convergent, f_1 is in $C^p(G)$. On the other hand, since all functions $\alpha_k f_{m_k,n_k}$ are non-negative,

$$|||f_1|||_q \geqslant \sup_k \alpha_k |||f_{m_k,n_k}|||_q = \infty$$

for all $q < p$ and so $f_1 \notin C^q(G)$.

For the exponent p', where $1/p+1/p' = 1$, let f_2 be a function constructed in the same way as f_1 with p' in place of p. Then $f_2^* \in C^p(G)$ but $f_2^* \notin C^q(G)$ for $q > p$. Consequently the function $f = f_1 + f_2^*$ has the property claimed in the theorem.

COROLLARY. *Let G be a non-commutative free group and let $1 < p < \infty$, $p \neq 2$. There exists a non-negative function f on G such that the operator $g \to f * g$ is bounded on $l^p(G)$ but the operator $g \to g * f$ is unbounded.*

Proof. The operator $g \to g * f$ is bounded on $l^p(G)$ if and only if $f^* \in C^p(G)$ which is equivalent to $f \in C^{p'}(G)$, where $1/p+1/p' = 1$.

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Analytic vectors and generation of one-parameter groups

by

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Abstract. We give a Hille–Yosida type condition for an operator to generate a one-parameter strongly continuous group on a Banach space, in terms of analytic vectors of the operator.

Introduction. Let E be a Banach space, and A a linear operator on E with domain $D(A)$. An element x in $\bigcap_{n=1} D(A^n)$ is called an *analytic vector for* A if for some $t > 0$

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \|A^n x\| < +\infty .$$

By a standard spectral projection argument one easily proves that the self-adjoint operator on a Hilbert space has a dense set of analytic vectors. Conversely, a result of Nelson [4], Lemma 5.1, states that the symmetric operator on a Hilbert space, with a dense set of analytic vectors, is essentially self-adjoint.

The main objective of the present paper is to establish Banach space counterparts to the above assertions. Our starting point is the observation that the skew-adjoint operators coincide with the generators of one-parameter strongly continuous unitary groups (Stone's theorem). We accordingly shift attention from self-adjoint operators to generators of one-parameter strongly continuous groups.

The generalization of the first assertion is routine. Indeed, given a Banach space E, if A generates a one-parameter strongly continuous group G on E, then for every $x \in E$

$$\sqrt{k/\pi} \int_R \exp(-kt^2) G(t) x \, dt \qquad (k > 0)$$

is an analytic vector for A which tends to x as $k \to +\infty$; accordingly, A has a dense set of analytic vectors.

The generalization of Nelson's result is much more involved. It takes the form of the following

THEOREM 1. *Let E be a Banach space and A an operator on E. Suppose that*

(i) *the set $\mathscr{A}(A)$ of analytic vectors for A is dense in E;*

(ii) *there exist $\omega, M > 0$ such that for all $\lambda \in R$ with $|\lambda| > \omega$, all $m \in N$, and all $x \in \mathscr{A}(A)$*

$$\|(\lambda - A)^m x\| \geqslant M^{-1}(|\lambda| - \omega)^m \|x\|.$$

Then the restriction of A to $\mathscr{A}(A)$ is closable and its closure generates a one-parameter strongly continuous group G on E such that

$$(*) \qquad \|G(t)\| \leqslant M \exp(\omega |t|).$$

Incidentally, the above theorem generalizes a certain unproved result of [1], Theorem 3.1.22.

1. Prerequisites. In the sequel, we make use of the following important

THEOREM (de Leeuw [3]). *Let G be a one-parameter strongly continuous semi-group on a Banach space E, with generator A. Let D be a G-invariant linear subspace of $D(A)$, dense in E. Then the closure of the restriction of A to D coincides with A.*

We also need the following lemma, which is of interest in its own.

LEMMA. *Let A be a densely defined operator on a Banach space E. Suppose there exist a constant $b > 0$ and a sequence λ_k of positive numbers diverging to infinity such that*

$$(1.1) \qquad \|(A - \lambda_k)x\| \geqslant b\lambda_k \|x\|$$

for all $x \in D(A)$. Then A is closable.

Proof. Let (x_n) be a sequence in $D(A)$ tending to zero with $\lim_{n\to\infty} Ax_n = x$. We shall show that $x = 0$.

By (1.1), we have for all $y \in D(A)$ and all $k, n \in N$

$$b\lambda_k \|y + \lambda_k x_n\| \leqslant \|A(y + \lambda_k x_n) - \lambda_k(y + \lambda_k x_n)\|.$$

Letting n tend to infinity, we get

$$b\lambda_k \|y\| \leqslant \|Ay + \lambda_k(x - y)\|$$

and further

$$\|y\| \leqslant b^{-1}(\lambda_k^{-1}\|Ay\| + \|x - y\|).$$

Hence

$$\|y\| \leqslant b^{-1}\|x - y\|,$$

and so

$$\|x\| \leqslant (1 + b^{-1})\|x - y\|.$$

Since y can be chosen arbitrarily close to x, it follows that $x = 0$. The proof is complete.

2. Proof of Theorem 1. Without loss of generality, we may assume that $D(A) = \mathscr{A}(A)$. The closability assertion is an immediate consequence of (ii) and the lemma.

For each $n \in N$, let D_n be the completion of $\mathscr{A}(A)$ under the norm

$$\sum_{i=0}^{n} \|A^i x\|, \qquad x \in \mathscr{A}(A).$$

By a standard argument, one can identify D_n with a subspace of $D(\bar{A}^n)$ endowed with the norm

$$\|x\|_n = \sum_{i=1}^{n} \|\bar{A}^i x\|, \qquad x \in D_n.$$

In view of (ii), we have for all $x \in D_n$ and all $\lambda \in R$ with $|\lambda| > \omega$

$$(2.1) \qquad \|(\bar{A} - \lambda)^n x\| \geqslant M^{-1}(|\lambda| - \omega)^n \|x\|.$$

Let $D_\infty = \bigcap_{n=1}^{\infty} D_n$. For each $r > 0$, let

$$\mathscr{A}_r = \{x \in D_\infty: \sum_{n=1}^{\infty} \frac{t^n}{n!} \|\bar{A}^n x\| < \infty \text{ for } 0 < t < r\}.$$

For each $x \in \mathscr{A}_r$ and each $t \in R$ with $|t| < r$, put

$$(2.2) \qquad G(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} \bar{A}^n x,$$

the sum being taken in the norm topology of E.

First we check that if $x \in \mathscr{A}_r$ and $|t| < r$ then

$$(2.3) \qquad G(t)x = \lim_{n\to\infty} \left(1 + \frac{t\bar{A}}{n}\right)^n x.$$

In fact, we have for any $n \in N$

$$\left(1 + \frac{t\bar{A}}{n}\right)^n x = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{t}{n}\right)^k \bar{A}^k x.$$

Since for each $k \in Z_+$, $\lim_{n\to\infty} n^{-k}\binom{n}{k} = 1/k!$, and for any $n \in N$ $n^{-k}\binom{n}{k} < 1/k!$, the result now follows from Lebesgue's dominated convergence theorem.

In a like manner we prove that if $x \in \mathcal{A}_r$ and $|t| < r/2$ then

$$\lim_{n\to\infty}\left(1-\frac{t\bar{A}}{n}\right)^n\left(1+\frac{t\bar{A}}{n}\right)^n x = x. \tag{2.4}$$

We claim that for all $x \in \mathcal{A}_r$ and all $t \in R$ with $|t| < r/2$

$$\|G(t)x\| \leqslant M\exp(\omega|t|)\|x\|. \tag{2.5}$$

Indeed, given $n \in N$, $x \in \mathcal{A}_r$, and $t \in R$ with $|t| < r/2$ we have by (2.1)

$$M^{-1}\left(1-\frac{\omega|t|}{n}\right)^n\left\|\left(1+\frac{t\bar{A}}{n}\right)^n x\right\| \leqslant \left\|\left(1-\frac{t\bar{A}}{n}\right)^n\left(1+\frac{t\bar{A}}{n}\right)^n x\right\|.$$

In view of (2.3) and (2.4), the claim now follows upon letting n tend to infinity.

Next, we show that if $|t| < r/2$ then $\mathcal{A}_r$ is $G(t)$-invariant and for all $x \in \mathcal{A}_r$ and all $s, t \in R$ with $|s|, |t| < r/2$, we have

$$G(s)\,G(t)x = G(s+t)x. \tag{2.6}$$

Indeed, it is easily seen that $\mathcal{A}_r$ is an invariant space for all powers of $\bar{A}$. Thus for $x \in \mathcal{A}_r$ the series in (2.2) converges in each norm $\|\ \|_m$. Consequently $G(t)x \in D_\infty$. Since $\bar{A}^m$ is a continuous operator from D_m to E, we have for $x \in \mathcal{A}_r$, $k \in N$, and $t \in R$ with $|t| < r$

$$\bar{A}^k G(t)x = G(t)\bar{A}^k x. \tag{2.7}$$

If $|t| < r/2$ and $0 < u < r$, then by (2.5) and (2.7)

$$\sum_{k=0}^{\infty}\frac{u^k}{k!}\|\bar{A}^k G(t)x\| = \sum_{k=0}^{\infty}\frac{u^k}{k!}\|G(t)\bar{A}^k x\| \leqslant M\exp(\omega|t|)\sum_{k=0}^{\infty}\frac{u^k}{k!}\|\bar{A}^k x\|,$$

which proves the invariance assertion. Moreover, if $|s| < r/2$, then

$$\sum_{k,l=0}^{\infty}\frac{|s|^k|t|^l}{k!\,l!}\|\bar{A}^{k+l}x\| = \sum_{p=0}^{\infty}\sum_{k+l=p}\frac{|s|^k|t|^l}{k!\,l!}\|\bar{A}^p x\| \leqslant \sum_{p=0}^{\infty}\frac{r^p}{p!}\|\bar{A}^p x\|,$$

and by (2.7) we can write

$$G(s)\,G(t)x = \sum_{k=0}^{\infty}\frac{s^k}{k!}G(t)\bar{A}^k x = \sum_{k=0}^{\infty}\frac{s^k}{k!}\sum_{l=0}^{\infty}\frac{t^l}{l!}\bar{A}^{k+l}x$$
$$= \sum_{p=0}^{\infty}\sum_{k+l=p}\frac{s^k t^l}{k!\,l!}\bar{A}^p x = G(s+t)x.$$

For each $r > 0$, let E_r be the closure of $\mathcal{A}_r$ in E. Given $t \in R$ with $|t| < r/2$, extend $G(t)$ by continuity to a bounded operator from E_r into itself, still denoted by $G(t)$, so that (2.5) and (2.6) continue to hold. Given $t \in R$, select $n \in N$ so that $|t|/n < r/2$ and set

$$G(t) = G\left(\frac{t}{n}\right)^n.$$

A routine verification yields that the right-hand side does not depend on n. Moreover, the operators $G(t)$ $(t \in R)$ form a locally equicontinuous one-parameter group G on E_r. Taking into account that for all $x \in \mathcal{A}_r$ the function $(-r, r) \ni t \to G(t)x \in E_r$ is continuous, we easily deduce that G is strongly continuous. A step by step application of the global $G(t)$-invariance of $\mathcal{A}_r$ for $t \in R$ with $|t| < r/2$ yields the global G-invariance of $\mathcal{A}_r$.

Let A_r denote the generator of G restricted to E_r. By de Leeuw's theorem, we have

$$A_r = \overline{A|\mathcal{A}_r} \tag{2.8}$$

By the Hille–Yosida theorem [2], Theorem 12.3.2, there exist $\omega_r, M_r > 0$ such that for all $\lambda \in R$ with $|\lambda| > \omega_r$

$$(A_r - \lambda)D(A_r) = E_r \tag{2.9}$$

and for all $m \in N$ and all $y \in E_r$

$$\|(A_r-\lambda)^{-m}y\| \leqslant M_r(|\lambda|-\omega_r)^{-m}\|y\|. \tag{2.10}$$

By (2.1), for all $m \in N$, all $\lambda \in R$ with $|\lambda| > \omega$, and all $y \in (A_r - \lambda)\mathcal{A}_r$

$$\|(A_r-\lambda)^{-m}y\| \leqslant M(|\lambda|-\omega)^{-m}\|y\|. \tag{2.11}$$

Since, by (2.8)–(2.10), $(A_r - \lambda)\mathcal{A}_r$ is dense in E_r for all $\lambda \in R$ with $|\lambda| > \omega_r$, it follows that (2.11) holds for $y \in E_r$ if $|\lambda| > \tilde{\omega}_r = \max(\omega_r, \omega)$. Thus, by the Hille–Yosida theorem, the group $G(t)$ as a group acting in E_r satisfies (*). The fact that $\tilde{\omega}_r \neq \omega$ is not essential, as can easily be seen on inspection of the proof of the Hille–Yosida theorem (cf. the remark following the proof of Theorem 12.3.1 in [2]).

Notice that, given $t \in R$, $G(t)$ on $E_r (r > 0)$ fit together to form a bounded operator on $\bigcup_{r>0} E_r$ being dense in E. Apparently, as t runs over R, $G(t)$ form a strongly continuous one-parameter group G on E satisfying (*). Since each $\mathcal{A}_r (r > 0)$ is G-invariant, so is $\bigcup_{r>0}\mathcal{A}_r$. Furthermore, a straighforward verification shows that $\bar{A}$ coincides with the generator of G on $D(\bar{A})$. In virtue of de Leeuw's theorem, $\bar{A}$ generates G.

The proof is complete.

3. Corollaries. As a first corollary to Theorem 1 we have

THEOREM 2. *Suppose that the assumptions of Theorem* 1 *are fulfilled and, moreover,*

(ii′) *there is a sequence* (λ_k) *with* $\lambda_k \to +\infty$ *or* $\lambda_k \to -\infty$ *such that for all* $x \in D(A)$

$$\|(\lambda_k - A)x\| \geqslant M^{-1}(|\lambda_k| - \omega)\|x\|.$$

Then A is closable and its closure generates a one-parameter strongly continuous group G on E satisfying (*).

Proof. The closability of A results immediately from the lemma.

Let $\tilde{A}$ denote the closure of the restriction of A to $\mathscr{A}(A)$. By Theorem 1, $\tilde{A}$ generates a strongly continuous one-parameter group G on E, satisfying (*). By the Hille–Yosida theorem, if $\lambda_k > \omega$, then $(\lambda_k - \tilde{A})$ maps $D(\tilde{A})$ in one-to-one manner onto E. By (ii′), $\lambda_k - \bar{A}$ is an injection on $D(\bar{A})$ that coincides with $\lambda_k - \tilde{A}$ on $D(\tilde{A})$. Thus $\bar{A} = \tilde{A}$, which ends the proof.

Our second corollary to Theorem 1 is

THEOREM 3. *Let E be a Banach space and A an operator on E. Suppose that*

(α) *the set of analytic vectors of A is dense in E,*

(β) *for all* $\lambda \in R - \{0\}$ *and all* $x \in D(A)$

$$\|(A - \lambda)x\| \geqslant |\lambda|\,\|x\|.$$

Then A has a closure generating a strongly continuous one-parameter group of operators.

Notice that the above theorem immediately yields Nelson's result, as (β) is equivalent to the simultanous dissipativity of A and $-A$ and to the skew-symmetry of A if E is a Hilbert space.

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Functional calculus and the Gelfand transformation

by

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Abstract. The commutativity of Taylor's functional calculus [8] with the Gelfand transformation is proved. It is shown as a corollary that each axiomatic joint spectrum in the sense of Żelazko [10] which is contained in Taylor's joint spectrum satisfies a spectral mapping property with respect to Taylor's analytic functional calculus.

1. Introduction. Let X be a complex Banach space and let us denote by $C^n(X)$ the set of all commutative n-tuples of linear bounded operators on X, $n \geqslant 1$. An *axiomatic joint spectrum* [10] is an assignment σ from each $C^n(X)$ into the closed subsets of C^n, respectively, which satisfies the following axioms:

(i) $\sigma(a)$ is the usual spectrum in the case of a single linear operator $a \in L(X) = C^1(X)$.

(ii) σ has the projection property, i.e., for each $a \in C^{n+1}(X)$ one has $\pi\sigma(a) = \sigma(\pi(a))$, denoting by π the natural projection onto the first n coordinates.

(iii) σ has the spectral mapping property with respect to the polynomial maps, that is, for each polynomial p in n indeterminates and for each $a \in C^n(X)$, the equality $p(\sigma(a)) = \sigma(p(a))$ holds true.

A direct improvement of [7] and [10] shows that to such an axiomatic joint spectrum σ is associated a closed subset $\sigma(A)$ in the maximal spectrum $M(A)$ of a commutative, unital Banach subalgebra A of $L(X)$, such that $\sigma(A)$ is minimal with respect to the property (a) below:

(a) The equality $\hat{a}(\sigma(A)) = \sigma(a)$ holds for every $a \in C^n(X)$ and $n \geqslant 1$.

We have denoted above by $\hat{b}$ the Gelfand transformation of $b \in A$. Then one proves that the set $\sigma(A)$ is functorial in A, in the following sense:

(b) If $A \subset B \subset L(X)$ are two algebras as above, then $i^*\sigma(B) = \sigma(A)$, denoting by i: $A \to B$ the inclusion map.

The set $\sigma(A)$ defined above is sufficiently large in $M(A)$ because of axiom (i). More exactly:

(c) If A is a maximal abelian Banach subalgebra of $L(X)$ and if $a \in A$, then $\hat{a}(\sigma(A)) = \hat{a}(M(A))$.