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A construction of convolution operators on free groups

by

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Abstract. For a non-commutative free group G and for any p, $1 < p < \infty$, we construct a non-negative function f on G which by convolution defines a bounded operator on $l^p(G)$ but unbounded on $l^q(G)$, $q \neq p$.

Let G be a discrete group. A complex function f on G is called a *convolution operator* on $l^p(G)$, $1 \leqslant p \leqslant \infty$, if $f*g$ belongs to $l^p(G)$ whenever $g \in l^p(G)$. The set $C^p(G)$ of all convolution operators on $l^p(G)$ equipped with the operator norm

$$|||f|||_p = \sup_{\|g\|_p \leqslant 1} \|f*g\|_p$$

is a Banach algebra. We note that $C^1(G)$ and $C^\infty(G)$ both coincide with the algebra $l^1(G)$ and that $f \in C^p(G)$ if and only if $f^* \in C^{p'}(G)$, where $1/p+1/p' = 1$, and f^* is the function on G defined by $f^*(x) = \overline{f(x^{-1})}$.

The aim of this note is to prove the following

THEOREM. *Let G be a non-commutative free group and let $1 < p < \infty$. There exists a non-negative function f on G such that $f \in C^p(G)$ but $f \notin C^q(G)$ for $1 < q < \infty$, $q \neq p$.*

Remarks. 1. The theorem is motivated by a result of Lohoué [3]. For any p, $1 < p < \infty$, and any connected semi-simple Lie group G he has constructed a positive measure μ on G which by convolution defines a bounded operator on L^p and unbounded on L^q for $q \neq p$.

2. It is known that for an amenable group G one always has $C^p(G) \subset C^q(G)$ for $1 \leqslant p \leqslant q \leqslant 2$ or $2 \leqslant q \leqslant p \leqslant \infty$ ([2], Theorem C) and that a non-negative function f belongs to $C^p(G)$ if and only if $f \in l^1(G)$ ([1], Theorem 3.2.2), which is in sharp contrast to the theorem above.

We start the proof of the theorem with the following two lemmas:

LEMMA 1. *Suppose f and g are two functions on G such that for x_1, x_2 in the support of f and y_1, y_2 in the support of g the equality $x_1 y_1 = x_2 y_2$ implies $x_1 = x_2$ and $y_1 = y_2$. Then*

$$\|f*g\|_p = \|f\|_p \|g\|_p, \quad 1 \leqslant p \leqslant \infty.$$

The proof is obtained by a simple verification.

LEMMA 2. *Let G be a non-commutative free group and let a and b denote two of the free generators in G. For a pair (m, n) of natural numbers let $f_{m,n}$ denote the characteristic function of the set*

$$\{a^i b^{-k}\colon\ i = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n\}.$$

Then

$$m^{1/p} n \leqslant |||f_{m,n}|||_p \leqslant m^{1/p} n + mn^{1-1/p}, \qquad 1 \leqslant p < \infty.$$

Proof. For $r \geqslant n$ define the function

$$g_r = r^{-1/p} \chi_{\{b^k\colon\ k=1,2,\ldots,r\}}$$

(we use the notation χ_A for the characteristic function of a set $A \subset G$ and δ_x for $\chi_{\{x\}}$). Then $\|g_r\|_p = 1$ and

$$f_{m,n} * g_r \geqslant nr^{-1/p} \chi_{\{a^i b^k\colon\ i=1,2,\ldots,m;\ k=0,1,2,\ldots,r-n\}}.$$

Therefore

$$\|f_{m,n} * g_r\|_p \geqslant m^{1/p} n\big(1-(n-1)/r\big)^{1/p}$$

and so

$$|||f_{m,n}|||_p \geqslant nm^{1/p}.$$

To show the second estimation we let g to be any function in $l^p(G)$ with $\|g\|_p = 1$. For a given integer j define $g_j(x) = g(b^{-j}x)$ if the first letter of the word $b^{-j}x$ is neither b nor b^{-1} and put $g_j(x) = 0$ elsewhere. The functions $\delta_{b^j} * g_j$ have pairwise disjoint supports and $g = \sum_j \delta_{b^j} * g_j$. Thus $\sum_j \|g_j\|_p^p = \|g\|_p^p = 1$. Let also φ be the characteristic function of the set $\{a^i\colon\ i = 1, 2, \ldots, m\}$. We have

$$f_{m,n} * g = \sum_{i=1}^{n} \sum_{j} \varphi * \delta_{b^{j-i}} * g_j = \sum_{i=1}^{n} \varphi * g_i + \sum_{j\neq 0} \varphi * \delta_{b^j} * \sum_{i=1}^{n} g_{i+j},$$

thus

$$\|f_{m,n} * g\|_p \leqslant \Big\|\sum_{i=1}^{n} \varphi * g_i\Big\|_p + \Big\|\sum_{j\neq 0} \varphi * \delta_{b^j} * \sum_{i=1}^{n} g_{i+j}\Big\|_p.$$

But

$$\Big\|\sum_{i=1}^{n} \varphi * g_i\Big\|_p \leqslant \|\varphi\|_1 \Big\|\sum_{i=1}^{n} g_i\Big\|_p \leqslant m\Big(\sum_{i=1}^{n} \|g_i\|_p\Big)$$

$$\leqslant mn^{1-1/p} \Big(\sum_{i=1}^{n} \|g_i\|_p^p\Big)^{1/p} \leqslant mn^{1-1/p}.$$

Also the functions $\varphi * \delta_{b^j} * \sum_{i=1}^{n} g_{i+j}$, $j \neq 0$, have pairwise disjoint supports, thus

$$\Big\|\sum_{j\neq 0} \varphi * \delta_{b^j} * \sum_{i=1}^{n} g_{i+j}\Big\|_p^p = \sum_{j\neq 0} \Big\|\varphi * \Big(\delta_{b^j} * \sum_{i=1}^{n} g_{i+j}\Big)\Big\|_p^p,$$

and since φ and $\delta_{b^j} * \sum_{i=1}^{n} g_{i+j}$, $j \neq 0$, satisfy the assumption of Lemma 1, we have

$$\Big\|\sum_{j\neq 0} \varphi * \delta_{b^j} * \sum_{i=1}^{n} g_{i+j}\Big\|_p^p = \sum_{j\neq 0} \|\varphi\|_p^p \Big\|\delta_{b^j} * \sum_{i=1}^{n} g_{i+j}\Big\|_p^p$$

$$= m \sum_{j\neq 0} \Big\|\sum_{i=1}^{n} g_{j+i}\Big\|_p^p \leqslant mn^{p-1} \sum_{j\neq 0} \sum_{i=1}^{n} \|g_{i+j}\|_p^p \leqslant mn^p.$$

All these together give

$$\|f_{m,n} * g\|_p \leqslant m^{1/p} n + mn^{1-1/p}.$$

Therefore

$$|||f_{m,n}|||_p = \sup_{\|g\|_p=1} \|f_{m,n} * g\|_p \leqslant m^{1/p} n + mn^{1-1/p}$$

and the lemma follows.

Proof of the theorem. We start the proof with a simple observation that for non-negative functions f_1 and f_2 on G

$$|||f_1 + f_2|||_p \geqslant \max\{|||f_1|||_p,\ |||f_2|||_p\}, \qquad 1 \leqslant p \leqslant \infty.$$

Thus $f_1 + f_2$ belongs to $C^p(G)$ if and only if f_1 and f_2 both are in $C^p(G)$.

Fix an exponent p and choose two sequences m_k, n_k of integers and a sequence α_k of positive numbers such that the sequence $\alpha_k m_k^{1/q} n_k$ is unbounded for every $q < p$ but the series

$$\sum_{k=1}^{\infty} \alpha_k \big(m_k^{1/p} n_k + m_k n_k^{1-1/p}\big)$$

is convergent (put for example $m_k = 2^k$, $n_k \geqslant 2^{kp}$ and $\alpha_k = k^{-2} m_k^{-1/p} n_k^{-1}$). Also define a function f_1 by

$$f_1 = \sum_{k=1}^{\infty} \alpha_k f_{m_k, n_k},$$

where $f_{m,n}$ are the functions from Lemma 2. Since the series $\sum_{k=1}^{\infty} \alpha_k |||f_{m_k,n_k}|||_p$ is convergent, f_1 is in $C^p(G)$. On the other hand, since all functions $\alpha_k f_{m_k,n_k}$ are non-negative,

$$|||f_1|||_q \geqslant \sup_k \alpha_k |||f_{m_k,n_k}|||_q = \infty$$

for all $q < p$ and so $f_1 \notin C^q(G)$.

For the exponent p', where $1/p+1/p' = 1$, let f_2 be a function constructed in the same way as f_1 with p' in place of p. Then $f_2^* \in C^p(G)$ but $f_2^* \notin C^q(G)$ for $q > p$. Consequently the function $f = f_1 + f_2^*$ has the property claimed in the theorem.

COROLLARY. *Let G be a non-commutative free group and let $1 < p < \infty$, $p \neq 2$. There exists a non-negative function f on G such that the operator $g \to f*g$ is bounded on $l^p(G)$ but the operator $g \to g*f$ is unbounded.*

Proof. The operator $g \to g*f$ is bounded on $l^p(G)$ if and only if $f^* \in C^p(G)$ which is equivalent to $f \in C^{p'}(G)$, where $1/p+1/p' = 1$.

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Analytic vectors and generation of one-parameter groups

by

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Abstract. We give a Hille–Yosida type condition for an operator to generate a one-parameter strongly continuous group on a Banach space, in terms of analytic vectors of the operator.

Introduction. Let E be a Banach space, and A a linear operator on E with domain $D(A)$. An element x in $\bigcap_{n=1}^{\infty} D(A^n)$ is called an *analytic vector for* A if for some $t > 0$

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \|A^n x\| < +\infty.$$

By a standard spectral projection argument one easily proves that the self-adjoint operator on a Hilbert space has a dense set of analytic vectors. Conversely, a result of Nelson [4], Lemma 5.1, states that the symmetric operator on a Hilbert space, with a dense set of analytic vectors, is essentially self-adjoint.

The main objective of the present paper is to establish Banach space counterparts to the above assertions. Our starting point is the observation that the skew-adjoint operators coincide with the generators of one-parameter strongly continuous unitary groups (Stone's theorem). We accordingly shift attention from self-adjoint operators to generators of one-parameter strongly continuous groups.

The generalization of the first assertion is routine. Indeed, given a Banach space E, if A generates a one-parameter strongly continuous group G on E, then for every $x \in E$

$$\sqrt{k/\pi} \int_R \exp(-kt^2)\, G(t)\, x\, dt \qquad (k > 0)$$

is an analytic vector for A which tends to x as $k \to +\infty$; accordingly, A has a dense set of analytic vectors.

The generalization of Nelson's result is much more involved. It takes the form of the following