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Banach S-algebras and conditional basic sequences in non-Montel Fréchet spaces

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Abstract. A non-associative multiplication is defined on Banach spaces with symmetric basis. This multiplication is continuous exactly when each basic sequence generated by one vector is equivalent to the original basis. Upper and lower l_p-estimates are proved for such algebra norms. As an application, these results are combined with the techniques of Figiel, Lindenstrauss and Milman to produce conditional basic sequences in a large class of non-Montel Fréchet spaces. This class includes subspaces of l_p-Köthe sequence spaces and subspaces of products of superreflective spaces. This partially answers a question of Pełczyński.

Altshuler, in [1], studied the class of Banach spaces X with a symmetric basis $\{x_n\}$ which have the further property that each basic sequence generated by one vector is equivalent to $\{x_n\}$. We show (Proposition 3.1) that such spaces X are exactly those which can be re-normed into a Banach S-algebra; that is, there is a (non-associative) multiplication which singles out this class of Banach spaces with a symmetric basis. The algebra norm of a Banach S-algebra must satisfy some l_p-estimates (Theorem 3.2). In fact, for each such X there is a p with $1 \leqslant p \leqslant \infty$, so that for each $q > p$ there is a constant C_q so that

$$C_q\left(\sum |\alpha_n|^q\right)^{1/q} \geqslant \left\|\sum \alpha_n x_n\right\| \geqslant \left(\sum |\alpha_n|^p\right)^{1/p}$$

for any scalar sequence $\{\alpha_n\}$. (The lower estimate is essentially in Altshuler [1].)

Thus a Banach S-algebra can replace some l_q and still preserve the ordering (as sets of sequence spaces) of the l_p-spaces. In Section 2, such spaces are defined to have index q. The techniques of Figiel, Lindenstrauss and Milman [8] applied to spaces of index $q < \infty$, yield "nearly" the same Dvoretzky-type results as obtained for l_q in Example 3.1 of [8] (Proposition 2.7).

In Section 4, these results are combined to affirmatively answer the following question of Pełczyński [13] for "most" non-Montel spaces:

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QUESTION. Does each non-nuclear Fréchet space have a conditional basic sequence?

This collection of "most" non-Montel spaces includes all subspaces of l_p-Köthe sequence spaces and all subspaces of a product of a collection of super-reflexive Banach spaces. Previous known results have answered the question in the affirmative for the special cases Banach spaces [14], l_p-Köthe sequence spaces [19], and Hilbertian spaces [14] and [19].

1. Notation and preliminaries. A *Fréchet* space is a complete metrizable locally convex topological vector space. The continuous dual of a space X will be denoted X'. A Fréchet space X is *Montel* if each $\sigma(X, X')$-convergent sequence converges in the original (strong) topology.

We will write $\{x_n\}$ for $\{x_n\}_{n=1}^{\infty}$, $\{x_n\}_j^k$ for $\{x_n\}_{n=j}^k$, $[x_n]_j^k$ for span $\{x_n\}_j^k$ and $[x_n]$ for the closed linear span of $\{x_n\}$. If $\{\alpha_n\}$ and $\{\beta_n\}$ are positive scalar sequences, $\alpha_n \sim \beta_n$ will denote that α_n/β_n is bounded and bounded away from zero.

Let l_p, $1 \leqslant p < \infty$ be the Banach space of scalar sequences $\{\xi_n\}$ with norm

$$\|\xi_n\|_p = (\sum |\xi_n|^p)^{1/p} < \infty.$$

Let l_∞ (respectively c_0) be the Banach space of bounded (respectively null) scalar sequences $\{\xi_n\}$ with norm $\|\xi_n\|_\infty = \sup_n |\xi_n|$. The spaces l_p^k, c_0^k are the k-dimensional spaces of sequences $\{\xi_n\}_1^k$, with norm $\|\cdot\|_p$, $\|\cdot\|_\infty$. If $1 \leqslant p \leqslant \infty$, let p' be the number so that $(1/p)+(1/p') = 1$, hence if $p < \infty$, $l_p' = l_{p'}$. If $\{X_n\}$ is a sequence of Banach spaces, the c_0-sum of $X_1 \oplus X_2 \oplus \ldots$ is the Banach space of sequences, $\{\xi_n\}$, so that $\xi_n \in X_n$, for each n, and $\lim_n \|\xi_n\| = 0$, with norm $\|\{\xi_n\}\| = \sup_n \|\xi_n\|$. If $1 \leqslant p < \infty$, and $\{a_n\}$ is a non-increasing null sequence of positive reals with $\sum a_n = \infty$, then the Lorentz sequence space, $d(a_n, p)$, is defined to be the Banach space of all scalar sequences $\{\xi_n\}$ with norm

$$\|\xi_n\| = \sup_\pi (\sum a_n |\xi_{\pi(n)}|^p)^{1/p} < \infty,$$

where π ranges over all permutations of the integers.

A sequence $\{x_n\}$ contained in the Fréchet space X is said to be a *basis* for X if for each $x \in X$ there is a unique scalar sequence $\{\alpha_n\}$ with $\sum \alpha_n x_n = x$. A sequence x is a *basic sequence* if it is a basis for $[x_n]$. A basis $\{x_n\}$ is *unconditional* if for each permutation of the integers π, $\{x_{\pi(n)}\}$ is also a basis; otherwise the basis is said to be *conditional.*

The basic sequence $\{x_n\}$ is *normalized* if it is bounded and bounded away from the origin (i.e., there is a neighborhood of the origin U, with $x_n \notin U$ for each n). Note that this is not the usual definition for Banach spaces. The basic sequence $\{x_n\}$ *dominates* the basic sequence $\{y_n\}$ if the convergence of $\sum \alpha_n x_n$ implies the convergence of $\sum \alpha_n y_n$. The basic sequences $\{x_n\}$ and $\{y_n\}$ are *equivalent* if each dominates the other.

If $\{x_n\}$ is a sequence contained in X and $\|\cdot\|$ is a semi-norm on X, so that

$$\|\textstyle\sum_1^p \beta_n \alpha_n x_{\pi(n)}\| \leqslant K \|\sum_1^{p+q} \alpha_n x_n\|, \tag{1}$$

is true for all integers p, q, scalars $\{\alpha_n\}$, scalars $\{\beta_n\}$ with $|\beta_n| \leqslant 1$ and permutations of the integers π, we will say $\{x_n\}$ is *K-symmetric with respect to* $\|\cdot\|$. If (1) is true under the same conditions except π is fixed to be the identity on the integers, we will say $\{x_n\}$ is *K-unconditional with respect to* $\|\cdot\|$. If (1) is true under the same conditions except π is the identity and $\beta_n = 1$ for each n, we will say $\{x_n\}$ is *K-basic with respect to* $\|\cdot\|$.

If X and Y are two isomorphic Banach spaces, the *Banach–Mazur distance* is defined to be:

$$d(X, Y) = \inf\{\|T\|\,\|T^{-1}\|:\ T\colon X \to Y \text{ an isomorphism}\}.$$

The Banach space X is said to be *finitely-representable* in the Banach space Y if for each $\varepsilon > 0$ and finite-dimensional subspace X_0 of X there is a subspace Y_0 of Y with $d(X_0, Y_0) < 1+\varepsilon$. A Banach space X is *super-reflexive* if only reflexive Banach spaces are finitely-representable in X.

For general references, we use [10] for Banach spaces and [10] and [18] for bases in Banach spaces. Although there is no reference for bases in Fréchet spaces, *correctly restated* results about bases in Banach spaces are true for Fréchet spaces.

Finally, we need the following equivalent condition for the statement, $\{x_n\}$ is an unconditional basic sequence in the Fréchet space X, where the topology of X is defined by the sequence of semi-norms $\{\|\cdot\|_k\}$:

For each k, there are j and M, so that, for each finite subsets of integers F and G, with $F \subset G$,

$$\|\textstyle\sum_{n\in F} \alpha_n x_n\|_k \leqslant M \|\sum_{n\in G} \alpha_n x_n\|_j$$

is true for any scalar sequence $\{\alpha_n\}$.

2. Banach spaces with index. A normalized basic sequence $\{x_n\}$ in a Banach space is said to have index p $(1 \leqslant p < \infty)$ if the following are true:

(i) $q < p$ and $\{\alpha_n\} \in l_q$ implies $\sum \alpha_n x_n \in [x_n]$, and

(ii) $q > p$ and $\sum \alpha_n x_n \in [x_n]$ implies $\{\alpha_n\} \in l_q$.

If $\|\cdot\|$ is the norm on $[x_n]$, the closed graph theorem implies that "having index p" is equivalent to "for all q, r, with $q < p < r$, there are constants K_q and K_r so that for each finite scalar sequence $\{\alpha_n\}$,

$$K_r (\textstyle\sum |\alpha_n|^r)^{1/r} \leqslant \|\sum \alpha_n x_r\| \leqslant K_q (\sum |\alpha_n|^q)^{1/q}\text{''}. \tag{*}$$

For a symmetric basis $\{x_n\}$, having index p is equivalent to (*) being true for each finite sequence $\{\alpha_n\}$ with $\alpha_n \equiv 1$ (Proposition 2.5).

Our main interest in Banach spaces with index p, $p < \infty$, is that the techniques of Figiel, Lindenstrauss and Milman [8] yield "large" finite-dimensional Hilbert subspaces in $[x_n]_1^k$. This in turn will allow construction of conditional basic sequences in certain Fréchet spaces. Many examples of Banach spaces with index are given in the next section.

It seems that having index is independent of most global properties a space may have. For instance, it is possible to construct uniformly convex Lorentz sequence spaces without index. Other examples are given in the remarks after Theorem 3.2.

Lemma 2.1 is given for later reference. The results are easy consequences of the closed graph theorem and standard facts about duality and l_p-spaces.

LEMMA 2.1. *If $\{x_n\}$ is a normalized basis for a Banach space and $\{x_n'\}$ are the coefficient functionals, then*

(a) *$\{\alpha_n\} \in l_q$ implies $\sum \alpha_n x_n \in [x_n]$ (respectively, $\sum \alpha_n x_n' \in [x_n']$) if and only if*

(b) *$\sum \alpha_n x_n' \in [x_n']$ (respectively, $\sum \alpha_n x_n \in [x_n]$) implies $\{\alpha_m\} \in l_q$.*

We now consider the case when $\{x_n\}$ is a symmetric basis. Suppose $\|\cdot\|$ is a 1-symmetric norm on $\{x_n\}$, define

$$\lambda(n) = \left\|\sum_1^n x_i\right\|, \quad \xi(n) = \log\lambda(n)/\log n,$$

$$S = \limsup \xi(n) \quad \text{and} \quad I = \liminf \xi(n).$$

Later, we will need to consider sequences of 1-symmetric norms $\{\|\cdot\|_k\}$ on $\{x_n\}$, in this case, we will distinguish the various $\lambda(n)$, $\xi(n)$, S and I, by use of sub or superscripts (i.e., $\lambda^k(n) = \|\sum_1^n x_i\|_k$, etc.).

If the sequence $\{\lambda(n+1)-\lambda(n)\}$ is non-increasing, we will say that $\|\cdot\|$ is a concave 1-symmetric norm. Although, it is not the case that each 1-symmetric norm $\|\cdot\|$ is concave, there is always an equivalent concave 1-symmetric $\|\cdot\|_0$ satisfying $\|\cdot\| \leqslant \|\cdot\|_0 \leqslant 2\|\cdot\|$ ([11], p. 119). As we will be dealing only with isomorphic properties of the Banach space and with asymptotical results about the sequences $\{\lambda(n)\}$ and $\{\xi(n)\}$, without loss of generality, we will assume that our 1-symmetric norms are concave.

If $\|\cdot\|$ is a concave 1-symmetric norm on $\{x_n\}$, define $a_1 = 1$ and $a_{n+1} = \lambda(n+1)-\lambda(n)$ for $n \geqslant 1$. The sequence $\{a_n\}$ is non-increasing and if $\{x_n\}$ is not equivalent to the usual basis of l_1 or c_0, then $\lim a_n = 0$ and $\sum a_n = \infty$. In this case, we define the *associated Lorentz norm* $[\![\cdot]\!]$ on span $\{x_n\}$ to be the norm of the Lorentz sequence space $d(\{a_n\}, 1)$, superpositioned onto the $\{x_n\}$. Note that if $\{\alpha_n\}$ is a finite sequence of scalars,

$$(**) \qquad \left\|\sum \alpha_n x_n\right\| \leqslant [\![\sum \alpha_n x_n]\!].$$

To see this, by symmetry it is enough to show $(**)$ when $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0$. And

$$\left\|\sum_1^m \alpha_n x_n\right\| = \left\|\sum_{n=1}^m (\alpha_n - \alpha_{n+1}) \sum_{j=1}^n x_j\right\|$$
$$\leqslant \sum_1^m (\alpha_n - \alpha_{n-1})\lambda(n) = \sum_1^m \alpha_n\big(\lambda(n)-\lambda(n-1)\big) = [\![\sum_1^m \alpha_n x_n]\!].$$

The following lemma is elementary and its proof is omitted.

LEMMA 2.2. *If $\{x_n\}$ is a symmetric basis with 1-symmetric norm $\|\cdot\|$, then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (i') $\Rightarrow$ (ii') $\Rightarrow$ (iii'), where*

(i) [(i')] $\delta > S$ $[\delta < I]$;

(ii) [(ii')] *There is a constant K so that for each n,*

$$\lambda(n) \leqslant Kn^{\delta}\,[\lambda(n) \geqslant Kn^{\delta}];$$

(iii) [(iii')] $\delta \geqslant S\,[\delta \leqslant I]$.

LEMMA 2.3. *If $\{x_n\}$ is a symmetric basic with 1-symmetric norm $\|\cdot\|$ and if $\delta > S$ with $1 < p = 1/\delta < \infty$, then there is a constant K so that for each finite scalar sequence $\{\alpha_n\}$,*

$$\left\|\sum \alpha_n x_n\right\| \leqslant K\left(\sum |\alpha_n|^p\right)^{1/p}.$$

Proof. Since the result is true if $\{x_n\}$ is equivalent to the usual basis of l_1 or c_0, it suffices to prove the lemma when the associated Lorentz norm $[\![\cdot]\!] = \|\cdot\|$. Let $q > p$ with $1/q > S$. Since $na_n \leqslant \lambda(n) \leqslant Cn^{1/q}$, $a_n \leqslant Cn^{-1/q'}$. If $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0$, then

$$[\![\sum \alpha_n x_n]\!] = \sum \alpha_n a_n \leqslant C\sum \alpha_n n^{-1/q'}.$$

By Hölder's inequality for p and p'

$$[\![\sum \alpha_n x_n]\!] \leqslant C\left(\sum |\alpha_n|^p\right)^{1/p}\left(\sum n^{-p'/q'}\right)^{1/p'}.$$

But $p'/q' > 1$, so that $K = C\left(\sum n^{-p'/q'}\right)^{1/p'} < \infty$ will satisfy the conclusion of the lemma, and the proof is complete.

If $\|\cdot\|$ is 1-symmetric on $\{x_n\}$ and $\|\cdot\|'$ is the dual norm on the coefficient functionals $\{x_n'\}$, then $\lambda(n)\lambda'(n) = n$([11], p. 118). Thus by duality, Lemmas 2.1 and 2.3, and the obvious relations between S' and I, we have the following lemma.

LEMMA 2.4. *If $\{x_n\}$ is a symmetric basic with 1-symmetric norm $\|\cdot\|$ and if $\delta < I$ with $1 < p = 1/\delta < \infty$, then there is a constant K so that for each finite scalar sequence $\{\alpha_n\}$,*

$$\left\|\sum \alpha_n x_n\right\| \geqslant K\left(\sum |\alpha_n|^p\right)^{1/p}.$$

PROPOSITION 2.5. *The symmetric basis $\{x_n\}$ has index p if and only if $S = I = 1/p$.*

Proof. If $S = I = 1/p$, then Lemmas 2.3 and 2.4 show that $\{x_n\}$ has index p. The converse follows easily from the upper and lower l_p-estimates in $(*)$ and Lemma 2.2.

Remark. If $\{x_n\}$ is a symmetric basis with index p and c_0 is not finitely representable in $[x_n]$, then l_p is block finitely represented (see [17]) in $\{x_n\}$. This follows from Theorem 3.3 of [17].

The following lemma, although ungainly, will prove to be the key to our results on conditional basic sequences.

LEMMA 2.6. *Suppose* $\|\cdot\|$ *is a concave* 1-*symmetric norm on* $\{x_n\}$ *and suppose there is a* B *so that for each* j, N, *there is an* $n \geqslant N$ *with* $\xi(m) \leqslant B$ *for* $n \leqslant m \leqslant n^j$; *then for each* p, $1 < p < 1/B$, *there is a constant* C *so that for each* $\delta > 0$, *there are infinitely many* l *with the property that for each scalar sequence* $\{\alpha_n\}_1^l$

$$\left\|\sum \alpha_n x_n\right\| \leqslant Cl^\delta \left(\sum |\alpha_n|^p\right)^{1/p}.$$

Proof. Since the result is true if $\{x_n\}$ is equivalent to the usual basis of l_1 or c_0, it suffices to prove the lemma when the associated Lorentz norm $[\![\cdot]\!] = \|\cdot\|$.

Let $p < 1/B$ be given and choose j so that $\delta > B/j$. Let N be given, choose $n \geqslant N$ so that $\xi(m) \leqslant B$, for $n \leqslant m \leqslant n^j$ and let $l = n^j$. The condition on $\xi(m)$ implies $\sum_1^n a_i \leqslant n^B$ and $a_m \leqslant m^{B-1}$, for $n \leqslant m \leqslant l$. Thus if the scalar sequence $\{\alpha_n\}_1^l$ satisfies $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant 0$,

$$\begin{aligned}[\![\textstyle\sum_1^l \alpha_n x_n]\!] &= \textstyle\sum_1^l a_n\alpha_n = \sum_1^n a_i\alpha_i + \sum_{n+1}^l a_i\alpha_i \\ &\leqslant \alpha_1 n^B + \left(\textstyle\sum_{n+1}^l m^{(B-1)p'}\right)^{1/p'} \left(\textstyle\sum_{n+1}^l |\alpha_i|^p\right)^{1/p} \\ &\leqslant \left(\textstyle\sum_1^l |\alpha_n|^p\right)^{1/p} \left(l^{B/j} + \left(\textstyle\sum_1^\infty m^{(B-1)p'}\right)^{1/p}\right)\end{aligned}$$

which completes the proof since $B/j < \delta$ and $(B-1)p' < -1$.

The results of Figiel, Lindenstrauss and Milman [8] will be applied next. We need some notation, which generally follows that of [8]. Let $|\cdot|$ be the l_2-norm and let $\{u_k\}$ be the usual basis of l_2. Let $E_k = [u_i]_1^k$, $S_{k-1} = \{x \in E_k: |x| = 1\}$ and μ_{k-1} be the normalized rotational invariant measure on S_{k-1}. For any norm $\|\cdot\|$, defined on span$\{u_k\}$, let $b(k)$, $M(k)$ be defined so that:

(1) $b(k)$ is the smallest number with $\|x\| \leqslant b(k)|x|$, for $x \in E_k$, and

$$\text{(2)}\qquad \begin{aligned}\mu_{k-1}\{x \in S_{k-1}: \|x\| \geqslant M(k)\} \geqslant 1/2,\\ \mu_{k-1}\{x \in S_{k-1}: \|x\| \leqslant M(k)\} \geqslant 1/2.\end{aligned}$$

If $\|\cdot\|$ is the l_p-norm, $1 \leqslant p < \infty$, Example 3.1 of [8] shows that:

(3) $b(k) = k^\alpha$, where $\alpha = p^{-1} - 2^{-1}$ for $p \leqslant 2$ and $\alpha = 0$ for $p \geqslant 2$, and

(4) $M(k) \sim k^\alpha$, where $\alpha = p^{-1} - 2^{-1}$.

Formula (4), for $p \geqslant 2$, is not explicitly stated; but it follows easily from the fact that the estimates in Example 3.1 of [8] are best possible. Suppose $\|\cdot\|$ is such that $\{u_k\}$ has finite index p. Since $\|\cdot\|_1 \leqslant K\|\cdot\|_2$ implies that $b_1(k) \leqslant Kb_2(k)$ and $M_1(k) \leqslant KM_2(k)$, for each $\delta > 0$ there are constants A and B so that

(5) $A^{-1}k^{\alpha-\delta} \leqslant M(k) \leqslant Ak^{\alpha+\delta}$, where $\alpha = p^{-1} - 2^{-1}$, and

(6) $B^{-1}k^\beta \leqslant b(k) \leqslant Bk^\gamma$, where $\beta = \gamma = 0$ $(p > 2)$ and $\gamma = p^{-1} - 2^{-1} + \delta$, $\beta = \gamma - 2\delta$ $(p \leqslant 2)$.

Thus (5), (6) and an application of Theorem 2.6 of [8] yields:

PROPOSITION 2.7. *If* $\|\cdot\|$ *is a norm so that* $\{u_k\}$ *has index* $p < \infty$ *and* $\varepsilon > 0$, *then there is a constant* C *so that* E_k *has an* n-*dimensional subspace* Y *with* $d(Y, l_2^n) \leqslant 2$, *where* $n = Ck^\alpha$ *and* $\alpha = 1 - \varepsilon$ *for* $p \leqslant 2$ *and* $\alpha = (2/p) - \varepsilon$ *for* $p \geqslant 2$.

The following proposition is needed for the proof of Theorem 4.1.

PROPOSITION 2.8. *For each* τ, $1 \leqslant \tau < 2$, *there is a constant* K, *so that if* $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \ldots \leqslant \|\cdot\|_N$ *are norms on* span$\{u_k\}$ *and* $\{u_k\}$ *has index* $p(i) < \infty$ *in the norm* $\|\cdot\|_i$, $1 \leqslant i \leqslant N$, *and if* $p(1)/p(N) \leqslant \tau$, *then*

(7) *for each* $M > 0$, *there is* $\{y_j\}_1^n \subset$ span$\{u_k\}$ *with* $\{y_j\}_1^n$ K-*basic with respect to* $\|\cdot\|_i$, $1 \leqslant i \leqslant N$, *with the further property that there is a scalar sequence* $\{\alpha_j\}_1^n$ *and* $F \subset \{1, 2, \ldots, n\}$ *so that* $\left\|\sum \alpha_j y_j\right\|_N = 1$, *but* $\left\|\sum_{j\in F} \alpha_j y_j\right\|_1 > M$.

Proof. Since $\tau < 2$, choose β so that $1 > 2\beta > \tau\beta > \tau - 1$. Then there is $\delta > 0$ so that

(8) $(1/p(1)) - (1/p(N)) + 2\beta/p(1) > 3\delta$ if $p(1) \geqslant 2$, or

(9) $(1/p(1)) - (1/p(N)) + \beta > 3\delta$ if $p(1) \leqslant 2$.

To see (9), use $p(1) \leqslant \tau$ and for $p(1) \geqslant \tau$ use $p(1) \leqslant \tau p(N)$.

Since $\beta < 1/2$, by [19, Lemma 3.3, p. 86], there are positive constants T, S and bases $\{y_j\}_1^n$ for l_2^n so that the basis constant of $\{y_j\}_1^n$ is $\leqslant T$ but there are scalars $\{\alpha_j\}_1^n$ and $F \subset \{1, 2, \ldots, n\}$ so that

$$\text{(10)}\qquad \left|\sum \alpha_j y_j\right| = 1 \quad\text{and}\quad \left|\sum_{j\in F} \alpha_j y_j\right| \geqslant Sn^\beta.$$

By the remarks at the end of Section 7 of [8], there is a constant C so that if

$$n = Ck \min_{1 \leqslant i \leqslant N} \left(M_i(k)/b_i(k)\right)^2,$$

then there is an n-dimensional subspace Y of E_k with:

$$\text{(11)}\qquad 2^{-1}|x| \leqslant M_i^{-1}(k)\|x\|_i \leqslant 2|x| \quad\text{for}\quad x \in Y,\ 1 \leqslant i \leqslant N.$$

From Lemma 2.1 (ii), $p(1) \geqslant p(2) \geqslant \ldots \geqslant p(N)$, thus (from (5) and (6)) the minimum is attained at $i = 1$. That is $n = Ck^{\gamma-\delta}$, where $\gamma = 1$, $p(1) \leqslant 2$ or $\gamma = 2/p(1)$ for $p(1) \geqslant 2$.

Let $\{y_j\}_1^n$ be the basis for Y given above. From (11), $\{y_j\}_1^n$ is $4T$-basic with respect to $\|\cdot\|_i$, $1 \leqslant i \leqslant N$. Now, from (10) and (11)

$$\left\|\sum_{j\in F} \alpha_j y_j\right\|_1 \Big/ \left\|\sum \alpha_j y_j\right\|_N \geqslant (1/4) M_1(k) M_N^{-1}(k) Sn^\beta.$$

From (5) and substituting for n: $\geqslant$ (constant) k^ν, where

$$\nu = (1/p(1)) - (1/2) - \delta - \left((1/p(N)) - (1/2) + \delta\right) + \beta(\gamma - \delta).$$

Since $\beta < 1/2$, and by the way γ is defined, either (8) or (9) imply $\nu > 0$. Thus for large enough k, this number is $> M$. This completes the proof.

Our final result of this section is an improvement on Proposition 2.8 when each of the norms is 1-symmetric on $\{u_k\}$. The proposition and its proof use the notation defined after Lemma 2.1.

PROPOSITION 2.9. *Let $\tau = 9/8$; there is a constant K so that if $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \ldots \leqslant \|\cdot\|_N$ are each concave 1-symmetric norms on $\{u_k\}$ and $I_N \leqslant I_\infty$ with $I_\infty/I_1 \leqslant \tau$ and if $\|\cdot\|_N$ satisfies the property:*

(12) *for each $B > I_\infty$ and positive integers j and m, there is an $n \geqslant M$ so that $\xi^N(m) \leqslant B$, for $n \leqslant m \leqslant n^j$.*

Then the conclusion of Proposition 2.8 (i.e. (7)) holds.

Proof. We will show how to modify the proof of Proposition 2.8 to handle this case as well. First note, by Lemma 2.4, the lower estimates on $M_i(k)$ and $b_i(k)$ in (5) and (6) are still valid with $p(i) = 1/I_i$. (Note that $I_1 \neq 0$ by hypothesis.) We now use Lemma 2.6 and (12) to provide a type of upper estimate on $M_N(k)$ and $b_N(k)$ (and hence on each $M_i(k)$ and $b_i(k)$).

For each $B > I_\infty$ and $\delta > 0$, there are infinitely many k so that $\|y\|_N \leqslant Ak^\delta \|y\|_{1/B}$, for $y \in [u_i]_1^k$, where $\|\cdot\|_{1/B}$ is the $l_{1/B}$-norm. Thus for infinitely many k, $M_N(k) \leqslant Ak^\delta M(k)$ and $b_N(k) \leqslant Ak^\delta b(k)$, where $M(k)$ and $b(k)$ are those given for the $l_{1/B}$-norm in (3) and (4). Therefore, for each $\delta > 0$, there is a constant A so that the following estimates hold for infinitely many k:

$$\begin{aligned} &A^{-1}k^r \leqslant M_1(k), && \text{where} \quad r = I_1 - (1/2) - \delta; \\ (13) \qquad &M_N(k) \leqslant Ak^s, && \text{where} \quad s = I_\infty - (1/2) + \delta; \\ &b_N(k) \leqslant Ak^t, && \text{where} \quad t = \delta \text{ if } I_\infty \leqslant 1/2 \quad \text{and} \\ & && \qquad\qquad t = s \text{ if } I_\infty \geqslant 1/2. \end{aligned}$$

Proceeding as in the proof of Proposition 2.8, for infinitely many k, we obtain an n-dimensional subspace Y of $[u_i]_1^k$ with

$$n \geqslant Ck \min_{1 \leqslant i \leqslant N} (M_i(k)/b_i(k))^2 \geqslant Ck (M_1(k)/b_N(k))^2.$$

If we let $\beta = 1/4$, then $\delta > 0$ can be chosen so that

$$I_1 - I_\infty + 2\beta I_1 > 4\delta \quad \text{if} \quad I_\infty \leqslant 1/2,$$

or

$$(1+2\beta)(I_1 - I_\infty) + \beta > 4\delta \quad \text{if} \quad I_\infty \geqslant 1/2.$$

Thus substitution of the estimates in (13) into the proof of Proposition 2.8 again yields that $\nu > 0$ and so the conclusion holds.

3. Banach and Fréchet S-algebras. If $\{\alpha_n\}$ and $\{\beta_n\}$ are scalar sequences, define $\{\alpha_n\} * \{\beta_n\}$, the *cut-Cauchy product* of $\{\alpha_n\}$ and $\{\beta_n\}$, to be the sequence

$\{\gamma_n\}$, where $\gamma_n = \alpha_i \beta_j$ if $n = 2^{-1}(i+j)(i+j-1)+i$. The name cut-Cauchy comes from the fact that $\{\gamma_n\}$ are the coefficients of the formal power series $(\sum \alpha_i x^i)(\sum \beta_j y^j)$ arranged in increasing degree with terms of the same degree arranged in increasing degree in x. The fact that $*$ is non-associative is not important for the sequel.

A symmetric basis $\{x_n\}$ of the Banach space X with 1-symmetric norm $\|\cdot\|$ is a *Banach S-algebra* and the norm is an *S-algebra norm* if $x = \sum \alpha_n x_n$ and $y = \sum \beta_n x_n$ are elements of X and if $\{\gamma_n\} = \{\alpha_n\} * \{\beta_n\}$ imply that $x * y = \sum \gamma_n x_n \in X$ and $\|x * y\| \leqslant \|x\| \cdot \|y\|$. That is, X is a Banach algebra with respect to the cut-Cauchy product. Although $*$ is non-commutative, we have $\|x * y\| = \|y * x\|$ because of the symmetric norm. This fact and its variations will be used in the sequel without further reference.

If $\{x_n\}$ is a symmetric basis for X, $\alpha = \sum \alpha_n x_n \in X$ and $k\colon N \times N \to N$ is a 1-1 function, let $u_n^\alpha = \sum_i \alpha_i x_{k(n,i)}$, $n = 1, 2, \ldots$ The sequence $\{u_n^\alpha\}$ was called the basic sequence generated by α in [1]. If $\alpha \neq 0$, it is easy to see that $\{u_n^\alpha\}$ is a symmetric basic sequence of X which dominates $\{x_n\}$.

PROPOSITION 3.1. *The symmetric basis x_n has an equivalent S-algebra norm if and only if, for each non-zero $\alpha \in \{x_n\}$, $\{u_n^\alpha\}$ is equivalent to $\{x_n\}$.*

Proof. If $\|\cdot\|$ is an S-algebra norm on $[x_n]$ and $\beta = \sum \beta_n x_n$, then

$$\|\sum \beta_n u_n^\alpha\| = \|\alpha * \beta\| \leqslant \|\alpha\| \cdot \|\beta\|.$$

Thus $\{x_n\}$ dominates $\{u_n^\alpha\}$ and so they are equivalent.

Conversely, it suffices to show that the cut-Cauchy product on $[x_n] \oplus [x_n]$ is continuous in each variable separately ([16], p. 5). This is a standard consequence of the equivalence of basic sequences ([10], p. 13).

THEOREM 3.2. *Banach S-algebras have index.*

Proof. Let $\|\cdot\|$ be an S-algebra norm on $\{x_n\}$. It follows that

$$\lambda(nk) = \|(\textstyle\sum_1^n x_i) * (\sum_1^k x_i)\| \leqslant \lambda(n)\lambda(k).$$

First, we show that for $0 \leqslant s \leqslant 1$, either $\inf \lambda(n) n^{-s} \geqslant 1$ or $\lim \lambda(n) n^{-s} = 0$. To see this, suppose for some m and s, $\lambda(m) < m^s$. There is a $t < s$ with $\lambda(m) = m^t$, and by induction, for $n = m^1, m^2, \ldots$ we have $\lambda(n) \leqslant n^t$. Hence for each n, $\lambda(n) \leqslant Kn^t$, where $K = m^t$. Therefore $\lambda(n) n^{-s} \leqslant Kn^{t-s}$ and $\lim \lambda(n) n^{-s} = 0$.

Now let $S = \inf\{s\colon \inf \lambda(n) n^{-s} \geqslant 1\}$ and let $p = 1/S$. Thus for $s > S$, $\lambda(n) \geqslant n^s$ and for $t < S$ there is a constant K with $\lambda(n) \leqslant Kn^t$. Thus by Lemma 2.2 and Proposition 2.5, $\{x_n\}$ has index p.

Remark. The lower inequality, $\|\sum \alpha_n x_n\| \geqslant (\sum |\alpha_n|^p)^{1/p}$, mentioned in the introduction, can be obtained from the proof of the Proposition in [1].

EXAMPLES of Banach S-algebras.

1. The norms of the spaces l_p, $1 \leqslant p < \infty$, and c_0 are S-algebra norms for their usual basis. In fact, for any of these norms, $\|\alpha * \beta\| = \|\alpha\| \cdot \|\beta\|$.

2. The Lorentz sequence space, $d(a_n, p)$, can be renormed to be an S-algebra if and only if there is a $K > 0$ with $s_{nk} \leqslant K s_n s_k$, for all n and k, where $s_n = \sum_1^n a_k$ [2].

3. There are conditions on $\{a_n\}$ which imply that $[f_n]$ has an equivalent S-algebra norm, where $\{f_n\}$ are the coefficient functionals of the usual basis of $d(a_n, p)$ (see [6]).

4. For each non-decreasing sequence $\{p(k)\}$ with $1 \leqslant p(n) \leqslant \infty$ we construct the S-algebra $S(p(k))$. Let X be the c_0-sum of $l^1_{p(1)} \oplus l^2_{p(2)} \oplus \ldots$ and let $\{x_i\}$ be the usual basis for X. Define $S(p(k))$ to be $\operatorname{sym}(X)$, the symmetrization of X, that is $\{\alpha_i\} \in \operatorname{sym}(X)$ if the norm

$$|\alpha_i| = \sup\{\|\sum \alpha_i x_{\pi(i)}\|_X\colon\ \pi \text{ a permutation}\} < \infty.$$

It is easy to check that $|\cdot|$ is an S-algebra norm. Let $p(\infty) = \lim_k p(k)$; it is easy to show that $S(p(k))$ has index $p(\infty)$.

5. Consider the following special cases of 2 above. Let $q < \infty$ be given and let $a_k = k^{1/q} - (k-1)^{1/q}$. The Lorentz sequence space $d(a_k, 1)$ can be renormed to be an S-algebra with index q. The space $d(a_k, 1)$ has the property, that for $\alpha = (\sum_1^m x_i)/\lambda(m)$ and scalars β_i, then $\|\sum \beta_i x_i\| = \|\sum \beta_i u_i^\alpha\|$.

6. There is no relationship between Banach spaces with given index and reflexivity. Indeed, the spaces $d(a_k, 1)$ above are non-reflexive spaces with index q. Let $b_n = \log_2(n+1) - \log_2 n$; then $d(b_n, 2)$ has index ∞, is reflexive and can be renormed to be an S-algebra. Thus the coefficient functionals of $d(b_n, 2)$ span a reflexive space with index 1 by Lemma 2.1. However, if $\{x_n\}$ has index q and if $\{x_n\}$ is super-reflexive, then $1 < q < \infty$ by [9].

7. Each example in 6 can be renormed to be an S-algebra with the exception of the dual of $d(b_n, 2)$. Indeed, by the Remark after Theorem 3.2, the only S-algebra with index 1 is l_1. Thus a reflexive S-algebra has index q with $1 < q \leqslant \infty$.

Proposition 3.1 suggests the following definition for a Fréchet S-algebra. The Fréchet space X with normalized symmetric basis $\{x_n\}$ is said to be a *Fréchet S-algebra* if for each non-zero $\alpha \in [x_n]$, $\{u_n^\alpha\}$ is equivalent to $\{x_n\}$. (The requirement that $\{x_n\}$ be normalized rules out exactly the space ω — the product of countably many copies of the scalars.) If, in the proof of Proposition 3.1, we use [20], p. 354, instead of [16], we obtain the first statement in the following proposition.

Proposition 3.3. *The normalized symmetric basis $\{x_n\}$ of the Fréchet space X spans a Fréchet S-algebra if and only if the cut-Cauchy product is a jointly continuous bilinear map $X \oplus X \to X$, and if and only if the topology on X can be defined by a sequence of norms $\{\|\cdot\|_k\}$ so that each $\|\cdot\|_k$ is 1-symmetric on $\{x_n\}$ and for each k and $\alpha, \beta \in [x_n]$, we have*

$$(\#) \qquad \|\alpha * \beta\|_k \leqslant \|\alpha\|_{k+1} \cdot \|\beta\|_{k+1}.$$

Proof. We complete the proof by showing how to renorm to obtain $(\#)$. Since $\{x_n\}$ is a normalized symmetric basis, the topology can be defined by an increasing sequence of norms $\{|\cdot|_k\}$ with each $|\cdot|_k$ 1-symmetric on $\{x_n\}$. Since the cut-Cauchy product is continuous, by passing to a subsequence if necessary, there are constants C_k so that

$$|\alpha * \beta|_k \leqslant C_k |\alpha|_{k+1} \cdot |\beta|_{k+1}.$$

Let $\|\cdot\|_k = |\cdot|_1$ and inductively construct $\|\cdot\|_{k+1}$ as follows:

$$\|\alpha\|_{k+1} = \sup\{|\alpha|_{k+1}\} \cup \{\|\alpha * \beta\|_k\colon\ |\beta|_{k+1} \leqslant 1\}.$$

It follows that each $\|\cdot\|_k$ is 1-symmetric on $\{x_n\}$ and $|\cdot|_k \leqslant \|\cdot\|_k$. By induction for $k \geqslant 2$, $\|\cdot\|_k \leqslant (\prod_{i=1}^{k-1} C_k)|\cdot|_k$ and $(\#)$ holds. Finally, since $\|\cdot\|_k$ is equivalent to $|\cdot|_k$, for each k, the norms $\{\|\cdot\|_k\}$ define the topology.

Examples of Fréchet S-algebras.

1. If $\{\|\cdot\|_k\}$ is an increasing sequence of S-algebra norms on $\{x_n\}$, then $\{\|\cdot\|_k\}$ generates a Fréchet topology in which $\{x_n\}$ is a Fréchet S-algebra. Each of these examples is an indexable Fréchet S-algebra, in the sense that the topology can be defined by a sequence of norms $\{\|\cdot\|_k\}$ in which $\{x_n\}$ has index $p(k)$. Let $q = \lim p(k)$, either for some k, $p(k) = q$, or the space is isomorphic to $\bigcup_{p>q} l_p$.

2. If $\|\cdot\|_k$ are Lorentz sequence space norms of $d(a_n^k, 1)$ with the property that for each k, there is a constant K and integer j so that $s_{mn}^k \leqslant K s_m^j s_n^j$, then $\{\|\cdot\|_k\}$ generate a Fréchet S-algebra. This can be proved by imitating the proof in [2] used in Example 2 after Theorem 3.2.

3. We now construct a particular example of 2 which will turn out to be an unindexable Fréchet S-algebra. We inductively (on j) construct $\{s_n^j\}_n$ so that $I_j = 0.3$ and $S_j = 0.5$ for each j; this will show that the space is unindexable.

Inductively pick integers $\{M(i)\}$ and $\{N(i)\}$ and the sequence s_n^1 with the following properties:

(1) $M(i) < N(i) < N(i)^{N(i)} < M(i+1)$;

(2) $n^{0.3} \leqslant s_n^1$, for all n;

(3) $s_{m(i)}^1 = (M(i))^{0.5}$, for each i; and

(4) $s_m^1 = n^{0.3}$, for $N(i) \leqslant n \leqslant N(i)^{N(i)}$ and each i.

Finally, we use the following lemma with $s_n = s_n^j$ to choose $s_n^{j+1} = v_n$ to complete the example.

Lemma 3.4. *Suppose $\{s_n\}$ is the $\{\lambda(n)\}$-sequence for a Lorentz sequence space. Let $t_0 = 0$, $t_1 = 1$ and inductively define $\{t_m\}$ by*

$$t_m = \max(\{s_{mj}/t_j\colon\ 1 \leqslant j \leqslant m-1\} \cup \{(s_{m^2})^{1/2}\});$$

and let

$$v_m = \sup\{\sum_{j=1}^{m}(t_{n(j)+1}-t_{n(j)}):\ 0 \leqslant n(1) < n(2) < \ldots < n(j) < \ldots\}.$$

Then $\{v_m\}$ *is the* $\{\lambda(n)\}$*-sequence for a Lorentz sequence space with* $s_m \leqslant v_m$ *and* $s_{mn} \leqslant v_m v_n$.

Furthermore if $s_n \geqslant n^{0.3}$ *and if for some constant* K *and integers* m *and* j, $s_n \leqslant Kn^{0.3}$ *for* $m \leqslant n \leqslant m^{2j}$, *then* $v_n \leqslant 3Kn^{0.3}$, *for* $m \leqslant n \leqslant m^j$.

Proof. By hypothesis, if $n \leqslant m$ then both $s_m/m \leqslant s_n/n$ and $s_m - s_n \leqslant (m-n)s_n/n$. By considering both possibilities of $t_m = s_{mj}/t_j$ or $t_m = (s_{m^2})^{1/2}$, it follows that if $n \leqslant m$, then $t_m/m \leqslant t_n/n$ and $t_{m+1}-t_m \leqslant 2t_n/n$. By definition of $\{t_n\}$ we have $s_{mn} \leqslant t_m t_n$. From the proof of Proposition 3.a.7 ([11], p. 119) it follows that $t_n \leqslant v_n \leqslant 3t_n$ and that $\{v_n\}$ is the $\{\lambda(n)\}$-sequence of a Lorentz sequence space. Finally, if $m \leqslant n \leqslant m^j$, then $m \leqslant ni \leqslant m^{2j}$ for $i = 1, \ldots, n$. Thus $t_n \leqslant K(n_i)^{0.3}/t_i \leqslant Kn^{0.3}$ or $(t_n)^2 \leqslant s_{n^2} \leqslant K(n^2)^{0.3}$ and therefore, if $m \leqslant n \leqslant m^j$, $v_n \leqslant 3Kn^{0.3}$. This completes the lemma and the example.

We turn our attention to S-algebras with index ∞. Proposition 3.6 provides a strong local condition that such spaces possess. The corresponding result for S-algebras with index q, $1 < q < \infty$, is false, as the space $d(a_k, 1)$ in Example 5 after Theorem 3.2 shows.

We need to define a rather strange collection of measure spaces. On the set $\{1, 2, \ldots, n\}$, let μ_n be the probability measure with

$$\mu_n(i) = (\log(i+1)-\log(i))/\log(n+1).$$

As before, if $\|\cdot\|$ is a 1-symmetric norm on $[x_n]$, let $\lambda(n) = \|\sum_1^n x_i\|$.

Lemma 3.5. *If* $\|\cdot\|$ *is an* S*-algebra norm on* $\{x_n\}$ *with infinite index, then for* $K > 1$ *and each integer* k, *the sets*

$$A(n) = \{m \leqslant n:\ \lambda(km) \geqslant K\lambda(m)\}$$

satisfy $\lim_n \mu_n(A(n)) = 0$.

Proof. Let $m(0) = 0$ and inductively choose $m(i+1)$ to be the first integer $\geqslant km(i)$ with $\lambda(km(i+1)) \geqslant K\lambda(m(i+1))$. If the induction terminates after a finite number of steps, then we are done. Otherwise, $\lambda(m(i+1)) \geqslant \lambda(km(i)) \geqslant K\lambda(m(i))$ and by induction $\lambda(m(i+1)) \geqslant K^i\lambda(m(1))$.

Since our space is of infinite index, for each $p < \infty$, $\lim_i \lambda(m(i))(m(i))^{-1/p} = 0$. Thus $\lim_i (K^p)^i(m(i))^{-1} = 0$. Since $K > 1$, we can choose $p < \infty$ to make K^p arbitrarily large and therefore $\lim_i i/\log(m(i)) = 0$.

Let n be given and choose i so that $m(i) \leqslant n < m(i+1)$. All the elements of $A(n)$ are between $m(j)$ and $km(j)$ for some $j \leqslant i$. Therefore

$$\mu_n(A(n)) \leqslant \sum_{j=1}^{i}(\log(1+km(j))-\log m(j)/\log(n+1))$$
$$\leqslant i(\log(k+1)/\log m(i)).$$

Since $\lim_i i\log(k+1)/\log m(i) = 0$, we are done.

Proposition 3.6. *Suppose* $\{\|\cdot\|_i\}_{i=1}^N$ *are each infinite-index* S*-algebra norms on* $\{x_n\}$; *then for each integer* k *and positive* ε, *there is an infinite subset of the integers* M *so that* $m \in M$, $1 \leqslant j \leqslant N$, *implies that if*

$$u_i^j = (x_{(i-1)m+1} + \ldots + x_{im})/\lambda^j(m), \qquad 1 \leqslant i \leqslant k,$$

then $(\{u_i^j\}_{i=1}^k, \|\cdot\|_j)$ *is* $(1+\varepsilon)$*-equivalent to the usual basis of* c_0^k.

Proof. For $\{\alpha_n\}_1^k$, let $\|\alpha_n\|_0 = \max_{1 \leqslant i \leqslant k}|\alpha_i|$ and suppose $\lambda^j(km) \leqslant (1+\varepsilon)\lambda^j(m)$; then

$$\|\alpha_n\|_0 \leqslant \|\sum_{i=1}^k \alpha_i u_i^j\|_j \leqslant \|\alpha_n\|_0 \|\sum_{i=1}^k u_i^j\|_j \leqslant \|\alpha_n\|_0(1+\varepsilon).$$

Thus, to prove the proposition, it suffices to show that for each L there is $m \geqslant L$ with $m \notin \bigcup_{j=1}^N A^j(n)$, where $A^j(n)$ is the set in Lemma 3.5 with $K = 1+\varepsilon$ and $\|\cdot\| = \|\cdot\|_j$. But this is a consequence of Lemma 3.5 for large enough n.

4. Non-Montel Fréchet spaces. Here we show that a large class of non-Montel Fréchet spaces have conditional basic sequences. To understand the scope of these results, some definitions are needed. A *prevariety* [3] is a collection of locally convex spaces, closed with respect to the formation of subspaces and arbitrary products. If B is a collection of Banach spaces, then $\varrho v(B)$ will denote the smallest prevariety containing B. A Fréchet space E is in $\varrho v(B)$ if and only if E is isomorphic to a subspace of a countable product of spaces in B ([7], Th. 4.1).

The Fréchet space E is said to have *property* ∞ if for each collection of continuous semi-norms A which define the topology on E, there is $\|\cdot\| \in A$ and a sequence $\{x_n\} \subset E$ with

(i) $\{\|x_n\|\}$ is bounded above and below; and

(ii) for each $p < \infty$ there is $\{\alpha_n\} \notin l_p$ so that $\sum \alpha_n x_n$ is unconditionally summable with respect to $\|\cdot\|$.

Note that if $X_1, \ldots, X_N$ each fail to have property ∞, then $X_1 \oplus \ldots \oplus X_N$ fails to have property ∞. Thus each Fréchet space in the collection $Q = \varrho v\{X:\ X$ is a Banach space which fails to have property $\infty\}$ does not possess property ∞. This collection Q is quite large, in fact it contains each of the following collections:

(1) Subspaces of l_p-Köthe sequence spaces, $p < \infty$.

(2) $\varrho v(X)$, where X is a super-reflexive Banach space, or X is a collection of such spaces (use [10], p. 130 and [15]).

(3) $\varrho v(X)$, where X is the collection of Banach spaces with finite cotype (see [8] for definition).

Finally, we come to Theorem 4.1, our main result. The proof of the theorem is partitioned into a collection of propositions and lemmas.

Theorem 4.1. *Each non-Montel Fréchet space has a conditional basic sequence or has property* ∞.

PROPOSITION 4.2. *Each non-Montel Fréchet space contains a conditional basic sequence or a normalized basic sequence spanning a Fréchet S-algebra.*

Proof. Let E be a non-Montel Fréchet space. By [12], Th. 3.5, E has a normalized basic sequence $\{x_n\}$. Let $X = [x_n]$. If $\{x_n\}$ is conditional we are done, otherwise X has a sequence of norms $\{\|\cdot\|_k\}$ which define the topology on X and which satisfy

(a) $\|x_n\|_k = 1$, for each n and k;

(b$_1$) $\{x_n\}$ is 1-unconditional with respect to $\|\cdot\|_k$, for each k; and

(c) $\|\cdot\|_k \leqslant \|\cdot\|_{k+1}$ on X, for each k.

This result is well-known for Banach spaces (i.e., see [10], p. 16), a similar proof works in Fréchet spaces. (To see (a), note that if $\{x_n\}$ is a normalized unconditional basis in $\|\cdot\|$, then

$$|||\sum \alpha_n x_n||| = \|\sum \alpha_n \|x_n\|^{-1} x_n\|$$

is an equivalent norm which satisfies (a).)

Next, we use the ideas of [14]. The statement A below is proved for Banach spaces in the proof of Proposition 3 of [14]. The proof can be generalized to prove statement A. (The fact that the basic sequence is normalized is essential.)

A. If $\{e_n\}$ is a normalized basic sequence in a Fréchet space and $\{y_n\}$ and $\{z_n\}$ are defined by:

$$y_{2n-1} = e_{2n-1}, \quad y_{2n} = e_{2n-1} + e_{2n}, \qquad \text{and} \qquad z_{2n-1} = e_{2n-1} + e_{2n}, \quad z_{2n} = e_{2n},$$

then $\{y_n\}$ and $\{z_n\}$ are basic sequences. Furthermore, if $\{y_n\}$ and $\{z_n\}$ are unconditional, then $\{e_{2n-1}\}_n$ and $\{e_{2n}\}_n$ are equivalent basic sequences.

Thus, Proposition 3 of [14] implies that X has a conditional basic sequence or (with possible renorming) (b$_1$) can be strengthened to:

(b$_2$) x_n is 1-symmetric with respect to $\|\cdot\|_k$, for each k.

Since for each non-zero $\alpha = \sum \alpha_i x_i \in X$, $\{u_n^\alpha\}$ is normalized, we can use the statement A. Therefore either X has a conditional basic sequence or by the comments before Proposition 3.3, (b$_2$) can be replaced with:

(b$_3$) $\{x_n\}$ spans a Fréchet S-algebra.

This completes the proof of the proposition.

LEMMA 4.3. *If $\{\|\cdot\|_k\}$ is an increasing sequence of norms each 1-symmetric on $\{x_n\}$, then either some $I_k > 0$ or $[x_n]$ in the topology generated by $\{\|\cdot\|_k\}$ has property ∞.*

Proof. Suppose $I_k \equiv 0$ and A be any collection of continuous semi-norms on $[x_n]$ which define the topology. Choose $\|\cdot\| \in A$ and an integer k so that there are constants B and C with

$$(*) \qquad \|y\|_1 \leqslant B\|y\| \leqslant C\|y\|_k, \quad \text{for} \quad y \in [x_n].$$

By $(*)$, $\{x_n\}$ is normalized in $\|\cdot\|$, and if there is a $p < \infty$ so that $\sum \alpha_n x_n$ unconditionally summable in $\|\cdot\|$ implies $\{\alpha_n\} \in l_p$, then $\sum \alpha_n x_n$ unconditionally summable in $\|\cdot\|_k$ also implies $\{\alpha_n\} \in l_p$. But this implies that the obvious map: $([x_n], \|\cdot\|_k) \to l_p$ is continuous and hence $\lambda^k(n) \geqslant Kn^{1/p}$. Thus Lemma 2.2 implies $I_k > 0$, a contradiction. Therefore $[x_n]$ has property ∞ and the proof is complete.

PROPOSITION 4.4. *An indexable Fréchet S-algebra has property ∞ or a conditional basic sequence.*

Proof. Let $\{x_n\}$ be the basis of the Fréchet S-algebra and let $\{\|\cdot\|_k\}$ be an increasing sequence of norms which define the topology and so that $\{x_n\}$ has index $p(k)$ in $\|\cdot\|_k$. By Lemma 4.3, for some k, $p(k) < \infty$. Since $p(k)$ is non-increasing, $q = \lim_k p(k)$ exists and $q < \infty$. By passing to a tail end of $\{\|\cdot\|_k\}$, we may assume $p(1) < \tau q$, where $\tau < 2$. Define, for $y \in \operatorname{span}\{x_n\}$, $\operatorname{support}(y) = \{m\colon y = \sum \alpha_n x_n \text{ and } \alpha_m \neq 0\}$.

Using Proposition 2.8, inductively define integers $N(0) = 0 < N(1) < \ldots$ $\ldots < N(k) < N(k+1) < \ldots$ and $\{y_n\} \subset \operatorname{span}\{x_n\}$ so that

(d) $\{y_n\colon N(k-1) < n \leqslant N(k)\}$ is K-basic with respect to $\|\cdot\|_j$, for $j \leqslant k$.

(e) If $m > N(k)$, then $y_m \in \operatorname{span}\{x_n\colon n > M\}$, where

$$M = \max \bigcup \{\operatorname{support}(y_n)\colon n \leqslant N(k)\}.$$

(f) There are scalars

$$\{\alpha_n\colon N(k-1) < n < N(k)\} \quad \text{and} \quad F(k) \subset \{N(k-1)+1, \ldots, N(k)\}$$

so that

$$\Big\|\sum_{i=N(k-1)+1}^{N(k)} \alpha_i y_i\Big\|_k = 1 \quad \text{but} \quad \Big\|\sum_{i\in F(k)} \alpha_i y_i\Big\|_1 \geqslant k.$$

Conditions (d) and (e) imply that $\{y_n\colon n > N(k-1)\}$ is K-basic with respect to $\|\cdot\|_k$. Since $\{y_n\}$ is independent, $\{y_n\}$ is basic in X. Condition (f) implies that $\{y_n\}$ is conditional (see Section 1). This completes Proposition 4.4.

Remarks. Two partial results for the case $\{\|\cdot\|_k\}$ are S-algebra norms on $\{x_n\}$, each with index ∞.

A. If for some k, $\sup_m \lambda^j(m)/\lambda^k(m)$ is bounded for each j, then $[x_n]$ has conditional basic sequences. The construction of the conditional basic sequence is similar to that of Theorem 4.4. Only Proposition 3.6 is used in place of Proposition 2.8.

B. If for some $\alpha > 1$, and some subsequence of integers $\{m(i)\}$, and for each k, there is a $j \geqslant k$ with

$$\inf_i \lambda^j(m(i))/\lambda^k([m(i)]^\alpha) > 0,$$

then $[x_n]$ has conditional basic sequences. Assume $[m(i)]^{\alpha-1} = k(i)$ is an integer for convenience sake. The conditional basic sequence is built out of blocks of the form:

$$y_s^i = \sum_{p(i)+1}^{p(i)+sm(i)} x_n, \qquad s = 1, 2, \ldots, k(i),$$

where $p(0) = 0$ and $p(i+1) = p(i)+k(i)m(i)$.

The inf condition implies that $\{z_n\}$ is basic, where $y_s^i = z_n$ if $n = \sum_{j=1}^{i-1} k(j)+s$. The proof that $\{z_n\}$ is conditional uses both that $k(i)$ is a power of $m(i)$ and $\|\cdot\|_k$ is of infinite index.

To complete the proof of Theorem 4.1 we need only drop the condition that the Fréchet S-algebra is indexable in Proposition 4.4. Since Proposition 2.9 can be used in place of Proposition 2.8 in the proof of Proposition 4.4, it suffices to show that if $\{\|\cdot\|_k\}$ are norms which satisfy the last statement of Proposition 3.3, then each $\|\cdot\|_k$ satisfies the hypothesis of $\|\cdot\|_N$ in Proposition 2.8. Thus the following sequence of lemmas will complete the proof of Theorem 4.1.

LEMMA 4.5. *If $\|\cdot\|$ is a concave 1-symmetric norm with $\xi(n)$ and $\lambda(n)$ defined as before Lemma 2.2 and if $0 \leqslant B \leqslant A \leqslant 1$ and $\varrho = (1-B)(1-A)^{-1}$ then $\xi(n) \leqslant B$ implies $\xi(m) \leqslant A$ for $n \leqslant m \leqslant n\varrho$.*

Proof. Since $\lambda(n)$ is concave, for $m \geqslant n$, $(m, \lambda(m))$ must lie on or below the line through (1,1) and (n, n^B). Thus, if $n^{\varrho A} \geqslant (n^B-1)(n-1)^{-1}(n^\varrho - n)+n^B$, the lemma is proved. We will show $n^{\varrho A}-n^B \geqslant n^{B-1}(n^\varrho - n)$, which is a stronger statement since $n^{B-1} \geqslant (n^B-1)(n-1)^{-1}$. It suffices to show $n^{\varrho A+1-B} - n \geqslant n^\varrho - n$, which follows from the definition of ϱ.

Suppose now that $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \ldots$ are concave 1-symmetric norms with $\xi^k(n)$, $\lambda^k(n)$ and I_k defined as before Lemma 2.2. Define I_∞ to be $\lim_k I_k$ which exists since $I_k \leqslant I_{k+1}$. Suppose further that

$$(\Delta) \qquad \|x * y\|_k \leqslant \|x\|_{k+1}\|y\|_{k+1},$$

for each k and $x, y \in [x_n]$.

LEMMA 4.6. *With the above notation, for each $\delta > 0$, and integers j, k, N, there is $n \geqslant N$ so that both*

$$\xi^k(n) \leqslant I_\infty+\delta \quad \text{and} \quad \xi^k(n^j) \leqslant I_\infty+\delta.$$

Proof. Suppose not, then there are k and N so that $n \geqslant N$ and $\xi^k(n) \leqslant I_\infty+\delta$ implies $\xi^k(n^j) > I_\infty+\delta$. By induction on (Δ) it follows that $\xi^k(n^j) \leqslant \xi^{k+j-1}(n)$. But this implies for $n \geqslant N$, $\xi^{k+j}(n) > I_\infty+\delta$; that is $I_{k+j} > I_\infty$; a contradiction.

LEMMA 4.7. *With the same notation, for each $\delta > 0$ with $1-I_\infty-\delta > 0$ and integers j, k, N, there is an $n \geqslant N$ so that $\xi^k(m) \leqslant I_\infty+\delta$ for $n \leqslant m \leqslant n^j$.*

Proof. Let $\varrho = (1-I_\infty-\delta/2)(1-I_\infty-\delta)^{-1} > 1$. Choose l large enough so that $l\varrho \geqslant l+1$ and hence for each i, $(l+i)\varrho \geqslant l+i+1$. Let $B = I+\delta/2$. By Lemma 4.6, we may choose $n \geqslant N$ so that both $\xi^{k+jl-l}(n) \leqslant B$ and $\xi^{k+jl-l}(n) \leqslant B$. Again by induction on (Δ) it follows that $\xi^k(n^{l+i}) \leqslant B$ for $0 \leqslant i \leqslant jl-l$. Thus by Lemma 4.5 it follows $\xi^k(m) \leqslant I_\infty+\delta$, for $n^l \leqslant m \leqslant n^{jl}$, which completes the proof of the lemma and Theorem 4.1.

A theorem weaker than Theorem 4.1 can be proved for non-Schwartz spaces (see [5]).

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