for kernels such as

$$K(\varrho, \varphi) = \frac{1}{\alpha \varrho^{2\alpha}(a+\lg^{\beta}\varrho)} \sin\varphi$$

where the family of ellipses is now

$$y_1 = \varrho^{\alpha}\cos\varphi,$$

$$y_2 = \varrho^{\alpha}(a+\lg^{\beta}\varrho)\sin\varphi, \qquad \beta \geqslant 1,\ \alpha > 0.$$

It is enough to take into account result (1.2) (Coifman–Guzmán) The proof can be seen in [16] if $\alpha = 1$ and in [14] if $\beta = 2$.

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An analogue of the argument theorem of Bohr and its application

by

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Abstract. An analogue of the argument theorem of Bohr is proved and used to establish the following result: given a real S^1 almost periodic function f on $\boldsymbol{R}$, the function $x \to \exp(i\int_0^x f(u)\,du)$ is W^1 almost periodic if and only if it is (uniformly) almost periodic, in which case the function $x \to \int_0^x f(u)\,du - \hat{f}(0)x$ is almost periodic. It is shown that the latter theorem fails if W^1 almost periodicity is replaced by what is here called E^2 almost periodicity.

1. Introduction. According to a well-known theorem of Bohr (cf. [3], [7]), given a real continuous function f on $\boldsymbol{R}$, the function $x \to \exp(if(x))$ is almost periodic if and only if there exists $a \in \boldsymbol{R}$ such that the function $x \to f(x)+ax$ is almost periodic.

Our main objective is to prove the following

THEOREM 1. *Suppose a real uniformly continuous function f on $\boldsymbol{R}$ satisfies the following cocycle condition:*

(co) *for every $t \in \boldsymbol{R}$, there exists $a_t \in \boldsymbol{R}$ such that the function $x \to f(x+t) - f(x) + a_t x$ is almost periodic.*

In order that there be $a \in \boldsymbol{R}$ such that the function $x \to f(x)+ax$ is almost periodic it is necessary and sufficient that the function $x \to \exp(if(x))$ be W^1 almost periodic.

Since given an S^1 almost periodic function f on $\boldsymbol{R}$, the function $x \to \int_0^x f(u)\,du$ is uniformly continuous (cf. [1], Th. 4.7.8), and, for any $t \in \boldsymbol{R}$, the function $x \to \int_x^{x+t} f(u)\,du$ is almost periodic (cf. [2], Th. 2.3.1), from Theorem 1 and the above-mentioned argument theorem of Bohr we easily deduce the following

THEOREM 2. *Let f be a real S^1 almost periodic function on $\boldsymbol{R}$. Then the function $x \to \exp(i\int_0^x f(u)\,du)$ is W^1 almost periodic if and only if it is almost* *periodic, in which case the function* $x \to \int_0^x f(u)\,du - \hat{f}(0)\,x$ *is almost periodic.*

There arises a natural question whether W^1 almost periodicity in the above theorem may be replaced by some other types of almost periodicity. We shall present a negative result in this respect, that will concern so-called E^2 almost periodicity, a stronger property than being simultaneously B^p almost periodic for all $p \geq 1$. The result will be somewhat ineffective as it is often the case of results employing a probabilistic argument.

2. Prerequisites. We utilize various classes of almost periodic functions on $\boldsymbol{R}$. Aside from the usual (uniformly) almost periodic functions on $\boldsymbol{R}$, there appear:

(i) the Stepanov S^p almost periodic functions on $\boldsymbol{R}$ $(1 \leq p < +\infty)$, i.e., those measurable functions on $\boldsymbol{R}$ that are limits of sequences of trigonometric polynomials in one of the seminorms

$$\|f\|_{S^p,T} = \sup\{(2T)^{-1} \int_{-T}^{T} |f(x+u)|^p\,du\colon\ x \in \boldsymbol{R}\} \qquad (T > 0),$$

no matter in which one;

(ii) the Weyl W^p almost periodic functions on $\boldsymbol{R}$ $(1 \leq p < +\infty)$, i.e., those measurable functions on $\boldsymbol{R}$ that are limits of sequences of trigonometric polynomials in the seminorm

$$\|f\|_{W^p} = \lim_{T \to \infty} \|f\|_{S^p,T};$$

(iii) the Besicovitch B^p almost periodic functions on $\boldsymbol{R}$ $(1 \leq p < +\infty)$, i.e., those measurable functions on $\boldsymbol{R}$ that are limits of sequences of trigonometric polynomials in the seminorm

$$\|f\|_{B^p} = \overline{\lim_{T \to \infty}} \left((2T)^{-1} \int_{-T}^{T} |f(u)|^p\,du\right)^{1/p};$$

(iv) the E^p almost periodic functions on $\boldsymbol{R}$ $(1 \leq p < +\infty)$, i.e., those measurable functions on $\boldsymbol{R}$ that are limits of sequences of trigonometric polynomials in the seminorm

$$\|f\|_{E^p} = \inf\{c > 0\colon\ \overline{\lim_{T \to \infty}}\,(2T)^{-1} \int_{-T}^{T} \exp\left(|f(u)|/c\right)^p du \leq 2\}.$$

It is easily verified that a measurable function f on $\boldsymbol{R}$ is E^p almost periodic $(1 \leq p < +\infty)$ if and only if there exists a sequence (p_n) of trigonometric polynomials such that for every $\alpha > 0$

$$\lim_{n \to \infty} \overline{\lim_{T \to \infty}}\,(2T)^{-1} \int_{-T}^{T} \exp(\alpha\,|f(u) - p_n(u)|^p)\,du = 1.$$

We have the following diagram of inclusions:

$$\begin{array}{ccc} S^pAP & \subset & W^pAP \\ \cup & & \cap \\ AP & \subset E^rAP \subset & B^pAP \end{array}$$

for any $1 \leq p, r < +\infty$.

Given an almost periodic (resp. S^p almost periodic, $1 \leq p < +\infty$, etc.) function f on $\boldsymbol{R}$ and $\mu \in \boldsymbol{R}$, $\hat{f}(\mu)$ stands for the μth Fourier coefficient of f, i.e.,

$$\hat{f}(\mu) = \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} f(x) \exp(-i\mu x)\,dx.$$

Given a probability space $(\Omega, \sigma, \boldsymbol{P})$, $\boldsymbol{E}$ denotes the expectation operator. If $\mathscr{A}$ is a σ-subalgebra of σ, $\boldsymbol{E}^{\mathscr{A}}$ denotes the conditional expectation operator relative to $\mathscr{A}$.

3. Proof of Theorem 1. The proof of the necessity part of the theorem is trivial.

The proof of the sufficiency part will be based on the following elementary

LEMMA. *Let f be a uniformly continuous W^1 almost periodic function on $\boldsymbol{R}$. Then $\hat{f}(0)$ is the uniform limit of convex combinations of translates of f.*

Proof. With T_s standing for the translation operator by s, let $\mathscr{S}$ be the convex hull of $\{T_s\colon s \in \boldsymbol{R}\}$. Clearly, $\mathscr{S}$ is closed under composition.

Observe that given a trigonometric polynomial p of the form

$$p(x) = \sum_{\lambda \in \Lambda} \hat{p}(\lambda) \exp(i\lambda x) \qquad (x \in \boldsymbol{R})$$

with a finite Λ, the composition S of all S_λ with $\lambda \in \Lambda - \{0\}$, each S_λ being defined as

$$\tfrac{1}{2}(T_{\pi/\lambda} + T_0),$$

satisfies $Sp = \hat{p}(0)$. Keeping this in mind, given any $\varepsilon > 0$, let p be a trigonometric polynomial such that $\|f - p\|_{W^1} < \varepsilon/3$. Of course $|\hat{f}(0) - \hat{p}(0)| < \varepsilon/3$. Select $S \in \mathscr{S}$ so that $Sp = \hat{p}(0)$. Clearly $\|Sf - \hat{f}(0)\|_{W^1} < 2\varepsilon/3$, and so $\|Sf - \hat{f}(0)\|_{S^1,T} < 2\varepsilon/3$ for some $T > 0$. Since $Sf - \hat{f}(0)$ is uniformly continuous, there exists $S' \in \mathscr{S}$ such that

$$\|S'(Sf - \hat{f}(0))\|_\infty < \|Sf - \hat{f}(0)\|_{S^1,T} + \varepsilon/3.$$

On account of the last two estimates and in view of $S'(Sf - \hat{f}(0)) = S'Sf - \hat{f}(0)$, we get $\|S'Sf - \hat{f}(0)\|_\infty < \varepsilon$, which ends the proof.

Proceeding to establish the sufficiency part of Theorem 1, suppose f is a real uniformly continuous function f on $\boldsymbol{R}$ that satisfies (co). Denote by g the function $x \to \exp(if(x))$. By the argument theorem of Bohr, the proof will be complete upon showing that g is almost periodic.

Since $\|g\|_{W^1} = 1$, g has at least one non-zero Fourier coefficient, say $\hat{g}(\mu)$ $(\mu \in \boldsymbol{R})$. Let $\varepsilon > 0$ be given. Applying the lemma to the function $g_\mu(x) = g(x)\exp(-i\mu x)$ $(x \in \boldsymbol{R})$, we see that there exist positive numbers a_i $(i = 1, \ldots, n)$ with $\sum_{i=1}^{n} a_i = 1$ and real numbers s_i $(i = 1, \ldots, n)$ such that

$$\Big\|\sum_{i=1}^{n} a_i T_{s_i} g_\mu - \hat{g}(\mu)\Big\|_\infty < \varepsilon.$$

Since the expression on the left side is equal to

$$\Big\|\sum_{i=1}^{n} a_i g T_{s_i} \bar{g}_\mu - \overline{\hat{g}(\mu)}\, g\Big\|_\infty,$$

and, in view of (co), each function $g T_{s_i} \bar{g}_\mu$ $(i = 1, \ldots, n)$ is almost periodic, we infer that $\overline{\hat{g}(\mu)}\, g$ is the uniform limit of almost periodic functions. This in turn implies immediately that g is almost periodic.

The proof is complete.

4. A negative result. Let $(\Omega, \sigma, \boldsymbol{P})$ be a probability space. Suppose there is given an ergodic flow on Ω, i.e., a one-parameter group $\{S_t: t \in \boldsymbol{R}\}$ of measure-preserving transformations of Ω onto itself, with the following properties:

(i) the map $\boldsymbol{R} \times \Omega \ni (t, \omega) \to S_t(\omega) \in \Omega$ is measurable relative to $(\mathscr{B}(\boldsymbol{R}) \otimes \sigma, \sigma)$, where $\mathscr{B}(\boldsymbol{R})$ denotes the Borel σ-algebra of $\boldsymbol{R}$;

(ii) given a random variable f on Ω, $f \circ S_t = f$ a.s. (almost surely) for all $t \in \boldsymbol{R}$ implies f is constant a.s.

Let (a_n) be a sequence in $l^2 - l^1$ of rationally independent positive numbers. Suppose the flow $\{S_t\}$ has, for each $n \in \boldsymbol{N}$, an eigenfunction θ_n corresponding to the eigenfrequency $a_n/2\pi$, such that

$$\theta_n \circ S_t = \exp(ia_n t)\,\theta_n$$

for all $t \in \boldsymbol{R}$. Suppose, moreover, that the eigenfunctions θ_n form a family of independent random variables each one uniformly distributed on $\boldsymbol{T}$ (the unit circle).

That the above assumptions can be fulfilled is seen as follows. Take $\boldsymbol{T}^{\boldsymbol{N}}$ for Ω with the Borel σ-algebra of $\boldsymbol{T}^{\boldsymbol{N}}$ as σ, and the direct product measure obtained from Lebesgue measure on each copy of $\boldsymbol{T}$ as $\boldsymbol{P}$. Define an ergodic flow on Ω by putting

$$S_t(\omega) = \big(\exp(ia_1 t)\,\omega_1, \exp(ia_2 t)\,\omega_2, \ldots\big)$$

for every $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$. Eventually realize an eigenfunction $\theta_n (n \in \boldsymbol{N})$ of $\{S_t\}$ as the projection from Ω onto the nth copy of $\boldsymbol{T}$.

Let

$$F = \sum_{k=1}^{\infty} a_k^2 \operatorname{Im} \theta_k.$$

Define a stochastic process $\{F_t\}$ by putting

$$F_t = F \circ S_t$$

for all $t \in \boldsymbol{R}$. Clearly, each sample path of $\{F_t\}$ is a real almost periodic function with mean value zero. Given $t \in \boldsymbol{R}$, set

$$X_t = \int_0^t F_u\, du.$$

In the sequel, when speaking about an almost periodic (resp. S^p almost periodic, $1 \leqslant p < +\infty$, etc.) stochastic process we shall mean that almost all trajectories of the process are almost periodic (resp. S^p almost periodic, $1 \leqslant p < +\infty$, etc.).

The main result of this section is

THEOREM 3. *The process $\{\exp(iX_t)\}$ is E^2 almost periodic. Almost none of its sample paths is almost periodic.*

Proof. For each $n \in \boldsymbol{N}$, set

$$Y^{(n)} = \sum_{k=1}^{n} a_k \operatorname{Re} \theta_k.$$

Let Y be the limit of $(Y^{(n)})$ a.s.; the existence of the limit follows from the three series theorem.

We claim that for any $\alpha > 0$

$$\lim_{n \to \infty} \boldsymbol{E} \exp(\alpha |\exp(iY) - \exp(iY^{(n)})|^2) = 1. \tag{1}$$

To prove the claim note first that the sequence in (1) is minorized by one. Thus we need only appropriate estimates from above. Since $|\exp(ix) - 1| \leqslant |x|$ for $x \in \boldsymbol{R}$, we may write

$$\boldsymbol{E} \exp(\alpha |\exp(iY) - \exp(iY^{(n)})|^2) = \boldsymbol{E} \exp\big(\alpha |\exp\big(i(Y - Y^{(n)})\big) - 1|^2\big) \leqslant \boldsymbol{E} \exp(\alpha |Y - Y^{(n)}|^2) \tag{2}$$

for every $\alpha > 0$ and every $n \in \boldsymbol{N}$.

Denote by (ε_n) a Bernoulli sequence, i.e., a sequence of independent identically distributed random variables each one taking the value plus and minus one with equal probability. $(\operatorname{Re} \theta_n)$ being a sequence of symmetric real-valued random variables not exceeding one in absolute value, the Kahane contraction principle (cf. [8], Th. 2.4.9) neatly applies so as to give

(3) $$E|Y-Y^{(n)}|^{2p} = E\Big|\sum_{k=n+1}^{\infty} a_k \operatorname{Re}\theta_k\Big|^{2p} \leqslant E\Big|\sum_{k=n+1}^{\infty} a_k \varepsilon_k\Big|^{2p}$$

for all $n, p \in \boldsymbol{N}$. Here the right-hand series converges a.s. by the three series theorem. On the other hand, by Khintchine's inequality (cf. [5]), we have for all $n, p \in \boldsymbol{N}$

$$E\Big|\sum_{k=n+1}^{\infty} a_k \varepsilon_k\Big|^{2p} \leqslant 2^p \pi^{-1/2} \Gamma(p+1/2)\Big(\sum_{k=n+1}^{\infty} a_k^2\Big)^p$$
$$\leqslant 2^p p!\Big(\sum_{k=n+1}^{\infty} a_k^2\Big)^p.$$

Hence by (3)

$$E\exp(\alpha|Y-Y^{(n)}|^2) = 1+\sum_{p=1}^{\infty} \frac{\alpha^p}{p!} E|Y-Y^{(n)}|^{2p}$$
$$\leqslant 1+\sum_{p=1}^{\infty}\Big(2\alpha\sum_{k=n+1}^{\infty} a_k^2\Big)^p.$$

From this estimate and from (2) one easily deduces (1).

Having established (1), we accomplish the proof of the first assertion of the theorem reasoning as follows.

Let $\{Y_t\} = \{Y \circ S_t\}$ and, for every $n \in \boldsymbol{N}$, $\{Y_t^{(n)}\} = \{Y^{(n)} \circ S_t\}$. By Birkhoff's ergodic theorem, for every $\alpha > 0$ and every $n \in \boldsymbol{N}$, the limit

$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} \exp\big(\alpha|\exp(iY_t)-\exp(iY_t^{(n)})|^2\big)\,dt$$

exists a.s. and equals

$$E\exp\big(\alpha|\exp(iY)-\exp(iY^{(n)})|^2\big).$$

Hence by (1)

$$\lim_{n\to\infty}\lim_{T\to\infty} (2T)^{-1}\int_{-T}^{T} \exp\big(\alpha|\exp(iY_t)-\exp(iY_t^{(n)})|^2\big)\,dt = 1 \text{ a.s.}$$

Since each of the processes $\{\exp(iY_t^{(n)})\}$ $(n \in \boldsymbol{N})$ is almost periodic, the latter equality implies that $\{\exp(iY_t)\}$ is E^2 almost periodic. On the other hand, the processes $\{X_t\}$ and $\{Y - Y_t\}$ are stochastically indistinguishable. Thus, almost every sample path of $\{\exp(iX_t)\}$ is, up to a random factor of absolute value one, the complex conjugate of the corresponding trajectory of $\{\exp(iY_t)\}$. As a result, the process $\{\exp(iX_t)\}$ is E^2 almost periodic.

We pass now to proving the second assertion of the theorem. Given $n \in \boldsymbol{N}$, let $\mathscr{A}_n = \sigma(\theta_k, 1 \leqslant k \leqslant n)$ be the σ-algebra generated by the θ_k shown. Let $\mathscr{A}_\infty = \sigma(\bigcup_{k=1}^{\infty} \mathscr{A}_k)$.

Since the process $\{X_t\}$ has continuous sample paths, $\sup\{X_t\colon t \in \boldsymbol{R}\}$ is a well-defined random variable Z on Ω. Z is non-negative because $X_0 = 0$. Clearly, Z is $\mathscr{A}_\infty$ adopted.

For each $t \in \boldsymbol{R}$ and each $n \in \boldsymbol{N}$, put

$$X_t^{(n)} = \sum_{k=1}^{n} a_k\big[\operatorname{Re}\theta_k - \operatorname{Re}\big(\exp(ia_k t)\theta_k\big)\big].$$

We see that for every $t \in \boldsymbol{R}$, X_t is the pointwise limit of $(X_t^{(n)})$.

Given $n \in \boldsymbol{N}$, let

$$Z_n = \sup\{X_t^{(n)}\colon t \in \boldsymbol{R}\}.$$

Since $\{a_k\colon k \in \boldsymbol{N}\}$ is a rationally independent set, it follows from Kronecker's theorem that

(4) $$Z_n = \sum_{k=1}^{n} a_k(1+\operatorname{Re}\theta_k).$$

Of course, Z_n is $\mathscr{A}_n$ adopted. Using the three series theorem, we easily derive from (4) that

(5) $$\lim_{n\to\infty} Z_n = +\infty \text{ a.s.}$$

We claim that given $n \in \boldsymbol{N}$, there exists an $\mathscr{A}_n$ adopted random variable τ_n on Ω such that

(6) $$X_{\tau_n}^{(n)} \geqslant Z_n - 1 \text{ a.s.}$$

Indeed, since for each $n \in \boldsymbol{N}$, the process $\{X_t^{(n)}\}$ is $\mathscr{B}(\boldsymbol{R}) \otimes \mathscr{A}_n$ measurable, the set

$$\{(t, \omega) \in \boldsymbol{R}\times\Omega\colon X_t^{(n)}(\omega) \geqslant Z_n(\omega) - 1\},$$

projecting along $\boldsymbol{R}$ onto Ω, is $\mathscr{B}(\boldsymbol{R}) \otimes \mathscr{A}_n$ measurable. Now the claim follows upon applying the section theorem of Meyer (cf. [4], Th. 2.44).

Given any $n, m \in \boldsymbol{N}$, put

$$\tau_n^{(m)} = \begin{cases} -m & \text{if } \tau_n \leqslant -m,\\ \tau_n & \text{if } -m < \tau_n < m,\\ m & \text{if } m \leqslant \tau_n. \end{cases}$$

Since $\|X_{\tau_n^{(m)}}^{(p+1)} - X_{\tau_n^{(m)}}^{(p)}\|_\infty \leqslant a_{p+1}^2 m$ for all $n, m, p \in \boldsymbol{N}$, $X_{\tau_n^{(m)}}$ is the $L^\infty(\Omega)$ limit of $(X_{\tau_n^{(m)}}^{(p)})$, and so $E^{\mathscr{A}_n}(X_{\tau_n^{(m)}})$ is the $L^\infty(\Omega)$ limit of $\big(E^{\mathscr{A}_n}(X_{\tau_n^{(m)}}^{(p)})\big)$. But $E^{\mathscr{A}_n}(X_{\tau_n^{(m)}}^{(p)}) = X_{\tau_n^{(m)}}^{(n)}$ a.s. for $p \geqslant n$. Therefore

(7) $$E^{\mathscr{A}_n}(X_{\tau_n^{(m)}}) = X_{\tau_n^{(m)}}^{(n)} \text{ a.s.}$$

On the other hand, we have

$$Z \geqslant X_{\tau_n^{(m)}} \text{ a.s.}$$

Hence by (7)

$$E^{\mathscr{A}_n}(Z) \geqslant X^{(n)}_{\tau_n^{(m)}} \text{ a.s.} \tag{8}$$

Here we have applied the generalized conditional expectation operator to the non-negative possibly non-integrable Z (a very readable discussion of generalized conditional expectations including generalized martingale theorems may be found in [6], § 20). Letting m in (8) tend to infinity, we get

$$E^{\mathscr{A}_n}(Z) \geqslant X^{(n)}_{\tau_n} \text{ a.s.}$$

Hence, in view of (5) and (6)

$$Z = \lim_{n \to \infty} E^{\mathscr{A}_n}(Z) = +\infty \text{ a.s.}$$

Since the sample paths of the process $\{X_t\}$ are integrals of almost periodic functions having mean value zero, it easily follows from the latter formula and the argument theorem of Bohr that almost no trajectory of the process $\{\exp(iX_t)\}$ is almost periodic.

The proof is complete.

We close the paper by remarking that the assumption made throughout that Ω be a state space for an ergodic flow may be dispensed with. By a standard argument currently used in the theory of stationary processes, we may easily widen the scope of Theorem 3 so as to yield the following

THEOREM 4. *Let (a_n) be a sequence in $l^2 - l^1$ of rationally independent positive numbers. Given a probability space $(\Omega, \sigma, \boldsymbol{P})$, suppose (θ_n) is a sequence of independent random variables on Ω each one uniformly distributed on $\boldsymbol{T}$. Let*

$$F_t = \sum_{k=1}^{\infty} a_k^2 \operatorname{Im}\bigl(\exp(ia_k t)\,\theta_k\bigr),$$

$$X_t = \int_0^t F_u \, du$$

for all $t \in \boldsymbol{R}$. Then each sample path of the process $\{F_t\}$ is a real almost periodic function with mean value zero. Moreover, the process $\{\exp(iX_t)\}$ is E^2 almost periodic while almost none of its sample paths is almost periodic.

The details of the proof of this theorem are left to the reader.

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