STUDIA MATHEMATICA, T. LXXIX. (1984)

Bounds for oscillatory integrals and L^2-theory of the corresponding singular integrals

by

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Abstract. Let $f(t) = at^{\alpha} e^{\lambda t^{\beta}}$ with $\alpha/\beta \geqslant 1$ or $f(t) = (at^{\alpha}+bt^{\beta}) e^{\lambda t^{\alpha}}$, $\beta = 2\alpha \neq 0$, where α, β, a, b, and λ are real numbers. We prove in this paper that

$$\left|\int_{\varepsilon}^{R} \frac{\sin f(t)}{t} dt\right| \leqslant C$$

where C is a constant independent of a, λ, ε and R or independent of $a, b, \varepsilon, \lambda$ and R, respectively.

The above estimate is used in the $L^2(\boldsymbol{R}^2)$ theory for singular integral operators with kernels whose polar coordinate expression is such as

$$K(\varrho, \varphi) = \frac{1}{\alpha \varrho^{3\alpha} e^{2\lambda\varrho^{\alpha}} (1+\lambda\varrho^{\alpha})} \sin \varphi.$$

These kernels do not satisfy any kind of homogeneity.

Estimates for some similar oscillatory integrals can be seen in [3]–[12].

§ 1. Basic lemmas and known results. We state now some of the known results and some basic general principles that we are going to use in the next section.

(1.1) THEOREM (Stein–Wainger). *Assume $a_1 < a_2 < \ldots < a_n$ are n non-negative fixed real numbers and $b_1, b_2, \ldots, b_n$ are real numbers. Then*

$$\left|\int_{-\infty}^{\infty} \exp\{i(b_1[x]^{a_1}+b_2[x]^{a_2}+ \ldots +b_n[x]^{a_n})\} \frac{dx}{x}\right| \leqslant k(a_1, a_2, \ldots, a_n)$$

where k does not depend upon $b_1, \ldots, b_n$.

(The integral is defined as

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\varepsilon}^{R} \frac{\exp\{\ \}}{x} dx,$$

$[x]^a$ means $\operatorname{sig} x \cdot |x|^a$ for a fixed real number a.)

The proof can be seen in [11]. This is an example of the type of integrand functions for which we are going to get estimates.

Another result improving the preceding one is the following, due to Coifman and Guzmán.

(1.2) THEOREM (Coifman–Guzmán). *Let* $F(t) = \sum_{i=1}^{n} t^{\alpha_i} P_i(\lg t)$ *where* $P_i(t) = \sum_{k=0}^{m_i} a_{i,k} t^k$ *and* $\alpha_{i+1} > \alpha_i > 0$, $i = 1, 2, \ldots, n-1$. *Then*

$$\left|\int_0^\infty \sin[F(t)]\frac{dt}{t}\right| \leqslant C, \quad 0 < R < \infty \tag{1.2a}$$

(C depending only upon α_i *and* m_i*) and for* $R > 1$

$$\left|\int_1^R \cos[F(t)]\frac{dt}{t}\right| \leqslant C\Big(1 + \min_{\substack{i=1,\ldots,n-1\\ k=0,\ldots,m_i}} [l_n |a_{i,k}|]\Big) \tag{1.2b}$$

with C depending only upon α_i *and* m_i, $i = 1, 2, \ldots, n$ (l_n = *Napierian logarithm).*

The proof can be seen in [14].

Observe that estimate (1.2a) with $m_i = 0$ $\forall i$ is the estimate of Stein–Wainger.

(1.3) LEMMA. *Assume* $F\colon [a, b] \to \boldsymbol{R}$, $a, b \in \boldsymbol{R}$, $a < b$, $F \in \mathscr{C}^1$ *and* F' *monotone.*

(i) *If* $|F'(t)| > \lambda > 0$, $t \in [a, b]$, *then*

$$\left|\int_a^b e^{iF(t)}\,dt\right| \leqslant 1/\lambda.$$

(ii) *If* $|F''(t)| > \varrho > 0$, $t \in [a, b]$, *then there exists a constant C independent of a and b such that*

$$\left|\int_a^b e^{iF(t)}\,dt\right| \leqslant C/\varrho.$$

This result is due to Van der Corput and the proof can be seen in [17], Vol. 1, p. 197. A generalization of this lemma is the following.

(1.4) LEMMA. *Let* $F\colon [a, b] \to \boldsymbol{R}$, $a, b \in \boldsymbol{R}$, $a < b$, $F \in \mathscr{C}^n$, $n > 1$ *be such that* $|F^{(n)}(t)| > \lambda > 0$ *if* $t \in [a, b]$. *Then there exists a constant* $C_n > 0$, *independent of a and b such that*

$$\left|\int_a^b e^{iF(t)}\,dt\right| \leqslant C_n/\lambda^{1/n}.$$

The proof can be seen in [14] and uses an induction argument. As a consequence we have

(1.5) COROLLARY. *Let f be in the hypothesis of the Van der Corput lemma over* $[1, R]$, $R \in \boldsymbol{R}$, $R \geqslant 1$. *Then there exists a constant* $C(\lambda)$ *independent of R such that*

$$\left|\int_1^R \frac{\sin f(t)}{t}\,dt\right| \leqslant C(\lambda).$$

In order to prove this, it is enough to apply Lemma (1.3) and integration by parts. Using the mean value theorem we obtain:

(1.6) LEMMA. *Let*

$$f\colon [0, 1] \times [0, 1] \to \boldsymbol{R}$$
$$(b, t) \to f(b, t)$$

be a $\mathscr{C}^1$-*function. For every* ε, $0 \leqslant \varepsilon \leqslant 1$, *we define the function*

$$g_\varepsilon(b) = \int_\varepsilon^1 \frac{\sin f(b, t)}{t}\,dt, \quad g_\varepsilon \in \mathscr{C}^1.$$

Assume that $|g'_\varepsilon(s)| < C_1$ *where* C_1 *is a constant independent of* ε *and* $s \in [0, 1]$ *and that* $|g_\varepsilon(0)| < C_2$ *(or* $|g_\varepsilon(1)| < C_3$*) with* C_2 *(or* C_3*) independent of* ε. *Then there exists a constant C independent of b and* ε *such that*

$$\left|\int_\varepsilon^1 \frac{\sin f(b, t)}{t}\,dt\right| \leqslant C.$$

(1.7) LEMMA. *Let* $f(t) = \sum_{i=1}^{n} t^{\alpha_i} P_i(\lg t)$ *where* $P_i(s) = \sum_{k=0}^{m_i} a_{i,k} s^k$ *and* $\alpha_{i+1} > \alpha_i > 0$, $i = 1, 2, \ldots, n-1$. *Then*

$$\left|\int_e^R \frac{\sin f(t)}{t \lg t}\,dt\right| \leqslant C, \quad e < R < \infty$$

where C depends only upon α_i *and* m_i, $i = 1, \ldots, n$.

Proof. By the hypothesis and Theorem (1.2) we know that

$$\left|\int_e^R \frac{\sin f(t)}{t}\,dt\right| \leqslant C_1, \quad e < R < \infty$$

where C_1 is a constant which depends only upon α_i and m_i, $i = 1, 2, \ldots, n$. Let us define

$$H(\varrho) = \int_e^\varrho \frac{\sin f(s)}{s}\,ds.$$

Then by integration by parts and taking absolute values we have

$$\left|\int_e^R \frac{\sin f(t)}{t\lg t}\,dt\right| \leqslant \left|\frac{H(t)}{\lg t}\right]_e^R\left| + \left|\int_e^R \frac{H(t)}{t\lg^2 t}\,dt\right| \leqslant C_1 + C_2\left|\frac{1}{\lg t}\right]_e^R\right| \leqslant C.$$

(1.8) Lemma. *Let $b \in \mathbf{R}$, $b \geqslant 1$ and $f_\alpha\colon [b, \infty) \to \mathbf{R}$ be a function depending upon the parameter α such that whenever $u \in [2^k b, 2^{k+1} b]$, $k = 0, 1, 2, \ldots$, the following inequalities hold:*

(i) $$|\sin f_\alpha(u)| \leqslant |f_\alpha(u)|,$$

(ii) $$\frac{|f_\alpha(u)|}{u} \leqslant \frac{C}{u^2} \quad (C \text{ independent of } b \text{ and } f).$$

Then

$$\left|\sum_{k=0}^{\infty} \int_{2^k b}^{2^{k+1} b} \frac{\sin f_\alpha(u)}{u}\,du\right| \leqslant C_1$$

with C_1 independent of b and α.

Proof. From the hypothesis it is immediate to prove the following inequalities:

$$\left|\sum_{k=0}^{\infty} \int_{2^k b}^{2^{k+1} b} \frac{\sin f_\alpha(u)}{u}\,du\right| \leqslant \sum_{k=0}^{\infty} \left|\int_{2^k b}^{2^{k+1} b} \frac{\sin f_\alpha(u)}{u}\,du\right|$$
$$\leqslant \sum_{k=0}^{\infty} \int_{2^k b}^{2^{k+1} b} \frac{|\sin f_\alpha(u)|}{u}\,du \leqslant \sum_{k=0}^{\infty} \int_{2^k b}^{2^{k+1} b} \frac{|f_\alpha(u)|}{u}\,du$$
$$\leqslant \sum_{k=0}^{\infty} \int_{2^k b}^{2^{k+1} b} \frac{C}{u^2}\,du \leqslant \sum_{k=0}^{\infty} \frac{C}{2^k b} \int_{2^k b}^{2^{k+1} b} \frac{du}{u} \leqslant C_1.$$

§ 2. **New results on bounds of integrals of Dirichlet type depending upon parameters.** In this section we will consider a family of functions $f_\alpha\colon [0, \infty) \to \mathbf{R}$ depending on a parameter α in a certain way that we specify in each case and we shall try to obtain a bound of the form

$$\left|\int_\varepsilon^R \frac{\sin f_\alpha(t)}{t}\,dt\right| \leqslant C$$

with C independent of ε, R and α. The bound will depend essentially on the oscillations at infinity and the behaviour at the origin of $\sin f_\alpha(t)$. Therefore it will depend on the distribution of the zeros of $f_\alpha(t)$ on the real line and on the increasing of the functions.

(2.1) Theorem. *Let $f(x) = bx^\alpha e^x$, where b and α are real numbers, with $\alpha \geqslant 1$. Then*

$$\left|\int_\varepsilon^R \frac{\sin f(x)}{x}\,dx\right| \leqslant C$$

where C is a constant independent of b, ε and R.

Proof. The convergence of the integral is immediate. We may assume $b \geqslant 0$ because the function sine is odd. Consider

$$\int_\varepsilon^R \frac{\sin f(x)}{x}\,dx = \int_\varepsilon^1 \frac{\sin f(x)}{x}\,dx + \int_1^R \frac{\sin f(x)}{x}\,dx.$$

We will obtain a bound for each term. Let $e^x = u$; then

(2.1a) $$\left|\int_1^R \frac{\sin f(x)}{x}\,dx\right| = \left|\int_e^{R'} \frac{\sin[bu\lg^\alpha u]}{u\lg u}\,du\right| \leqslant C_1$$

by (1.7), with C_1 independent of b and R'.

If $0 \leqslant b \leqslant 1$, by Lemma (1.6) we have

(2.1b) $$\left|\int_\varepsilon^1 \frac{\sin f(x)}{x}\,dx\right| \leqslant C_2$$

because the function

$$g_\varepsilon(b) = \int_\varepsilon^1 \frac{\sin(bx^\alpha e^x)}{x}\,dx$$

satisfies $g_\varepsilon(0) = 0$ and $|g'_\varepsilon(s)| \leqslant e$ for every $s \in [0, 1]$.

If $b \geqslant 1$, let $x = yb^{-1/\alpha}$; then

$$\int_\varepsilon^1 \frac{\sin f(x)}{x}\,dx = \int_{\varepsilon'}^{b^{1/\alpha}} \frac{\sin y^\alpha e^{y/b^{1/\alpha}}}{y}\,dy = \int_{\varepsilon'}^1 \frac{\sin h(y)}{y}\,dy + \int_1^{b^{1/\alpha}} \frac{\sin h(y)}{y}\,dy$$

where $h(y) = y^\alpha e^{y \cdot b^{-1/\alpha}}$. By Lemma (1.5)

(2.1c) $$\left|\int_1^{b^{1/\alpha}} \frac{\sin h(y)}{y}\,dy\right| \leqslant C_3$$

due to the fact that the function h satisfies the hypothesis of the Van der Corput lemma, with $|h'(y)| \geqslant \alpha e^{1/b^{1/\alpha}} \geqslant \alpha \geqslant 1$.

If we apply Lemma (1.6), we also have

(2.1d) $$\left|\int_\varepsilon^1 \frac{\sin h(y)}{y}\,dy\right| \leqslant C_4.$$

In fact, if

$$g_{\varepsilon'}(a) = \int_{\varepsilon'}^{1} \frac{\sin y^\alpha e^{ay}}{y} dy,$$

then $|g'_{\varepsilon'}(s)| \leqslant e$ $\forall s \in [0, 1]$ and

$$|g_{\varepsilon'}(0)| = \left| \int_{\varepsilon'}^{1} \frac{\sin y^\alpha}{y} dy \right| \leqslant C_5.$$

The last inequality holds because of Theorem (1.1). The theorem follows from inequalities (2.1a) to (2.1d).

As a consequence we have

(2.2) THEOREM. *Let $f(t) = at^\alpha e^{\lambda t^\beta}$ where a, α, β and λ are real numbers and $\alpha/\beta \geqslant 1$; then*

$$\left| \int_{\varepsilon}^{R} \frac{\sin f(t)}{t} dt \right| \leqslant C$$

with C independent of a, λ, ε and R.

Proof. Let $x = \lambda t^\beta$. Then, using Theorem (2.1) we have

$$\left| \int_{\varepsilon}^{R} \frac{\sin f(t)}{t} dt \right| = \left| \frac{1}{\beta} \int_{\varepsilon'}^{R'} \frac{\sin (bx^\gamma e^x)}{x} dx \right| \leqslant \frac{1}{|\beta|} C$$

with $\gamma = \alpha/\beta$ and $b = a/\lambda^\gamma$ for $\lambda \neq 0$. If $\lambda = 0$, this result is Theorem (1.1).

Another type of result is the following one:

(2.3) THEOREM. *Let $f(x) = (ax+bx^2)e^x$ where a and b are real number: Then*

$$\left| \int_{\varepsilon}^{R} \frac{\sin f(x)}{x} dx \right| \leqslant C$$

with C independent of ε, R, a and b.

Proof. Consider the following decomposition:

$$\int_{\varepsilon}^{R} \frac{\sin f(x)}{x} dx = \int_{\varepsilon}^{1} \frac{\sin f(x)}{x} dx + \int_{1}^{R} \frac{\sin f(x)}{x} dx$$

and let us prove the theorem for each term.

If we apply Lemma (1.7), we have

$$\text{(23.a)} \qquad \left| \int_{1}^{R} \frac{\sin f(x)}{x} dx \right| \leqslant C_1$$

with C_1 independent of a, b and R. In fact, letting $e^x = t$, we have

$$\left| \int_{1}^{R} \frac{\sin f(x)}{x} dx \right| = \left| \int_{e}^{R'} \frac{\sin [(a \lg t + b \lg^2 t) t]}{t \lg t} dt \right| \leqslant C_1.$$

Let us estimate

$$\int_{\varepsilon}^{1} \frac{\sin f(x)}{x} dx.$$

Since the function sine is odd, we may assume $a \geqslant 0$ and $b \in \boldsymbol{R}$. If $0 \leqslant a \leqslant 1$ and $0 \leqslant b$,

$$\text{(2.3b)} \qquad \left| \int_{\varepsilon}^{1} \frac{\sin f(x)}{x} dx \right| \leqslant C_2$$

with C_2 independent of a, b and ε. In fact, the function

$$g_\varepsilon(a) = \int_{\varepsilon}^{1} \frac{\sin [(ax+bx^2) e^x]}{x} dx$$

satisfies $|g'_\varepsilon(s)| \leqslant e$ $\forall s \in [0, 1]$ and

$$|g_\varepsilon(0)| = \left| \int_{\varepsilon}^{1} \frac{\sin bx^2 e^x}{x} dx \right| \leqslant C.$$

The last inequality follows from Theorem (2.1), with $\alpha = 2$. Now, if we apply Lemma (1.6), we obtain estimate (2.3b).

If $a \geqslant 0$ and $0 \leqslant b \leqslant 1$ by the same kind of argument we have

$$\text{(2.3c)} \qquad \left| \int_{\varepsilon}^{1} \frac{\sin f(x)}{x} dx \right| \leqslant C_3.$$

Assume now $a \geqslant 1$ and $b \geqslant 1$. Let $x = y/\sqrt{b}$, then

$$\int_{\varepsilon}^{1} \frac{\sin f(x)}{x} dx = \int_{\varepsilon'}^{\sqrt{b}} \frac{\sin[((a/\sqrt{b}) y + y^2) e^{y/\sqrt{b}}]}{y} dy = \int_{\varepsilon'}^{1} \frac{\sin h(y)}{y} dy + \int_{1}^{\sqrt{b}} \frac{\sin h(y)}{y} dy$$

where $h(y) = \left(\frac{a}{\sqrt{b}} y + y^2\right) e^{y/\sqrt{b}}$. By Lemma (1.5) we obtain

$$\text{(2.3d)} \qquad \left| \int_{1}^{\sqrt{b}} \frac{\sin h(y)}{y} dy \right| \leqslant C_4$$

because the function h satisfies the hypothesis of the Van der Corput lemma with $h'(y) \geqslant 2$.

It remains to prove the bound for

$$\int_{\varepsilon'}^{1} \frac{\sin [((a/\sqrt{b}) y + y^2) e^{y/\sqrt{b}}]}{y} dy = \int_{\varepsilon'}^{1} \frac{\sin [(my + y^2) e^{ny}]}{y} dy,$$

with $m = a/\sqrt{b}$ and $n = 1/\sqrt{b}$. From the hypothesis on a and b we can conclude $0 \leqslant n \leqslant 1$ and $0 \leqslant m$. If $0 \leqslant m \leqslant 1$ and $0 \leqslant n \leqslant 1$,

(2.3e)
$$\left| \int_{\varepsilon'}^{1} \frac{\sin[(my+y^2)e^{ny}]}{y} dy \right| \leqslant C_5,$$

with C_5 independent of m, n and ε', by Lemma (1.6).

If $m \geqslant 1$, $0 \leqslant n \leqslant 1$ we use the change of variables $my = t$:

$$\int_{\varepsilon'}^{1} \frac{\sin[(my+y^2)e^{ny}]}{y} dy = \int_{\varepsilon''}^{m} \frac{\sin[(t+t^2/m^2)e^{nt/m}]}{t} dt$$
$$= \int_{\varepsilon''}^{1} \frac{\sin h(t)}{t} dt + \int_{1}^{m} \frac{\sin h(t)}{t} dt, \quad \text{with } h(t) = (t+t^2/m^2)e^{nt/m}.$$

The function h satisfies the hypothesis of the Van der Corput lemma with $|h'(t)| \geqslant 1$. Applying Lemma (1.5) we have

(2.3f)
$$\left| \int_{1}^{m} \frac{\sin h(t)}{t} dt \right| \leqslant C_6.$$

For the other term, let $t = m/u$; then

$$\int_{\varepsilon''}^{1} \frac{\sin h(t)}{t} dt = \int_{m}^{m/\varepsilon''} \frac{\sin[(m/u+1/u^2)e^{n/u}]}{u} du$$
$$\leqslant \sum_{k=0}^{\infty} \int_{2^k m}^{2^{k+1} m} \frac{\sin \varrho(u)}{u} du = \int_{m}^{4m} \frac{\sin \varrho(u)}{u} du + \sum_{k=2}^{\infty} \int_{2^k m}^{2^{k+1} m} \frac{\sin \varrho(u)}{u} du$$

with $\varrho(u) = (m/u+1/u^2)e^{n/u}$. Now, we have

$$\left| \int_{m}^{4m} \frac{\sin \varrho(u)}{u} du \right| \leqslant \int_{m}^{4m} \frac{du}{u} = \lg 4.$$

On the other hand, for any $k \geqslant 2$, $|\sin \varrho(u)| \leqslant |\varrho(u)|$ and $|\varrho(u)|/u \leqslant e/u^2$, therefore using Lemma (1.8) we may write

$$\left| \sum_{k=2}^{\infty} \int_{2^k m}^{2^{k+1} m} \frac{\sin \varrho(u)}{u} du \right| \leqslant e \lg 2 \sum_{k=2}^{\infty} (1+1/2^k) \leqslant C_7.$$

So

$$\left| \int_{\varepsilon''}^{1} \frac{\sin h(t)}{t} dt \right| \leqslant \lg 4 + C_7.$$

We have proved the bound

$$\left| \int_{\varepsilon}^{1} \frac{\sin[(ax+bx^2)e^{x}]}{x} dx \right| \leqslant C$$

assuming that a and b are non-negative.

If we assume that, for example, b is negative,

$$\left| \int_{\varepsilon}^{1} \frac{\sin u(x)}{x} dx \right| = \left| \int_{\varepsilon}^{1} \frac{\sin[(ax-bx^2)e^{x}]}{x} dx \right| \leqslant C$$

with C independent of a, b and ε. Observe that the bound

$$\left| \int_{1}^{R} \frac{\sin u(x)}{x} dx \right| \leqslant C$$

has been proved in the first part of the proof of this theorem, as an application of Lemma (1.7)

If $0 \leqslant a \leqslant 1$ and $0 \leqslant b$ or $a \geqslant 0$ and $0 \leqslant b \leqslant 1$, by Lemma (1.6) we have

(2.3g)
$$\left| \int_{\varepsilon}^{1} \frac{\sin u(x)}{x} dx \right| \leqslant C_8.$$

If $a \geqslant 1$ and $b \geqslant 1$, and $x = t/\sqrt{b}$, then

$$\int_{\varepsilon}^{1} \frac{\sin[(ax-bx^2)e^{x}]}{x} dx = \int_{\varepsilon'}^{\sqrt{b}} \frac{\sin\left[\left(\left(\frac{a}{\sqrt{b}}\right)t - t^2\right)e^{t/\sqrt{b}}\right]}{t} dt$$
$$= \int_{\varepsilon'}^{1/n} \frac{\sin[(mt-t^2)e^{nt}]}{t} dt$$

with $m = a/\sqrt{b}$ and $n = 1/\sqrt{b}$. Obviously, $0 \leqslant n \leqslant 1$ and $m \geqslant 0$.

If $0 \leqslant m \leqslant 1$ and $0 \leqslant n \leqslant 1$, we may write

$$\int_{\varepsilon'}^{1/n} \frac{\sin[(mt-t^2)e^{nt}]}{t} dt = \int_{\varepsilon'}^{2} \frac{\sin h(t)}{t} dt + \int_{2}^{1/n} \frac{\sin h(t)}{t} dt$$

where $h(t) = (mt-t^2)e^{nt}$.

By Lemma (1.6) with $0 \leqslant \varepsilon' \leqslant 2$, we have

$$\left| \int_{\varepsilon'}^{2} \frac{\sin h(t)}{t} dt \right| \leqslant C'$$

with C' independent of m, n and ε'.

On the other hand,

$$\left| \int_{2}^{1/n} \frac{\sin h(t)}{t} dt \right| \leqslant C''$$

with C'' independent of m and n, by Lemma (1.5).

Assume now that either $m \geqslant 1$ and $n \geqslant 1$, or $a/\sqrt{b} \geqslant 1$ and $1/\sqrt{b} \leqslant 1$. If $a \geqslant b \geqslant \sqrt{b} \geqslant 1$, we have

$$\left|\int_{\varepsilon}^{1} \frac{\sin[(ax-bx^2)e^x]}{x}\,dx\right| \leqslant C_9 \tag{2.3h}$$

with C_9 independent of a, b and ε. With the change of variables $x = t/a$, we obtain

$$\int_{\varepsilon}^{1} \frac{\sin[(ax-bx^2)e^x]}{x}\,dx = \int_{\varepsilon'}^{1} \frac{\sin h(t)}{t}\,dt + \int_{1}^{a} \frac{\sin h(t)}{t}\,dt$$

with $h(t) = \left(t-(b/a)t^2\right)e^{t/a}$. From Lemma (1.6) since $c = b/a^2 \leqslant 1$, because $a \geqslant b$, the following inequality holds:

$$\left|\int_{\varepsilon'}^{1} \frac{\sin h(t)}{t}\,dt\right| \leqslant k_1 \tag{2.3i}$$

with k_1 independent of a, b and ε'.

On the other hand,

$$\int_{1}^{a} \frac{\sin h(t)}{t}\,dt = \int_{1}^{a/3} \frac{\sin h(t)}{t} + \int_{a/3}^{a} \frac{\sin h(t)}{t}\,dt$$

if $a \geqslant 3$ (otherwise the bound is immediate). Since h verifies $|h'(t)| \geqslant 2/9\ \forall t \in [1, 9/3]$, we may apply Lemma (1.5). Hence

$$\left|\int_{1}^{a/3} \frac{\sin h(t)}{t}\,dt\right| \leqslant k_2. \tag{2.3j}$$

Also

$$\left|\int_{a/3}^{a} \frac{\sin h(t)}{t}\,dt\right| \leqslant \int_{a/3}^{a} \frac{dt}{t} = \lg 3. \tag{2.3k}$$

Therefore (2.3h) is now a consequence of (2.3 i, j, k). Assume now that $b \geqslant a \geqslant \sqrt{b} \geqslant 1$. Then

$$\left|\int_{\varepsilon}^{1} \frac{\sin[(ax-bx^2)e^x]}{x}\,dx\right| \leqslant C_{10} \tag{2.3l}$$

with C_{10} verifying the conditions of the theorem.

In order to prove the last inequality, let $x = t/b$. Then

$$\int_{\varepsilon}^{1} \frac{\sin[(ax-bx^2)e^x]}{x}\,dx = \int_{\varepsilon'}^{1} \frac{\sin v(t)}{t}\,dt + \int_{1}^{b} \frac{\sin v(t)}{t}\,dt$$

with $v(t) = \left((a/b)t - \frac{t^2}{b}\right)e^{t/b}$. By Lemma (1.6) we obtain

$$\left|\int_{\varepsilon'}^{1} \frac{\sin v(t)}{t}\,dt\right| \leqslant k_3 \tag{2.3ll}$$

with k_3 independent of a, b and ε'. Let $t = au$; then

$$\int_{1}^{b} \frac{\sin v(t)}{t}\,dt = \int_{1/a}^{1} \frac{\sin h(u)}{u}\,du + \int_{1}^{b/a} \frac{\sin h(u)}{u}\,du$$

with $h(u) = (a^2/b)(u-u^2)e^{au/b}$. If we apply Lemma (1.6), we have

$$\left|\int_{1}^{b/a} \frac{\sin h(u)}{u}\,du\right| \leqslant k_4. \tag{2.3m}$$

On the other hand,

$$\int_{1/a}^{1} \frac{\sin h(u)}{u}\,du = \int_{1/a}^{1} \frac{\sin[c(u-u^2)e^{\lambda u}]}{u}\,du$$

with $c = a^2/b \geqslant 1$ and $\lambda = a/b \leqslant 1$. If we use the change of variable $u = t/c$, we obtain the following equality:

$$\int_{1/a}^{1} \frac{\sin[c(u-u^2)e^{\lambda u}]}{u}\,du = \int_{c/a}^{1} \frac{\sin v(t)}{t}\,dt + \int_{1}^{c} \frac{\sin v(t)}{t}\,dt$$

with $v(t) = (t-t^2/c)e^{\lambda t/c}$. From Lemma (1.6),

$$\left|\int_{c/a}^{1} \frac{\sin v(t)}{t}\,dt\right| \leqslant k_5 \tag{2.3n}$$

with k_5 as in the theorem. Finally we have

$$\left|\int_{1}^{c} \frac{\sin v(t)}{t}\,dt\right| = \left|\int_{1}^{a^2/b} \frac{\sin[(t-(b/a^2)t^2)e^{t/a}]}{t}\,dt\right| \leqslant k_6 \tag{2.3o}$$

with k_6 independent of a and b. In fact, if $a^2/3b \leqslant 1$, we may write

$$\left|\int_{1}^{a^2/b} \frac{\sin v(t)}{t}\,dt\right| \leqslant \int_{1}^{3} \frac{dt}{t} = \lg 3.$$

On the other hand, if $a^2/3b \geqslant 1$,

$$\int_{1}^{a^2/b} \frac{\sin v(t)}{t}\,dt = \int_{1}^{a^2/3b} \frac{\sin v(t)}{t}\,dt + \int_{a^2/3b}^{a^2/b} \frac{\sin v(t)}{t}\,dt.$$

We may apply Lemma (1.5) for the first integral. Therefore

$$\left|\int_{1}^{a^2/3b} \frac{\sin v(t)}{t}\,dt\right| \leqslant k_7. \tag{2.3p}$$

For the second one, we have

$$\left|\int_{a^2/3b}^{a^2/b} \frac{\sin v(t)}{t}\,dt\right| \leqslant \int_{a^2/3b}^{a^2/b} \frac{dt}{t} = \lg 3. \tag{2.3q}$$

Inequality (2.3l) follows from (2.3ll, m, n, o). The theorem is proven by taking into account (2.3a) up to (2.3l).

As a consequence we have

(2.4) THEOREM. *Let* $f(t) = (at^\alpha + bt^\beta) e^{\lambda t^\alpha}$ *where* $\beta = 2\alpha \neq 0$, a, b *and* λ *are real numbers. Then*

$$\left| \int_\varepsilon^R \frac{\sin f(t)}{t} dt \right| \leqslant C$$

with C *independent of* $a, b, \lambda, \varepsilon$ *and* R.

Proof. Let $\lambda t^\alpha = x$ and apply Theorem (2.3). Then

$$\left| \int_\varepsilon^R \frac{\sin f(t)}{t} dt \right| = \left| \frac{1}{\alpha} \int_{\varepsilon'}^{R'} \frac{\sin [(cx + dx^2) e^x]}{x} dx \right| \leqslant \frac{1}{|\alpha|} C$$

where $c = a/\lambda$ and $d = b/\lambda^2$ when $\lambda \neq 0$. If $\lambda = 0$, the theorem coincides with Theorem (1.1).

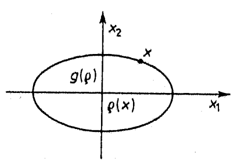
§ 3. **L^2-theory of the corresponding singular integrals.** In this section we will use the theorems of § 2 to study the L^2-theory of some singular integral operators, defined through a convolution, in $\boldsymbol{R}^2$. The kernels of this convolution will be truncated by a certain family of ellipses, contracting to the origin.

Before we go on we will give an idea of how a family of ellipses generates a metric in $\boldsymbol{R}^2$. Let the family of ellipses be defined by

(3a) $$x_1 = \varrho \cos\varphi, \quad \varrho \in (0, \infty), \qquad x_2 = g(\varrho) \sin\varphi, \quad \varphi \in (0, 2\pi]$$

where ϱ and $g(\varrho)$ are the semiaxes, g is an increasing continuous function, such that $g(\varrho) = o(\varrho)$, $\varrho \to 0$ and $g(\varrho) \to \infty$, $\varrho \to \infty$. Therefore we have a nested family, contracting to the origin.

For every $x \in \boldsymbol{R}^2$ we define $\varrho(x) = \varrho$ (with $\varrho(0) = 0$) where ϱ is x_1-semiaxis of the ellipse passing through x.



Then a sufficient condition for $\varrho(x) = \varrho$ to be a quasi-metric is that

(*) $$\frac{g(\varrho)}{g(M\varrho)} \underset{M \to \infty}{\to} 0 \quad \text{uniformly on } \varrho.$$

In fact, a constant M, $M < \infty$, $M \geqslant 1$ can then be found such that $\forall x, y \in \boldsymbol{R}^2$

$$\varrho(x+y) \leqslant M[\varrho(x) + \varrho(y)]$$

whenever (*) holds (see [14]). Among others, the following functions; $g(\varrho) = \varrho^\beta e^{\lambda \varrho^\alpha}$, $\beta \geqslant 2\alpha$ and $g(\varrho) = \varrho^\alpha (a + \lg^\beta \varrho)$, $\beta \geqslant 1$, $\alpha > 0$, are examples of functions verifying the sufficient condition (*).

Now we go back to the problem of studying in $\boldsymbol{R}^2$ the L^2 bound of the operator $T_{\varepsilon,\eta}$, defined by

$$T_{\varepsilon,\eta} f(x) = K_{\varepsilon,\eta} * f(x), \quad f \in L^2(\boldsymbol{R}^2)$$

where

$$K_{\varepsilon,\eta} = \begin{cases} K(x) & \text{for} \quad \varepsilon \leqslant \varrho(x) \leqslant \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\varrho(x)$ is a metric defined through a family of ellipses that we will specify later. K is a kernel that verifies

(1) K is odd, i.e., $K(-y) = -K(y)$;

(2) K admits the following factorization:

$$K(y_1, y_2) = h(\varrho)\, g(\varphi)$$

where

(**) $$y_1 = \Phi(\varrho) \cos\varphi, \qquad y_2 = \Psi(\varrho) \sin\varphi$$

gives the family of ellipses contracting to the origin;

(3) $\int_\Sigma |K(\bar{y})|\, d\bar{y} < \infty$

where Σ = unit sphere in $\boldsymbol{R}^2$; $\Sigma = \{x \in \boldsymbol{R}^2\colon |x| = 1\}$.

Therefore $K_{\varepsilon,\eta}$ is given through the truncation of K by the family of ellipses (**). Note that these kernels do not satisfy, in general, any type of homogeneity, therefore the techniques are different from the ones used in [1] and [2].

In order to prove the L^2-boundedness it will be enough to show that there exists a constant $A > 0$ such that $|\hat{K}_{\varepsilon,\eta}(x)| \leqslant A$ for any ε, η and $x \in \boldsymbol{R}^2$. In fact, if such an inequality holds, using the theorem of Plancherel, we obtain

$$\|T_{\varepsilon,\eta} f\|_2 = \|\hat{T}_{\varepsilon,\eta} f\|_2 = \|\hat{K}_{\varepsilon,\eta} f\|_2 \leqslant A \|\hat{f}\|_2 = A \|f\|_2 \quad \text{or} \quad \|T_{\varepsilon,\eta}\|_2 \leqslant A.$$

So we may define Tf for $f \in L^2$ as

$$Tf = \lim_{\substack{\varepsilon \to 0 \\ \eta \to \infty}} K_{\varepsilon,\eta} f$$

and T is bounded in $L^2(\boldsymbol{R}^2)$ with $\|T\|_2 \leqslant A$. The problem lies in how to study the behaviour of h and g in such a way that the inequality $|\hat{K}_{\varepsilon\eta}(x)| \leqslant A$ holds (independently of ε, η and $x \in \boldsymbol{R}^2$).

In order to show that the inequality holds, we are going to use the results of Section 2.

Let us take in $\boldsymbol{R}^2$ the following family of ellipses contracting to the origin

$$\text{(3b)}\qquad \begin{aligned} y_1 &= \varrho^\alpha e^{\lambda\varrho^\alpha}\cos\varphi \equiv \Phi(\varrho)\cos\varphi,\\ y_2 &= \varrho^\beta e^{\lambda\varrho^\alpha}\sin\varphi \equiv \Psi(\varrho)\sin\varphi \end{aligned}$$

with $\beta = 2\alpha$. Consider (in polar coordinates) the following kernel

$$\text{(3c)}\qquad K(\varrho, \varphi) = \frac{1}{\alpha\varrho^{3\alpha} e^{2\lambda\varrho^\alpha}(1+\lambda\varrho^\alpha)} \sin\varphi \equiv h(\varrho)\, g(\varphi).$$

Then we have

(3.1) THEOREM. *For* $x\in\boldsymbol{R}^2$ *let*

$$K_{\varepsilon,\eta}(x) = \begin{cases} K(x) & \text{if } \varepsilon \leqslant \varrho(x) \leqslant \eta,\\ 0 & \text{otherwise} \end{cases}$$

be the truncation of a kernel K *by the family of ellipses* (3b), *using a change to polar coordinates. Here the kernel* K *verifies*:

(i) K *is odd, i.e.,* $K(-y) = -K(y)$;

(ii) K *can be expressed as* $K(y_1, y_2) = h(\varrho)\, g(\varphi)$, *like in* (3c);

(iii) $\int_\Sigma |K(\bar{y})|\, d\bar{y} < \infty$.

Then there exists a constant $A > 0$ *such that* $|\hat{K}_{\varepsilon,\eta}(x)| \leqslant A$ *for every* ε, η *and* x.

If for any $f\in L^2(\boldsymbol{R}^2)$ *we define* $T_{\varepsilon,\eta} f = K_{\varepsilon,\eta} * f$, *then* $T_{\varepsilon,\eta}$ *is a bounded operator in* $L^2(\boldsymbol{R}^2)$ *uniformly in* ε *and* η.

Proof. As we saw, it is enough to prove that $\hat{K}$ (the symbol of the operator T) is a bounded function. We are going to calculate the Fourier transform of the kernel

$$\hat{K}_{\varepsilon,\eta}(x) = \int_{\boldsymbol{R}^2} K_{\varepsilon,\eta}(y)\, e^{-2\pi i(x,y)}\, dy$$

$$= \int_{\varepsilon\leqslant\varrho(y)\leqslant\eta} K(y)\, e^{-2\pi i(x,y)}\, dy = -i \int_{\varepsilon\leqslant\varrho(y)\leqslant\eta} K(y)\sin[2\pi(x, y)]\, dy$$

where (x, y) is the scalar product in $\boldsymbol{R}^2$. If we use polar coordinates, we may write

$$\hat{K}_{\varepsilon,\eta}(x) = -i\int_{-\pi}^{\pi}\int_{\varepsilon}^{\eta} h(\varrho)\, g(\varphi)\, H(y_1, y_2;\ \varrho, \varphi)\sin[2\pi(x, y)]\, d\varrho\, d\varphi$$

where $H(y_1, y_2;\ \varrho, \varphi) = \alpha\varrho^{3\alpha-1} e^{2\lambda\varrho^\alpha}[(1+\lambda\varrho^\alpha)+\sin^2\varphi]$ is the Jacobian of the change of variables. So

$$\hat{K}_{\varepsilon,\eta}(x) = -i\int_{-\pi}^{\pi}\int_{\varepsilon}^{\eta} h(\varrho)\, g(\varphi)\sin[(a\varrho^\alpha+b\varrho^\beta)e^{\lambda\varrho^\alpha}]\, \alpha\varrho^{3\alpha-1} e^{2\lambda\varrho^\alpha}(1+\lambda\varrho^\alpha)\, d\varrho d\varphi -$$

$$-i\int_{-\pi}^{\pi}\int_{\varepsilon}^{\eta} h(\varrho)\, g(\varphi)\sin[(a\varrho^\alpha+b\varrho^\beta)e^{\lambda\varrho^\alpha}]\, \alpha\varrho^{3\alpha-1} e^{2\lambda\varrho^\alpha}\sin^2\varphi\, d\varrho d\varphi$$

where

$$\sin[(a\varrho^\alpha+b\varrho^\beta)e^{\lambda\varrho^\alpha}] = \sin 2\pi(x_1\varrho^\alpha e^{\lambda\varrho^\alpha}\cos\varphi + x_2\varrho^{2\alpha} e^{\lambda\varrho^\alpha}\sin\varphi).$$

If $h(\varrho) = [\alpha\varrho^{3\alpha} e^{2\lambda\varrho^\alpha}(1+\lambda\varrho^\alpha)]^{-1}$, we have

$$|\hat{K}_{\varepsilon,\eta}(x)| \leqslant \int_{-\pi}^{\pi} |g(\varphi)| \left|\int_{\varepsilon}^{\eta}\frac{\sin u(\varrho)}{\varrho}\, d\varrho\right| d\varphi + \int_{-\pi}^{\pi} |g(\varphi)\sin^2\varphi| \left|\int_{\varepsilon}^{\eta} n(\varrho)\frac{\sin u(\varrho)}{\varrho}\, d\varrho\right| d\varphi$$

with

$$u(\varrho) = (a\varrho^\alpha+b\varrho^\beta)e^{\lambda\varrho^\alpha} \text{ and } n(\varrho) = 1/(1+\lambda\varrho^\alpha).$$

By Theorem (2.4) we obtain

$$\left|\int_{\varepsilon}^{\eta}\frac{\sin u(\varrho)}{\varrho}\, d\varrho\right| \leqslant C$$

with C independent of a, b, λ, ε and η. Using integration by parts, since

$$n(\varrho)\xrightarrow[\varrho\to 0]{} 1,\quad n(\varrho)\xrightarrow[\varrho\to\infty]{} 0 \quad\text{and}\quad n'(\varrho) = -\frac{\lambda\alpha\varrho^{\alpha-1}}{(1+\lambda\varrho^\alpha)^2},$$

we have

$$\int_{\varepsilon}^{\eta}\eta(\varrho)\frac{\sin u(\varrho)}{\varrho}\, d\varrho = [F(\varrho)\, n(\varrho)]_\varepsilon^\eta - \int_{\varepsilon}^{\eta} F(\varrho)\, n'(\varrho)\, d\varrho$$

where

$$F(x) = \int_0^x \frac{\sin u(s)}{s}\, ds.$$

So

$$\left|\int_{\varepsilon}^{\eta} n(\varrho)\frac{\sin u(\varrho)}{\varrho}\, d\varrho\right| \leqslant C'$$

with C' independent of λ, ε, η, a and b.

On the other hand, $g(\varphi) = \sin\varphi$. Hence

$$|\hat{K}_{\varepsilon,\eta}(x)| \leqslant C\int_{-\pi}^{\pi}|g(\varphi)|\, d\varphi + C'\int_{-\pi}^{\pi}|g(\varphi)\sin^2\varphi|\, d\varphi \leqslant A$$

with A independent of ε, η and x. We note that for any odd g satisfying

$$\int_{-\pi}^{\pi}|g(\varphi)|\, d\varphi < \infty$$

we obtain the same result.

Remark. The same kind of proof we have used in Theorem (3.1) works for kernels such as

$$K(\varrho, \varphi) = \frac{1}{\alpha \varrho^{2\alpha}(a+\lg^{\beta}\varrho)} \sin\varphi$$

where the family of ellipses is now

$$y_1 = \varrho^{\alpha}\cos\varphi,$$
$$y_2 = \varrho^{\alpha}(a+\lg^{\beta}\varrho)\sin\varphi, \qquad \beta \geqslant 1,\ \alpha > 0.$$

It is enough to take into account result (1.2) (Coifman–Guzmán) The proof can be seen in [16] if $\alpha = 1$ and in [14] if $\beta = 2$.

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Received November 23, 1982 (1844)

STUDIA MATHEMATICA, T. LXXIX. (1984)

An analogue of the argument theorem of Bohr and its application

by

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Abstract. An analogue of the argument theorem of Bohr is proved and used to establish the following result: given a real S^1 almost periodic function f on $\boldsymbol{R}$, the function $x \to \exp(i\int_0^x f(u)\,du)$ is W^1 almost periodic if and only if it is (uniformly) almost periodic, in which case the function $x \to \int_0^x f(u)\,du - \hat{f}(0)x$ is almost periodic. It is shown that the latter theorem fails if W^1 almost periodicity is replaced by what is here called E^2 almost periodicity.

1. Introduction. According to a well-known theorem of Bohr (cf. [3], [7]), given a real continuous function f on $\boldsymbol{R}$, the function $x \to \exp(if(x))$ is almost periodic if and only if there exists $a \in \boldsymbol{R}$ such that the function $x \to f(x) + ax$ is almost periodic.

Our main objective is to prove the following

THEOREM 1. *Suppose a real uniformly continuous function f on $\boldsymbol{R}$ satisfies the following cocycle condition:*

(co) *for every $t \in \boldsymbol{R}$, there exists $a_t \in \boldsymbol{R}$ such that the function $x \to f(x+t) - f(x) + a_t x$ is almost periodic.*

In order that there be $a \in \boldsymbol{R}$ such that the function $x \to f(x) + ax$ is almost periodic it is necessary and sufficient that the function $x \to \exp(if(x))$ be W^1 almost periodic.

Since given an S^1 almost periodic function f on $\boldsymbol{R}$, the function $x \to \int_0^x f(u)\,du$ is uniformly continuous (cf. [1], Th. 4.7.8), and, for any $t \in \boldsymbol{R}$, the function $x \to \int_x^{x+t} f(u)\,du$ is almost periodic (cf. [2], Th. 2.3.1), from Theorem 1 and the above-mentioned argument theorem of Bohr we easily deduce the following

THEOREM 2. *Let f be a real S^1 almost periodic function on $\boldsymbol{R}$. Then the function $x \to \exp(i\int_0^x f(u)\,du)$ is W^1 almost periodic if and only if it is almost*