On normal inertia

by

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The object of this note is to extend to the case of non-singular bounded normal matrices the inertia theorem, previously proved by the writer in the particular hermitian case.

Definitions. In what follows, all capitals A, T,... will denote infinite matrices which are bounded in the sense of Hilbert and are non-singular, that is to say such as to have unique bounded reciprocal matrices A^{-1}, T^{-1},.... Denoting by an asterisk the transition to the transposed matrix of the conjugate complex elements, the properties which define a hermitian H, a unitary U and a normal N are $H^* = H$, $U^* = U^{-1}$ and $N^* N = NN^*$ respectively. The definition of an N may be expressed in terms of either of the more particular types H, U. In fact, it is easily verified that an arbitrary A is an N if and only if $i(A - A^*)(A + A^*)$ is an H, and also if and only if $A^* A^{-1}$ is a U.

Denoting by E the unit matrix and by λ any point in the complex plane, the spectrum of any A is defined as the set of those λ for which $\lambda E - A$ is not non-singular. The spectrum of A^{-1} consists of the reciprocal values of the numbers contained in the spectrum of A. The spectrum of any A is a set which is closed, bounded and not empty [1]). Conversely, any given set having these three properties is the spectrum of a suitable A, which may, as a fact, be chosen as an N. Furthermore, an N is a U or an H if and only if the spectrum lies on the boundary of the unit circle or on the real axis respectively. If an A is an H and only contains positive λ in its spectrum, then A is called positive definite and will be denoted by P.

[1]) A. Wintner, Math. Ztschr. 30 (1929), § 2.

Polar decomposition. The basis of the following considerations is a result[2]) which, apparently, can only be proved by means of the transcendental tool of HILBERT's spectral resolution, and states that for a given P there exists[3]) exactly one Q which is again a P and has the property that $QQ = Q^2$ is the given P; and that this unique Q, which will be denoted by $P^{1/2}$, is commutable with every A which is commutable with $Q^2 = P$ itself.

Now, the unique existence of Q implies[4]) that there exists for every A exactly one P and exactly one U such that $A = PU$. This theorem, which holds with restriction to the real field also, is an extension to Hilbert's space of a fact well known in the kinematics of continua, the linear deformation defined by an arbitrary non-singular A being decomposed[5]) into a unique rotation U and a unique dilatation P. Correspondingly, the unique P and unique U may be called the polar components of A, and will be denoted by $|A|$ and $\exp i\{A\}$ respectively.

There also exist for every A a unique $\tilde{P}$ and a unique $\tilde{U}$ such that $A = \tilde{U}\tilde{P}$ (instead of $A = PU$). However, one then can write $A = (\tilde{U}\tilde{P}\tilde{U}^{-1})\tilde{U}$, so that, since $\tilde{U}\tilde{P}\tilde{U}^{-1}$ clearly is a P, the uniqueness of the decomposition $A = PU$ implies that $\tilde{U} = U$, while $\tilde{P} = U^{-1}PU$. It is also seen that $\tilde{P} = P$ if and only if $UP = PU$. Since $A = PU$, this condition is satisfied if and only if $AA^* = A^*A$. In other words, an arbitrary A is an N if and only if $|A|$ and $\exp i\{A\}$ are commutable, that is, if and only if in the unique polar decompositions PU, $\tilde{U}\tilde{P}$ of A one not only has $U = \tilde{U}$ but also $P = \tilde{P}$. This implies that an N is an H if and only if $U = \exp i\{N\}$ is an H. In fact, the product of two H is again an H if and only if the two given H are commutable.

Transformations. If A_1 and A_2 are equivalent, that is, if $A_1 = T^*A_2T$ for a suitable T, write $A_1 \asymp A_2$. If A_1 and A_2 are similar, that is, if $A_1 = T^{-1}A_2T$ for a suitable T, write $A_1 \sim A_2$.

[2]) Loc. cit. [1]), § 7.

[3]) It should be mentioned that what is „transcendental" in this result is the uniqueness, and not the existence, of Q (Toeplitz). Cf. also F. J. Wecken, Math. Annalen 110 (1935).

[4]) A. Wintner, Amer. Journ. of Math. 54 (1932).

[5]) For finite matrices, the uniqueness of the polar decomposition has implicitly been proved by L. Autonne (1902). Cf. A. Wintner, Annali di Mat., ser. IV, 3 (1934) p. 108, footnote.

Finally, if A_1 and A_2 are unitarily equivalent, that is, if both conditions $A_1 = T^*A_2T$, $A_1 = T^{-1}A_2T$ may be satisfied by the same $T (= U)$, write $A_1 \cong A_2$. It is easily verified from the formal rules of HELLINGER and TOEPLITZ that any of these three notions is a class notion; that is, one has

$$(A_1 = A_2) \to (A_1 \| A_2),\ (A_1 \| A_2) \to (A_2 \| A_1),$$
$$(A_1 \| A_2 \text{ and } A_2 \| A_3) \to (A_1 \| A_3),$$

where $\|$ is any of the three signs $\asymp$, $\sim$, $\cong$, and the arrow denotes implication.

If $A_1 \sim A_2$, then A_1 and A_2 have the same spectrum. If $A \cong B$, where B denotes any of the four letters N, U, H, P, then A also is a B. If $A \asymp C$, where C denotes either of the letters H, P, then A also is a C. On the other hand, if $A \asymp U$, then A need not be unitary (and not even normal). Actually, two unitary matrices cannot be equivalent unless they are unitarily equivalent. This fact, which will be needed later on, may be proved as follows:

Let V and U be unitary and $T^*UT = V$ for some T. Put $T = WP$, where W is unitary and P positive definite. Then $PW^*UWP = V$. This may be written in the form

$$P(W^*WU) = (VP^{-1}V^{-1})V.$$

But P and $VP^{-1}V^{-1}$ are positive definite; while W^*UW and V are unitary, since so are U, V, W. It follows, therefore, from the uniqueness of a polar decomposition that $W^*UW = V$; *q. e. d.*

The problem of equivalence. If $x, u, \ldots$ denote points of the complex Hilbert space and $A(x, \bar{y})$ the bilinear form which belongs to a matrix A, the conjugate complex, cogredient linear substitutions $x = Tu$, $\bar{y} = \bar{T}\bar{v}$ transform $A(x, \bar{y})$ into a bilinear form in u and $\bar{v}$ which has the matrix T^*AT, since $A(x, \bar{y}) = y^*Ax$. If, in particular, A is an H and the Hilbert space is replaced by a vector space with a finite number of dimensions, the question as to the complete system of invariants of an A under the transformation group of equivalence is answered by the signature criterion of the classical inertia theorem of Jacobi-Sylvester. It was recently shown[6]) that the same criterion holds in case of Hilbert's

[6]) A. Wintner, Math. Ztschr. 37 (1933).

space also, although H may then have a continuous spectrum. This is not a contradiction, since the spectrum of an A is invariant under similarity, but not under equivalence, transformations of A. In other words, while the usual proofs of the Jacobi-Sylvester theorem break down in case of Hilbert's space, the theorem itself is valid in this case also.

There now arises the question as to the complete system of invariants of an A under the transformation group of equivalence, if A is not restricted to be an H. Apparently, this problem has never been attacked even in the case of finite matrices. It will, however, be shown that, also in case of Hilbert's space, the problem can completely be solved if the A are restricted to be normal. In, fact, it will turn out that the problem of equivalence can then be reduced to the unitary equivalence of unitary matrices. Now, while, the latter problem depends on the structure of the spectra which are, in general, continuous, it is known[7]) that the problem as to the complete system of the unitary invariants of a U may be reduced to the problem as to the complete system of unitary invariants of an H. But the latter problem has been solved by HELLINGER[8]). Accordingly, the equivalence problem of the N is completely solved by the following theorem:

Two bounded non-singular normal matrices N_1, N_2 *are equivalent if and only if their unitary polar components* $\exp i\{N_1\}$, $\exp i\{N_2\}$ *are unitarily equivalent.*

The signature criterion[6]) mentioned above, is a particular case of this theorem. In fact, if N_1, hence also $N_2 (\asymp N_1)$, is an H, then so are their unitary components (cf. above). Hence, the spectra of both unitary components lie on the boundary of the unit circle and also on the real axis. Consequently, the unitary components cannot have continuous spectra, and are unitarily equivalent if and only if they are unitarily equivalent to one and the same diagonal matrix, in which all diagonal elements are ± 1.

It is similarly shown that the non-existence[9]) of invariants of equivalency for real skew-symmetric A also is a corollary. In fact, the general theorem and its proof are valid in the real field also.

[7]) Loc. cit. [1]), § 8.

[8]) E. Hellinger, Dissertation, Göttingen 1907.

[9]) A. Wintner Proc. Edinburgh Math. Soc. 1937.

The known cases of a (real or complex) $H = H^*$ and a (real or complex) $S = iH$ present two essential simplifications. For in case of a general N, which may be real or complex,

(I) the problem of equivalence can only be solved by means of a Hellinger analysis of the continuous spectra (which never occur if $N = H$ or $N = S = iH$);

(II) it is no longer true that T^*NT is an N for every T.

Proof of the theorem. The theorem announced before states that

$$N_1 \asymp N_2 \text{ if and only if } \exp i\{N_1\} \cong \exp i\{N_2\}.$$

But it was proved above that if U and V are unitary and $U \asymp V$, then $U \cong V$. On the other hand, it is obvious that if $U \cong V$, then $U \asymp V$. Consequently, the theorem to be proved is equivalent to the statement that

$$N_1 \asymp N_2 \text{ if and only if } \exp i\{N_1\} \asymp \exp i\{N_2\}.$$

Since the symbol $\asymp$ has the class properties of consistency, symmetry and transitivity, it follows that the theorem to be proved is equivalent to the statement that

$$N_1 \asymp \exp i\{N_1\} \text{ and } N_2 \asymp \exp i\{N_2\}.$$

In other words, one has merely to show that $N \asymp \exp i\{N\}$ for any N.

Now, the polar components $|N|$, $\exp i\{N\}$ of any N were seen to be commutable. Furthermore, it was pointed out that $P^{1/2}$ is, for any P, commutable with any A which is commutable with P. Placing $P = |N|$ and $A = \exp i\{N\}$, it follows that the polar decomposition $N = |N| \exp i\{N\}$ may be written as $N = |N|^{1/2} (\exp i\{N\}) |N|^{1/2}$. But $|N|^{1/2}$ is positive definite, hence hermitian. Thus, the condition $N = T^* (\exp i\{N\}) T$ for $N \asymp \exp i\{N\}$ is satisfied by $T = |N|^{1/2}$.

Remark. On replacing the problem of equivalence, treated above, by the corresponding problem of similarity, one might expect that

$$N_1 \cong N_2 \text{ whenever } N_1 \sim N_2.$$

Actually, this is known to be true in the particular cases $N_1 = H_1$, $N_2 = H_2$ and $N_1 = U_1$, $N_2 = U_2$ only, in which cases it may be proved by above methods [10]). In case of an arbitrary pair $N_k = P_k U_k$, where $k = 1, 2$, there arises a difficulty, although $P_k U_k = U_k P_k$ by assumption. In case of finite matrices, it follows, of course, by transformation to the diagonal form, that similar normal matrices are unitarily equivalent.

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[10]) Loc. cit. [4]).

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