$$\geqslant \frac{(r_1'^3 \varkappa_k)^{1/2}\sqrt{2}\lambda}{8L^2\zeta r_1'}\left(\frac{5}{4}\int_0^{\zeta}\varphi - \int_0^{\infty}\varphi\right), \quad \lambda = \frac{1}{(r_1'\varkappa_1)^{1/2}},$$

$$\geqslant \frac{\sqrt{2}}{64L^2\zeta}\int_0^{\infty}\varphi.$$

For the same reason

$$\int_{B_j^-}\int_{x-R_j}\varphi(\xi_2 - r(\xi_1))\cos((x_j-\xi)\cdot y)\,dy\,d\xi \geqslant \frac{\sqrt{2}}{64L^2\zeta}\int_{-\infty}^{0}\varphi.$$

Therefore

$$|T_j^\lambda f_j(x)| \geqslant \frac{\sqrt{2}}{64L^2\zeta}\|\varphi\|_1.$$

Then T is $L^p \to L^p$ bounded operator if and only if $p = 2$.

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Invertibility of some second order differential operators

by

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Abstract. The authors examine L^p invertibility of second order linear partial differential operators with constant coefficients. Invertibility of such operators is shown to depend upon the geometric structure of the level surfaces associated to the symbol of the operator.

The purpose of this paper is to examine the invertibility of second order linear partial differential operators with constant coefficients. In general dimension we shall treat operators with level hyper-surfaces; in two dimensions we shall give a complete classification. Following the remarks of Kenig–Tomas [2], we shall see that the invertibility of such operators depends upon the geometric structure of the level surfaces. The main tools in our approach are the Kakeya counterexample of C. Fefferman, the classical multiplier theorems of Marcinkiewicz and Hörmander, and a multiplier theorem of Littman, McCarthy and Rivière.

Let $d(x)$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, denote a second order polynomial on $\mathbf{C}$. $d(x)$ can be expressed as

$$d(x) = P(x) + iQ(x),$$
$$P(x) = X^T AX + \alpha\cdot X + t,$$
$$Q(x) = X^T BX + \beta\cdot X + s,$$

where A, B are real symmetric matrices, α, β are real vectors, t and s are real constants.

THEOREM 1. *Assume* $d(x)$, $\dfrac{1}{d(x)} \in L^\infty(\mathbf{R}^n)$, *has* $(n-1)$-*dimensional level hyper-surface. Then* $\dfrac{1}{d(x)}$ *is a Fourier multiplier of* L^p, $1 < p < \infty$, *except in the following two cases*:

(1) α *is not an eigenvector of* A.

(2) *The rank of A is at least three and the restriction of $X^T AX$ to the eigenspace is not positive definite.*

In these cases, $\frac{1}{d(x)}$ *is a Fourier multiplier of* L^p *if and only if* $p = 2$.

THEOREM 2. *Let* $n = 2$, $\frac{1}{d(x)} \in L^\infty(\mathbf{R}^2)$.

(1) *If* $d(x)$ *has level curves, then* $\frac{1}{d(x)}$ *is a Fourier multiplier of* L^p, $1 < p < \infty$, *except when the level curve is a parabola. In this case,* $\frac{1}{d(x)}$ *is a Fourier multiplier of* L^p *if and only if* $p = 2$.

(2) *If* $d(x)$ *has finite level set, then* $\frac{1}{d(x)}$ *is a Fourier multiplier of* L^p, $1 < p < \infty$.

Preliminaries.

LEMMA 1 [4] (Marcinkiewicz multiplier theorem). *Let* $m(x)$, $x = (x_1, \ldots, x_n)$, *be a bounded function on* $\mathbf{R}^n$. *Suppose for each* $0 < k \leqslant n$

$$\sup_{\{x_1,\ldots,x_n\}\setminus\{x_{i_1},\ldots,x_{i_k}\}} \int_{\varrho} \left|\frac{\partial^k m}{\partial x_{i_1}\,\partial x_{i_2}\ldots\partial x_{i_k}}\right| dx_{i_1}\ldots dx_{i_k} \leqslant B$$

as ϱ *ranges over dyadic rectangles of* $\mathbf{R}^k$, B *is a constant. Then* $m(x)$ *is a Fourier multiplier of* L^p, $1 < p < \infty$.

LEMMA 2 [4] (Hörmander multiplier theorem). *Suppose that* $m(x)$ *is of class* C^k *in the complement of the origin of* $\mathbf{R}^n$, *where* k *is an integer* $> n/2$. *Assume that for every differential monomial* $\left(\frac{\partial}{\partial x}\right)^\alpha$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, *with* $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, *we have*

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha m(x)\right| \leqslant B|x|^{-|\alpha|}, \quad \text{where} \quad |\alpha| \leqslant K,\ B \text{ is constant}.$$

Then $m(x)$ *is a Fourier multiplier of* L^p, $1 < p < \infty$.

LEMMA 3 [3]. *Let* $h(u)$ *be a complex valued function of a real variable* $u \in \mathbf{R}^1$. *Assume that*

$$\left|u^k \frac{d^k h}{du^k}(u)\right| \leqslant C_k \quad \text{for} \quad u \neq 0,\ k = 0, 1, \ldots, r.$$

Let $\{l_k(x)\}_{k=1}^r$ *be a family of affine functionals from* $\mathbf{R}^n$ *to* $\mathbf{R}^1$. *Then*

$$m(x) = h\Big(\prod_{k=1}^{r} l_r(x)\Big)$$

is a Fourier multiplier of $L^p(\mathbf{R}^n)$, $1 < p < \infty$.

LEMMA 4 [2]. *Let* $m(x, y)$ *be a Fourier multiplier for* $L^p(\mathbf{R}^{i+j})$, $x \in \mathbf{R}^i$, $y \in \mathbf{R}^j$. *Then for almost every* $x \in \mathbf{R}^i$, $m_x(y) = m(x, y)$ *is a Fourier multiplier of* $L^p(\mathbf{R}^j)$.

LEMMA 5 [2]. *If* $m(x)$, $x \in \mathbf{R}^n$, *is a Fourier multiplier for* L^p, *then both the conjugate* $\bar{m}$ *and powers of* m *are Fourier multipliers.*

LEMMA 6 [5]. *If* $P(D)$ *is a hypoelliptic pseudo-differential operator, then for every differential monomial* $\left(\frac{\partial}{\partial x}\right)^\alpha$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, *with* $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$,

$$\left|\left[\left(\frac{\partial}{\partial x}\right)^\alpha P(x)\right] \Big/ P(x)\right| \leqslant \frac{C}{(1+|x|)^{|\alpha|}} \quad \text{for} \quad |x| \text{ large},$$

where C *is a constant.*

LEMMA 7. *Assume* A *is a real symmetric matrix, and* α *is an eigenvector of* A. *There is an orthogonal matrix* M *such that* $M^T A M$ *is diagonal. Then* $M^T\alpha$ *is an eigenvector of* $M^T A M$.

Proof. Since A is symmetric, $M^T = M^{-1}$. Assume α is an eigenvector corresponding to the eigenvalue λ, i.e. $A\alpha = \lambda\alpha$. Then we have

$$(M^T A M)(M^T\alpha) = M^T A\alpha = M^T(\lambda\alpha) = \lambda(M^T\alpha).$$

LEMMA 8. *If* $d(x) = P(x) + iQ(x)$ *has* $(n-1)$*-dimensional level hyper-surface, then after a proper affine transformation and multiplication by a constant, we have either*

$$d(x) = X^T D X + r\cdot X + \delta,$$

where D *is a diagonal matrix,* $D_{ll} = 1$, $1 \leqslant l \leqslant \mu$, $D_{ll} = -1$, $\mu+1 \leqslant l \leqslant \nu$, $D_{ll} = 0$, $l > \nu$, $\nu \leqslant n$, *and*

$$r = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$\delta \in \mathbf{C}$ *or*

$$d(x) = x_1^2 + i[a(x_1 - b)^2 + s]$$

or

$$d(x) = x_1 + t + i(ax_1^2 + s).$$

Proof. Since $d(x) = P(x) + iQ(x)$ has $(n-1)$-dimensional level hyper-surface $P(x) = a$, $Q(x) = b$, we claim that $P(x)$ and $Q(x)$, as polynomials of degree 2, involve the same variables. If all $x_1, \ldots, x_n$ appear both in P and Q, the claim is obvious. Now suppose $P(x)$ involves $x_1, \ldots, x_m$, $1 \leqslant m < n$. If $P(x) = a$ at $x_1^0, \ldots, x_m^0$, then $Q(x) = b$ at $x_1^0, \ldots, x_m^0, x_{m+1}, \ldots, x_n$ for every $x_{m+1}, \ldots, x_n \in \boldsymbol{R}^{n-m}$. So $Q(x)$ involves at most $x_1, \ldots, x_m$. Symmetrically $P(x)$ and $Q(x)$ involve the same variables.

Case 1. $P(x)$ and $Q(x)$, as polynomials of degree 2, involve at least two variables, for instance $x_1, \ldots, x_m$, $2 \leqslant m \leqslant n$. In this case $\frac{\partial P(x)}{\partial x_j}$, $\frac{\partial Q(x)}{\partial x_j}$, $j = 1, 2$, are not zero in $\boldsymbol{R}^n$ away from at most four hyper-planes π_i.

Since $d(x) = P(x) + iQ(x)$ has an $(n-1)$-dimensional hyper-surface, that is, there exists a region $D \subset \boldsymbol{R}^n$ and constants a and b such that the hyper-surface $P(x) = a$ coincides with the hyper-surface $Q(x) = b$ in D and $D \cap \pi_i = \emptyset$. By the implicit function theorem, there are four functions $x_j = \varphi_{aj}(x_{j^0}, x_3, \ldots, x_n)$ and $x_j = \varphi_{bj}(x_{j^0}, x_3, \ldots, x_n)$, $j = 1, 2$, $j^0 \cdot j = 2$, as implicit functions of $P(x) = a$, $Q(x) = b$, satisfying $P(\varphi_{a1}, x_2, \ldots, x_n) = a$, $P(x_1, \varphi_{a2}, x_3, \ldots, x_n) = a$,

$$Q(\varphi_{b1}, x_2, \ldots, x_n) = b, \qquad Q(x_1, \varphi_{b2}, x_3, \ldots, x_n) = b$$

and $\varphi_{aj} = \varphi_{bj}$, $j = 1, 2$.

Therefore $\nabla P(x) = \nabla Q(x)$, i.e., $kA = B$, $k\alpha = \beta$. Then $d(x) = (k+i) \times \times [X^T AX + \alpha \cdot X] + t + is$. After a proper affine transformation and multiplication by a constant, $d(x)$ can be considered as

$$d(x) = X^T DX + r \cdot X + \delta$$

where D, r and δ are as described.

Case 2. $P(x)$ and $Q(x)$ only involve one variable, for instance, x_1. When $P(x) = x_1 + t$, we have $Q(x) = x_1 + s$ or $Q(x) = x_1^2 + s$. When $P(x) = x_1^2 + t$, we have $Q(x) = x_1 + s$ or $Q(x) = a(x_1 - b)^2 + s$.

Summarizing, by affine transformation and multiplication by a constant, $d(x)$ can be written as in Lemma 8.

Lemma 9. *When* $n = 2$, *if* $\frac{1}{d(x)} \in L^\infty(\boldsymbol{R}^2)$ *has finite level set, then either* $d(D)$ *is elliptic or after affine transformation* $d(x)$ *can be one of the following*:

$$d(x) = x_1^2 - x_2^2 + t + i[(x_1 - a)^2 - (x_2 - b)^2 + s] \qquad (|a| \neq |b|),$$

$$d(x) = x_1^2 - x_2^2 + t + i(ax_1 + s),$$

$$d(x) = (x_1 - a)^2 - x_2 + i(x_1^2 + s) \qquad (s > 0),$$

$$d(x) = x_1^2 - x_2 + i(ax_2 + s).$$

It is obvious that in the other cases $d(x)$ has zero or level curve.

Proof of Theorem 1. (1) If α is not an eigenvector of A, from Lemmas 7 and 8,

$$\nu < n, \qquad r = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and is not an eigenvector of D. Then the restriction of $\frac{1}{d(x)}$ to $R^2(x_1, x_n)$ is $\frac{1}{x_1^2 + x_n + t + is}$. By Lemma 5, if it is a multiplier of L^p, setting $\varphi(y) = \left(\frac{2u}{u^2 + s^2}\right)^2$ and $F(x) = x_1^2 + x_2 + t$, $\varphi(F(x))$ should be a multiplier of L^p. But using the result in [6], $\varphi(F(x))$ is a Fourier multiplier if and only if $p = 2$. Then so is $\frac{1}{d(x)}$.

(2) If the rank of A is at least three and the restriction of $X^T AX$ to the eigenspace is not positive definite, from Lemma 8, the restriction of $\frac{1}{d(x)}$ to $R^{n-1}(x_1, \ldots, x_{n-1})$ is $\frac{1}{X^T DX + t + is}$, $\mu \geqslant 2$, $\nu \geqslant \mu + 1$, that is,

$$\frac{1}{\sum_{j=1}^{\mu} x_j^2 - \sum_{j=\mu+1}^{\nu} x_j^2 + t + is}.$$

By Kenig–Tomas [2] this is a Fourier multiplier of $L^p(R^\nu)$ if and only if $p = 2$. From Lemma 4, so is $\frac{1}{d(x)}$.

(3) The remainder is to prove for the cases:

1. $D_{ll} = 1$, $1 \leqslant l \leqslant \mu$; $D_{ll} = 0$, $l > \mu$.
2. $D_{11} = 1$, $D_{22} = -1$; $D_{ll} = 0$, $l > 2$.
3. $D_{11} = 1$; $D_{ll} = 0$, $l > 1$.
4. $D = [0]$, $r = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.
5. $d(x) = x_1^2 + i[a(x_1 - b)^2 + s]$.
6. $d(x) = x_1 + t + i(ax_1^2 + s)$.

Case 1. Using Hörmander multiplier theorem (Lemma 2) since $\frac{\partial d(x)}{\partial x_l} = 0$, $l > \mu$, we only need to prove

$$\left|\frac{\partial^k (1/d(x))}{\partial x_{i_1} \dots \partial x_{i_k}}\right| \leqslant \frac{C}{(1+|x|_\mu)^k}, \quad i_k \leqslant \mu,\ k \leqslant \mu,\ |x|_\mu = |(x_1 \dots x_\mu)|.$$

It is sufficient to show

$$\left|\frac{d^{(\alpha)}(x)}{d(x)}\right| \leqslant \frac{C}{(1+|x|_\mu)^{|\alpha|}}.$$

Now $d(x)$ is elliptic in $\boldsymbol{R}^\mu$, then by Lemma 6 we obtain the required result.

The remaining cases 2–6 are essentially two-dimensional, and are treated in the proof of Theorem 2.

Proof of Theorem 2. (1) When $d(x)$ has level curve, by Lemma 8, it must be in the form of one of the following:

(a) $d(x) = x_1^2 + x_2^2 + \delta$,
(b) $d(x) = x_1^2 - x_2^2 + \delta$,
(c) $d(x) = x_1^2 - x_2 + \delta$,
(d) $d(x) = x_1 + \delta$,
(e) $d(x) = x_1 + iax_1^2 + \delta$,
(f) $d(x) = x_1^2 + i[a(x_1 - b)^2 + s]$.

Let us prove all above $\frac{1}{d(x)}$ are Fourier multipliers of L^p, $1 < p < \infty$, except case (c).

(a) $d(x) = x_1^2 + x_2^2 + \delta$, $d(D)$ is elliptic of order 2. By Hörmander multiplier theorem (Lemma 2), we need to prove

$$\left|\left(\frac{\partial}{\partial x}\right)^{(\alpha)} \left(\frac{1}{d(x)}\right)\right| \leqslant B|x|^{-|\alpha|}, \quad 1 \leqslant |\alpha| = |(\alpha \cdot \alpha)| \leqslant 2.$$

It is enough to prove

$$\left|\frac{\partial^{(\alpha)} d(x)}{d(x)}\right| \leqslant \frac{C}{(1+|x|)^{|\alpha|}},$$

which is obvious from Lemma 6.

(b) & (d) $d(x) = x_1^2 - x_2^2 + \delta$ or $d(x) = x_1 + \delta$, $\delta = t + is$. In Lemma 3, choosing $h(u) = 1/(u+\delta)$, we have

$$\left|u\,\frac{dh}{du}\right| = \frac{|u|}{(u+t)^2 + s^2} \leqslant \begin{cases} \dfrac{k}{s^2}, & \text{where } |u| \leqslant \max\{1, |2t|\} = k, \\[2mm] \dfrac{|u|}{|u/2|^2} = 4, & \text{where } |u| > k, \end{cases}$$

$$\left|u^2\,\frac{d^2 h}{du^2}\right| = \frac{2u^2}{[(n+t)^2 + s^2]^{3/2}} \leqslant \begin{cases} \dfrac{2k^2}{s^3}, & \text{where } |u| \leqslant k, \\[2mm] \dfrac{2u^2}{(u^2/4)^{3/2}} = 16, & \text{where } |u| > k. \end{cases}$$

So by Lemma 3, $\frac{1}{x_1^2 - x_2^2 + \delta}$ and $\frac{1}{x_1 + \delta}$ are Fourier multipliers of L^p, $1 < p < \infty$.

(e) & (f) $d(x) = x_1 + iax_1^2 + \delta$ or $d(x) = x_1^2 + i[a(x_1 - b)^2 + s]$. It is easy to see that $\frac{1}{d(x)}$ satisfies Marcinkiewicz multiplier theorem (Lemma 1).

As to (c), $\frac{1}{x_1^2 - x_2 + \delta}$ is a multiplier of L^p, if and only if $p = 2$. The proof is the same to that in case 1 of Theorem 1. Then Theorem 2 is established.

(2) If $d(x)$ has only a finite level set, Lemma 9 shows that $d(x)$ will be one of the following:

(a) $d(D)$ is elliptic,
(b) $d(x) = x_1^2 - x_2^2 + t + i[(x_1 - a)^2 - (x_2 - b)^2 + s]$,
(c) $d(x) = x_1^2 - x_2^2 + t + i[ax_1 + s]$,
(d) $d(x) = (x_1 - a)^2 - x_2 + i(x_1^2 + s)$,
(e) $d(x) = x_1^2 - x_2 + i(ax_2 + s)$.

By direct computation, all cases but (b) satisfy the hypothesis of the Marcinkiewicz multiplier theorem. Case (b) satisfies the hypothesis of the Hörmander multiplier theorem. We omit the details.

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