1/2^i,\\ 0 \text{ „ } & 0\leqslant t\leqslant 1/2^{i+1},\ 1/2^i\leqslant t\leqslant 1.\end{cases}$$

[7]) B. J. Pettis, A Note on Regular Banach Spaces, Bull. Am. Math. Soc. 44 (1938) p. 420–428, Theorem 4'.

It is readily verified that

$$\sum_{i=1}^{\infty}|f_i(x)|\leqslant\int_0^{1/2}|x(t)|\,dt.$$

Therefore the sequence $\{f_i\}$ satisfies the conditions of the theorem [8]). Yet since $\bar{E}$ is equivalent to M, and

$$\|\sum_{i=m}^{n}f_i\|=\max|\sum_{i=m}^{n}\varphi_i(t)|=1 \text{ where } m<n,$$

the series $\sum_{i=1}^{\infty}f_i$ cannot be convergent. The above considerations also prove that the unit sphere in L is not weakly compact — a fact which may be shown by a variety of other methods.

[8]) In connection with this example see Orlicz, loc. cit. where it is used to prove that M is not weakly complete.

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Editorial remark. The paper of S. Banach and S. Mazur, Zur Theorie der linearen Dimension, Studia Math. 4 (1933) p. 100–112, contains the following theorem (p. 108, b) and p. 109 ref. [14])): *If the space E of type (B) is such that the space $\bar{E}$ is weakly complete, a necessary and sufficient condition for the unconditional convergence of the series $\sum_{i=1}^{\infty}f_i$ of elements of the space $\bar{E}$ is that the series $\sum_{i=1}^{\infty}|f_i(x)|$ converge for each element $x\in E$.* If the unit sphere in the space E is weakly compact, then the unit sphere in the space $\bar{E}$ is also weakly compact and therefore the space $\bar{E}$ is weakly complete (see l. c. [3])). Hence the above result of Banach and Mazur contains the theorem 1 of Mr. Taylor and even more. From the weak completness of the space $\bar{E}$ does not follow the weak compactness of a unit sphere in the space E, as for instance in the case of the space (C) of continuous functions. Similarly the lemma used in the proof of theorem 1 remains true if the space $\bar{E}$ is weakly complete (see the proof of b), p. 108–109); the same is to be said concerning theorem 2. The proofs of Mr. Taylor are direct and therefore interesting.