THEOREM 2. *Suppose that the assumptions of Theorem* 1 *are fulfilled and, moreover,*

(ii′) *there is a sequence* (λ_k) *with* $\lambda_k \to +\infty$ *or* $\lambda_k \to -\infty$ *such that for all* $x \in D(A)$

$$\|(\lambda_k - A)x\| \geqslant M^{-1}(|\lambda_k| - \omega)\|x\|.$$

Then A is closable and its closure generates a one-parameter strongly continuous group G on E satisfying (*).

Proof. The closability of A results immediately from the lemma.

Let $\tilde{A}$ denote the closure of the restriction of A to $\mathscr{A}(A)$. By Theorem 1, $\tilde{A}$ generates a strongly continuous one-parameter group G on E, satisfying (*). By the Hille–Yosida theorem, if $\lambda_k > \omega$, then $(\lambda_k - \tilde{A})$ maps $D(\tilde{A})$ in one-to-one manner onto E. By (ii′), $\lambda_k - \bar{A}$ is an injection on $D(\bar{A})$ that coincides with $\lambda_k - \tilde{A}$ on $D(\tilde{A})$. Thus $\bar{A} = \tilde{A}$, which ends the proof.

Our second corollary to Theorem 1 is

THEOREM 3. *Let E be a Banach space and A an operator on E. Suppose that*

(α) *the set of analytic vectors of A is dense in E,*

(β) *for all* $\lambda \in R - \{0\}$ *and all* $x \in D(A)$

$$\|(A - \lambda)x\| \geqslant |\lambda|\,\|x\|.$$

Then A has a closure generating a strongly continuous one-parameter group of operators.

Notice that the above theorem immediately yields Nelson's result, as (β) is equivalent to the simultanous dissipativity of A and $-A$ and to the skew-symmetry of A if E is a Hilbert space.

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STUDIA MATHEMATICA, T. LXXIX. (1984)

Functional calculus and the Gelfand transformation

by

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Abstract. The commutativity of Taylor's functional calculus [8] with the Gelfand transformation is proved. It is shown as a corollary that each axiomatic joint spectrum in the sense of Żelazko [10] which is contained in Taylor's joint spectrum satisfies a spectral mapping property with respect to Taylor's analytic functional calculus.

1. Introduction. Let X be a complex Banach space and let us denote by $C^n(X)$ the set of all commutative n-tuples of linear bounded operators on X, $n \geqslant 1$. An *axiomatic joint spectrum* [10] is an assignment σ from each $C^n(X)$ into the closed subsets of C^n, respectively, which satisfies the following axioms:

(i) $\sigma(a)$ is the usual spectrum in the case of a single linear operator $a \in L(X) = C^1(X)$.

(ii) σ has the projection property, i.e., for each $a \in C^{n+1}(X)$ one has $\pi\sigma(a) = \sigma(\pi(a))$, denoting by π the natural projection onto the first n coordinates.

(iii) σ has the spectral mapping property with respect to the polynomial maps, that is, for each polynomial p in n indeterminates and for each $a \in C^n(X)$, the equality $p(\sigma(a)) = \sigma(p(a))$ holds true.

A direct improvement of [7] and [10] shows that to such an axiomatic joint spectrum σ is associated a closed subset $\sigma(A)$ in the maximal spectrum $M(A)$ of a commutative, unital Banach subalgebra A of $L(X)$, such that $\sigma(A)$ is minimal with respect to the property (a) below:

(a) The equality $\hat{a}(\sigma(A)) = \sigma(a)$ holds for every $a \in C^n(X)$ and $n \geqslant 1$.

We have denoted above by $\hat{b}$ the Gelfand transformation of $b \in A$. Then one proves that the set $\sigma(A)$ is functorial in A, in the following sense:

(b) If $A \subset B \subset L(X)$ are two algebras as above, then $i^*\sigma(B) = \sigma(A)$, denoting by i: $A \to B$ the inclusion map.

The set $\sigma(A)$ defined above is sufficiently large in $M(A)$ because of axiom (i). More exactly:

(c) If A is a maximal abelian Banach subalgebra of $L(X)$ and if $a \in A$, then $\hat{a}(\sigma(A)) = \hat{a}(M(A))$.

Let us remark now that if we postulate a closed subset $\sigma(A)$ of the Gelfand spectrum $M(A)$ of every maximal abelian subalgebra A in $L(X)$ such that condition (c) holds true and the coherence derived from (b) is satisfied, i.e., $i^*\sigma(A) = j^*\sigma(B)$ where A and B are maximal abelian subalgebras and $i\colon A\cap B \to A$, $j\colon A\cap B \to B$ are the inclusion maps, then (a) defines an axiomatic joint spectrum, and conversely. It is interesting to note that this set-theoretic alternative definition of the joint spectrum implies analytic properties of it.

It is known that there are many joint spectra [6], but among them Taylor's joint spectrum has a privileged place, not only because it seems to be maximal (this is an open question raised by Żelazko), but also because of the existence in its neighbourhoods of the analytic functional calculus with nice functorial properties, [8]. We shall prove below that each axiomatic joint spectrum which is contained in Taylor's spectrum is still functorial with respect to analytic base changes given by the functional calculus of J. L. Taylor. This fact decreases the gap between the two spectra from the point of view of the analytic functional calculus.

2. The main result. Let us denote by $\mathrm{Sp}(a)$ the joint spectrum of an n-tuple $a\in C^n(X)$, in the sense of [7], (the space X is fixed in that note, so that it is implied in the notations). For every closed commutative subalgebra A of $L(X)$ we have correspondingly the set $\mathrm{Sp}(A)$ in its maximal spectrum. If $f\in\mathcal{O}(\mathrm{Sp}(a))$ is an analytic function in a neighbourhood of $\mathrm{Sp}(a)$, then $f(a)$ will be the operator given by the functional calculus constructed in [8].

THEOREM. *Let X be a complex Banach space and let a be a commutative n-tuple of linear bounded operators on X.*

If A is a commutative, unital Banach subalgebra of $L(X)$ which contains the range of Taylor's functional calculus of the n-tuple a, then $\widehat{f(a)}(x) = f(\hat{a})(x)$ for every $f\in\mathcal{O}(\mathrm{Sp}(a, X))$ and every $x\in\mathrm{Sp}(A, X)$.

Proof. Let f be a holomorphic function in a neighbourhood U of $\mathrm{Sp}(a, X)$ and let us consider the functional calculus map

$$\Theta\colon \mathcal{O}(U)\to A.$$

It is a continuous morphism of Fréchet algebras, hence we have an induced map between the spaces of continuous characters:

$$\Theta^*\colon M(A)\to \tilde{U}.$$

The set $\tilde{U}$ above can be identified with the disjoint union of the envelopes of holomorphy of the connected components of the open set U. Thus $\tilde{U}$ is a finite-dimensional Stein manifold, [3], Theorem 5.4.5, so that it can be immersed in a numerical space, [3], Theorem 5.3.9, that is, there exists an N-tuple of analytic functions $F\in\mathcal{O}(U)^N$ such that the extension $\tilde{F}\colon \tilde{U}\to C^N$ is one to one.

Then the spectral mapping theorem of [8] asserts that $F(\mathrm{Sp}(a)) = \mathrm{Sp}(F(a))$ or $F(\hat{a}(\mathrm{Sp}(A))) = \widehat{F(a)}(\mathrm{Sp}(A))$. Denoting by $z = (z_1, \ldots, z_n)$ the system of coordinates, the last equality becomes $(\tilde{F}\circ\tilde{z}\circ\Theta^*)(\mathrm{Sp}(A)) = (\tilde{F}\circ\Theta^*)(\mathrm{Sp}(A))$ and because of the injectivity of $\tilde{F}$ we obtain $\Theta^*(\mathrm{Sp}(A)) = \mathrm{Sp}(a)\subset U\subset\tilde{U}$.

Let us now consider a polynomial p on C^n and a character $x\in\mathrm{Sp}(A)$. Then

$$p(\Theta^*(x)) = \widehat{\Theta(p)}(x) = \widehat{p(a)}(x) = p(\hat{a}(x)).$$

But the polynomial functions separate the points of U, therefore $\Theta^*(x) = \hat{a}(x)$ and consequently

$$\widehat{f(a)}(x) = \widehat{\Theta(f)}(x) = f(\Theta^*(x)) = f(\hat{a}(x)). \blacksquare$$

The above theorem has many corollaries derived directly from the functional calculus theory in Banach algebras, like the superposition property and the implicit function theorem (see [2] for comparison and [4] for other proofs of them in the framework of Taylor's functional calculus). We state here only the following

COROLLARY. *Let σ be an axiomatic joint spectrum contained in Taylor's joint spectrum. Then the equality*

$$f(\sigma(a)) = \sigma(f(a))$$

holds true for every $f\in\mathcal{O}(\mathrm{Sp}(a, X))^m$, $a\in C^n(X)$ and $n, m\geqslant 1$.

Proof. Let A be a commutative Banach subalgebra of $L(X)$ which contains the range of the analytic functional calculus of the n-tuple a. By applying the theorem to the algebra A,

$$f(\sigma(a)) = f(\hat{a}(\sigma(A))) = \widehat{f(a)}(\sigma(A)) = \sigma(f(a))$$

and the proof is complete.

It would be interesting to know whether in the above conditions an arbitrary function $f\in\mathcal{O}(\sigma(a))$ admits a holomorphic extension to a neighbourhood of $\mathrm{Sp}(a, X)$. Note that $g(\mathrm{Sp}(a)) = g(\sigma(a))$ for every $g\in\mathcal{O}(\mathrm{Sp}(a))$.

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A note on criteria of Le Page and Hirschfeld-Żelazko for the commutativity of Banach algebras

by

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Abstract. A theorem of Le Page [6] states that a unital complex Banach algebra A is commutative if and only if there exists a constant $k > 0$ such that $\|xy\| \leqslant k\|yx\|$ for all $x, y \in A$. Hirschfeld and Żelazko [5] proved that a complex Banach algebra A is commutative if there exists a constant $k > 0$ such that $\|x\|^2 \leqslant k\|x^2\|$ for all $x \in A$; in the case that A possesses a unit this result is an easy consequence of the theorem of Le Page. In [3] Bonsall and Duncan showed that a spectral state f of a unital complex Banach algebra A has the following property: $f(xy) = f(yx)$ $(x, y \in A)$. We shall now demonstrate how all these results and even generalizations of these results can be obtained as immediate consequences of a single theorem which is also valid for Banach algebras without unit or without approximate unit. The case that a bounded approximate identity exists has been studied by Baker and Pym [2].

In the following A denotes a complex Banach algebra. Let E be a normed vector space over the field of complex numbers.

THEOREM. *For a continuous linear operator T: $A \to E$ the following statements are equivalent*:

(i) *There exists $k > 0$ such that* $\|T(xy+y)\| \leqslant k\|yx+y\|$ *for all* $x, y \in A$.

(ii) $T(xy) = T(yx)$ *for all* $x, y \in A$.

Proof. The implication (ii)$\Rightarrow$(i) is obvious (with $k = \|T\|$).

(i)$\Rightarrow$(ii): If A is unital, define $\tilde{A} := A$; if A is not unital, let $\tilde{A}$ be the Banach algebra obtained by adjoining an identity to A. Let $x, y \in A$ be fixed. Then we consider the function G: $\boldsymbol{C} \to E$ which is defined in the following way:

$$G(\lambda) := T[\exp(-\lambda x)\, y \cdot \exp(\lambda x)] \quad (\lambda \in \boldsymbol{C}).$$

Note that A is a two-sided ideal in $\tilde{A}$ and that $\exp(-\lambda x)\, y \cdot \exp(\lambda x)$ lies in A even if A is not unital. G is an entire (vector-valued) function, and – as we shall see now – G is bounded.