2° We have $\sup\{a_\infty(g)\colon g\in Z_M^0\} = M$; the supremum is not attained. One can easily see that the set $\{g\in Z_M^0\colon \|g\|_\infty = 1\}$ is dense in Z_M^0 in the L^2-norm. Thus

$$\sup\{a_\infty(g)\colon g\in Z_M^0\} = \sup\left\{\frac{1}{M\|g\|_2^2}\colon g\in Z_M^0\right\} = (\inf\{M\|g\|_2^2\colon g\in Z_M^0\})^{-1}$$
$$\leqslant (M\cdot\inf\{\|g\|_1^2\colon g\in Z_M^0\})^{-1} = M.$$

On the other hand, $\lim_{a=+0} a_\infty(f_a) = M$.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
81-825 Sopot, ul. Abrahama 18
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
00-950 Warsaw, ul. Śniadeckich 8

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Multipliers along curves

by

YANG-CHUN CHANG (Peking) and P. A. TOMAS (Austin, Tex.)

Abstract. The authors analyse boundedness of Fourier multiplier operators which are constant along curves. Boundedness is shown to depend upon a balance between the curvature of the level curves and the lack of smoothness of the multiplier function.

Introduction. In this paper we shall give an analysis of the L^p boundedness of Fourier multiplier operators which are constant along curves in the plane. The three fundamental examples for our study are the following:

(A) $m(x, y) = \varphi(x^2+y^2)$, φ smooth and compactly supported. Then m gives a multiplier operator bounded on all L^p, $1\leqslant p\leqslant\infty$. If φ has a discontinuity of the first type away from origin, the work of C. Fefferman [1] shows m gives a multiplier bounded on L^p if and only if $p=2$.

(B) $m(x, y) = \varphi(x^2-y^2)$, φ smooth and compactly supported. Then m gives a multiplier operator bounded on all L^p, $1<p<\infty$, by the Hörmander–Mihlin multiplier theorem [4].

(C) $m(x, y) = \varphi(y-x^2)$. If φ has any reasonable growth properties, m gives a multiplier of L^p if and only if $p=2$, from the work of Kenig and Tomas [2].

These examples show that the L^p boundedness of such multipliers depends on a balance between the curvature of the level curves, and the "bumpiness" of the multiplier function. It is these intuitive ideas we shall make precise. Such questions have already been considered by Ruiz [3].

In Section one of the paper we shall use some techniques of Ruiz [3] to give a different geometric characterization of the level curves. In Section two we shall follow Ruiz' [3] proof and show certain restrictions on φ can be removed.

Section one. We shall analyse the $L^p(R^2)$ boundedness of Fourier multiplier operators T, where

$$\widehat{Tf}(\xi) = m(\xi)\hat f(\xi)$$

and m is constant along level curves of a function $F\colon R^2\to R^1$, that is, $m(\xi) = \varphi\circ F(\xi)$. We shall consider a restricted class of level curves, which we shall call regular.

DEFINITION. The level curves $F(\xi) = c$ are *regular* if for each c there is a parametric realization ψ_c: $R^1 \to R^2$, parametrized by arclength s with $s \in (a, +\infty)$ (a is finite or $-\infty$), satisfying the following: if $\partial F/\partial n$ denotes the normal derivative of F along ψ_c, $G = 1 \Big/ \dfrac{\partial F}{\partial n}$, and $\varkappa$ the curvature of ψ_c, then for $s > 0$

(1) $$lG(s) \leqslant s^{-1/2} \int_b^{b+s^{1/2}} G(t)\,dt \leqslant LG(s)$$

for every b, $s < b < b+s^{1/2} < 2s$,

(2) $$M_1 \int_0^c \frac{G(t)}{\bar{n}_s \cdot \bar{n}(t)}\,dt \leqslant \int_s^{2s} \varkappa(t)\,dt \leqslant M_2 G(s),$$

where l, L, M_1, M_2 are constants, $\bar{n}_s$ is the unit vector normal to $F(\xi) = 0$ at s and $\bar{n}(t)$ the unit vector normal to $F(\xi) = t$ at the point P_t as shown in Figure 0. $G(t)$ is valued at P_t.

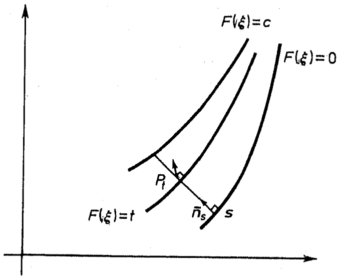

Fig. 0

To clarify these conditions, we translate them into geometric terms. If we draw the level curve $F(\xi) = 0$, then $\theta = \int_s^{2s} \varkappa(t)\,dt$ is the angle subtended by the normals at s and $2s$ (Figure 1). If we draw the curve $F(\xi) = c$, $d = \int_0^c \frac{G(t)}{\bar{n} \cdot \bar{n}_t}\,dt$ is the distance between the curves $F(\xi) = c$ and $F(\xi) = 0$, where the distance is measured at s, along the direction of the normal to $F(\xi) = 0$ at s (Figure 2).

Condition (1) is clearly a technical condition on the regularity of the behavior of the normal direction, and could in fact be replaced by other technical conditions.

The inequalities of condition (2)

$$M_1 d \leqslant \theta \leqslant M_2 G(s)$$

lie at the heart of the matter. A multiplier whose derivatives are well controlled will satisfy the Marcinkiewicz conditions, and therefore the negative results we obtain require a certain "bumpiness" in the part of the multiplier. The thickness of the bump is d, and condition (2) is an upper bound, which corresponds to a large derivative.

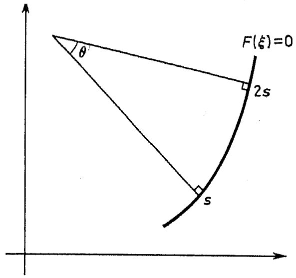

Fig. 1

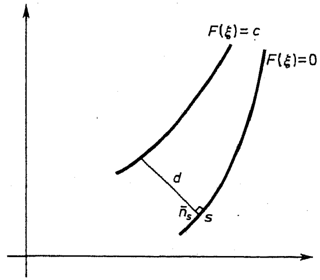

Fig. 2

The other essential condition concerns curvature. As the characteristic function of a half plane shows, a multiplier may have large derivative if it is concentrated along straight lines. Even multipliers of the form $\varphi(y^2 - x^2)$, whose large derivatives are only asymptotically concentrated along straight lines, have L^p boundedness. To obtain the negative results we seek, the large derivatives must be concentrated along curves with a significant amount of curvature. The curvature is measured by θ, and condition (2) puts a lower bound on θ.

Condition (2), then, expresses in a precise manner the necessary relationship between curvature and size of derivative.

To clarify the meaning of these conditions, Table 1 below lists some standard curves and their relation to the various conditions.

$F(x, y)$	$y - \log x$	$y - x^\alpha (\alpha \leqslant 1)$	$y - x^\alpha (\alpha > 1)$	$y - 1/x$	xy	$x^2 - y^2$	$x^2 + y^2 - 1$
$\theta \leqslant M_2 G(s)$	yes	yes	yes	yes	yes	yes	no
$M_1 d \leqslant \theta$	no	no	yes	no	no	no	yes
regularity	no	no	yes	no	no	no	no

Table 1

With this concept of regularity we can now state the principal result.

THEOREM 1. *Assume* $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$, *where* $m(\xi) = \varphi\circ F(\xi)$ *and* F *is regular. If* $\varphi \in L^q \cap L^\infty(R)$ *for some* $q > 0$, *and* φ *is not identically zero, then* T *is* $L^p(R^2)$ *bounded if and only if* $p = 2$.

The proof proceeds similarly to that of Ruiz [3] and makes use of the Kakeya set construction of C. Fefferman [1]. We require two preliminary lemmas.

LEMMA 1 (Kakeya's set). *For positive numbers* A, B, C, *there exist a set* $E \subset R^2$ *and a family of pairwise disjoint rectangles* $\{R_j\}$, *with size* $\frac{1}{2C^{1/2}} \times \frac{A}{4B}$ *whose directions are within an angle* $\frac{B}{A}$, *such that*

$$|E| \leqslant \frac{M}{\log\log C}\sum |R_j|, \quad \text{where } M \text{ is a constant,}$$

and such that $|E \cap \tilde{R}_j| \geqslant \frac{1}{4}|R_j|$, *where* $\tilde{R}_j$ *is the usual rectangle adjacent to* R_j (Figure 3).

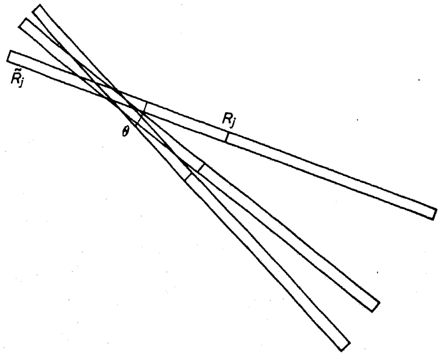

Fig. 3

LEMMA 2 [3]. *Assume* $\{B_j\}$ *is a family of parallel pairwise disjoint strips with width* λ, *and*

$$\widehat{T_j^\lambda f}(\xi) = m(\xi)\chi_{B_j}(\xi)\hat{f}(\xi).$$

If $\|Tf\|_p \leqslant C_p\|f\|_p$ *for every* $f \in L^p$, $p \geqslant 2$, *then for any* $\{f_j\} \in L^p(l^2)$,

$$\|(\sum |T_j^\lambda f_j|^2)^{1/2}\|_p \leqslant C'_p\|(\sum |f_j|^2)^{1/2}\|_p.$$

Proof of Theorem 1. We may assume $\varphi \geqslant 0$ and $\varphi \in L^1 \cap L^\infty$. Choose c so that

$$\int_0^c \varphi \geqslant \frac{L+l/4}{L+l/2}\int_0^\infty \varphi, \qquad \int_{-c}^0 \varphi \geqslant \frac{L+l/4}{L+l/2}\int_{-\infty}^0 \varphi.$$

For any fixed s let A, B and C (in Lemma 1) be M_1, θ and s. Now let us set the strips $\{B_j\}$ with the same direction as the normal at s, as shown in Figure 4, such that the normal direction at x_j, $\bar{n}_j$, is the direction of suitably rotated Kakeya's rectangle R_j (in Lemma 1), and the width of B_j is $s^{1/2}$.

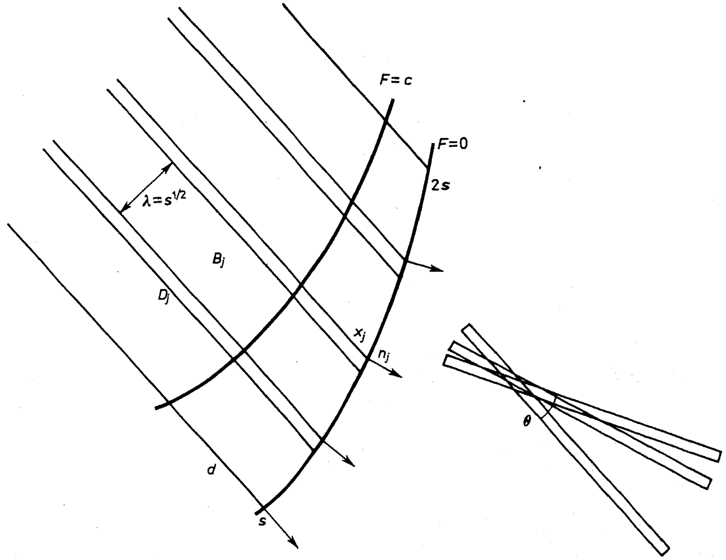

Fig. 4

If $\|Tf\|_p \leqslant C_p\|f\|_p$, and

$$T_s f(\xi) = \sum_s^{2s} [\chi_{B_j}(\xi) + \chi_{D_j}(\xi)]\,\varphi[F(\xi)]\hat{f}(\xi),$$

then $\|T_s f\|_p \leqslant C_p\|f\|_p$. Therefore, by Lemma 2,

$$\|(\sum |T_j^\lambda f_j|^2)^{1/2}\|_p \leqslant C'_p\|(\sum |f_j|^2)^{1/2}\|_p,$$

where $\widehat{T_j^\lambda f}(\xi) = \chi_{B_j}(\xi)\widehat{T_s f}(\xi) = \chi_{B_j}(\xi)\varphi[F(\xi)]\hat{f}(\xi)$, C_p' independent of s.

We shall prove that C_p' cannot be bounded as $s \to \infty$, if $p > 2$. Set $f_j(\xi) = \chi_{R_j}(\xi)e^{ix_j\xi}$, then

$$\begin{aligned} T_j^\lambda f_j(x) &= \{[\varphi(F(\xi))\chi_{B_j}(\xi)]^{\check{}} * \chi_{R_j}(\xi)e^{ix_j\xi}\}(x) \\ &= \int_y \Big[\int_{B_j} \varphi(F(\xi))e^{i\xi y}\,d\xi\Big]\chi_{R_j}(x-y)e^{ix_j(x-y)}\,dy \\ &= \int_{B_j}\int_y \varphi(F(\xi))e^{ix_jx}e^{-i(x_j-\xi)\cdot y}\chi_{R_j}(x-y)\,dy\,d\xi \\ &= e^{ix_jx}\int_{B_j}\int_{x-R_j} \varphi(F(\xi))e^{-i(x_j-\xi)\cdot y}\,dy\,d\xi. \end{aligned}$$

Hence

$$|T_j^\lambda f_j(x)| \geqslant \Big|\int_{B_j}\int_{x-R_j} \varphi(F(\xi))\cos[(x_j-\xi)\cdot y]\,dy\,d\xi\Big|.$$

The curve separates the strip B_j into two parts B_j^+ and B_j^- (Figure 5).

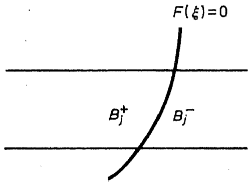

Fig. 5

Since $\int_{B_j} = \int_{B_j^+} + \int_{B_j^-}$, it is enough to estimate one of these two integrals.

$$\int_{B_j^+}\int_{x-R_j} \varphi(F(\xi))\cos((x_j-\xi)\cdot y)\,dy\,d\xi$$

$$= \Big(\int_{B_j^0+B_j^c}\Big)\varphi(F(\xi))\int_{x-R_j}\cos((x_j-\xi)\cdot y)\,dy\,d\xi,$$

where B_j^0 is the subset of the strip B_j between the curve $F(\xi) = 0$ and $F(\xi) = c$, and $B_j^c = B_j^+ \setminus B_j^0$ (Figure 6).

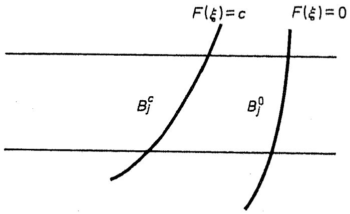

Fig. 6

We claim, for every $x \in \tilde{R}_j$ and $y \in x - R_j$,

$$\cos((x_j-\xi)\cdot y)\begin{cases} \geqslant 1/2 & \text{if } \xi \in B_j^0, \\ \leqslant 1 & \text{otherwise.}\end{cases}$$

This is because, when $\xi \in B_j^0$, $(x_j-\xi)\cdot y = |x_j-\xi|\,|y|\cos(\pi/2-\alpha-\beta)$, where α is the angle between the vector y and the direction of R_j, β is the angle between the vector $x_j-\xi$ and the direction perpendicular to R_j (Figure 7). Then

$$\begin{aligned} |(x_j-\xi)\cdot y| &= |x_j-\xi|\,|y|\,|\sin\alpha\cos\beta + \cos\alpha\sin\beta| \\ &\leqslant \big||x_j-\xi|\cos\beta\big|\cdot\big||y|\sin\alpha\big| + \big||x_j-\xi|\sin\beta\big|\cdot\big||y|\cos\alpha\big| \\ &\leqslant \lambda\cdot(\text{width of } R_j) + d\cdot(2 \text{ length of } R_j) \\ &= s^{1/2}\frac{1}{s^{1/2}} + d\frac{M_1}{2\theta}. \end{aligned}$$

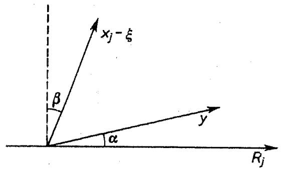

Fig. 7

By condition (2) of the regularity of the curve $F(\xi) = 0$, $|(x_j-\xi)\cdot y| \leqslant 1$. Hence $\cos((x_j-\xi)\cdot y) > \frac{1}{2}$ as $\xi \in B_j^0$. Therefore,

$$\begin{aligned} &\int_{B_j^+}\varphi(F(\xi))\int_{x-R_j}\cos((x_j-\xi)\cdot y)\,dy\,d\xi \\ &= \Big(\int_{B_j^0} + \int_{B_j^c}\Big)\int_{x-R_j}\varphi(F(\xi))\cos((x_j-\xi)\cdot y)\,dy\,d\xi \\ &\geqslant \tfrac{1}{2}\int_{B_j^0}\int_{x-R_j}\varphi(F(\xi))\,dy\,d\xi - \int_{B_j^c}\int_{x-R_j}\varphi(F(\xi))\,dy\,d\xi \\ &= |R_j|\Big(\tfrac{1}{2}\int_{B_j^0} - \int_{B_j^c}\Big)\varphi(F(\xi))\,d\xi. \end{aligned}$$

Let ϱ be the rotation of angle ν, where ν is the angle between the ξ_1 axis and the normal to $F(\xi) = 0$ at point s. Let $\xi = \varrho\xi'$, $\varphi[F(\xi)] = \varphi[F(\varrho\xi')]$. Changing variables $F(\varrho\xi') = u$, $\xi_2' = v$, the Jacobian is

$$\frac{\partial(u, v)}{\partial(\xi_1', \xi_2')} = \frac{\partial F}{\partial\xi_1'} = \frac{\partial F}{\partial\xi_1}\cos\nu + \frac{\partial F}{\partial\xi_2}\sin\nu = \frac{\partial F}{\partial n}(\bar{n}\cdot\bar{n}_s).$$

Since

$$ds = \left[\sqrt{\left(\frac{\partial F}{\partial \xi_1'}\right)^2 + \left(\frac{\partial F}{\partial \xi_2'}\right)^2} \Big/ \left|\frac{\partial F}{\partial \xi_1'}\right|\right] dv$$
$$= \left[\frac{\partial F}{\partial n} \Big/ \frac{\partial F}{\partial n} |\bar{n}\cdot\bar{n}_s|\right] dv = dv/|\bar{n}\cdot\bar{n}_s|,$$

we obtain

$$\tfrac{1}{2}|R_j| \int_{B_j^0} \varphi[F(\xi)]\,d\xi = (|R_j|/2) \int_{B_j^0} \varphi(u) \frac{1}{\frac{\partial F}{\partial n}|\bar{n}\cdot\bar{n}_s|}\,dv\,du$$
$$\geqslant (|R_j|/2) \int_0^c \varphi(u) \int_{b_j}^{b_j+s^{1/2}} \frac{1}{\partial F/\partial n}\,dt\,du$$
$$= \frac{M_1}{16\theta s^{1/2}} \int_{b_j}^{b_j+s^{1/2}} G(t)\,dt \int_0^c \varphi(u)\,du.$$

By condition (1) of regularity of curve $F(\xi) = c$ we have

$$|R_j|/2 \int_{B_j^0} \varphi[F(\xi)]\,d\xi \geqslant \frac{M_1 l}{16\theta} G(s) \int_0^c \varphi(u)\,du.$$

Similarly

$$|R_j| \int_{B_j^c} \varphi[F(\xi)]\,d\xi \leqslant |R_j| \int_c^\infty \varphi(u) \int_{b_j}^{b_j+s^{1/2}} G(t)\,dt\,du$$
$$= \frac{M_1}{8\theta s^{1/2}} \int_{b_j}^{b_j+s^{1/2}} G(t)\,dt \int_c^\infty \varphi(u)\,du \quad \text{(by condition (1))}$$
$$\leqslant \frac{M_1 L}{8\theta} G(s) \int_c^\infty \varphi(u)\,du.$$

So,

$$\int_{B_j^+} \geqslant \frac{M_1}{8\theta} G(s) \left[\frac{l}{2}\int_0^c - L\int_c^\infty\right] \varphi(u)\,du$$
$$= \frac{M_1}{8\theta} G(s) \left[\left(\frac{l}{2}+L\right)\int_0^c - L\int_c^\infty\right] \varphi(u)\,du.$$

We have chosen c so that

$$\int_0^c \varphi \geqslant \frac{L+l/4}{L+l/2} \int_0^\infty \varphi.$$

Therefore

$$\int_{B_j^+} \geqslant \frac{M_1}{8\theta} G(s) \frac{l}{4} \int_0^\infty \varphi.$$

Then from condition (2) of regularity,

$$\int_{B_j^+} \geqslant \frac{M_1 l}{32 M_2} \int_0^\infty \varphi.$$

By the same estimate,

$$\int_{B_j^-} \geqslant \frac{M_1 l}{32 M_2} \int_{-\infty}^0 \varphi.$$

So

$$\left|\int_{B_j}\right| \geqslant \frac{M_1 l}{32 M_2} \|\varphi\|_1.$$

Using Fefferman's method ([1]), it follows that if $\|(\sum |T_j^\lambda f_j|^2)^{1/2}\|_p \leqslant C_p' \|(\sum |f_j|^2)^{1/2}\|_p$, C_p' independent of s, then $1 \leqslant C_p/(\log\log s)^{1-2/p}$. This contradicts the boundedness of T on L^p, $p > 2$ (the case $p > 2$ yields that of $p < 2$ by duality), and then completes the proof of Theorem 1 (the case $p = 2$ is trivial).

Section two. In this section we prove a result of Ruiz on multipliers of the form $m(\xi_1, \xi_2) = \varphi(\xi_2 - r(\xi_1))$. We shall use the same class of curves r as Ruiz [3], but remove Ruiz' restriction on φ.

DEFINITION. A curve $y = r(x)$ is *regular in the sense of Ruiz* [3], if

(a) The function r is C^∞, $r'(x) \to \infty$ as $x \to \infty$, and $r''(x) > 0$ for x large enough;

(b) If $k > k_0$ and $x_1 > x_2$, $x_i \in [2^k, 2^{k+1}]$, $i = 1, 2$, then

$$1 < \frac{r'(x_1)}{r'(x_2)} < L, \qquad 1 < \frac{\varkappa(x_2)}{\varkappa(x_1)} < M,$$

$\varkappa(x)$ is the curvature of curve $y = r(x)$;

(c) The length of r in $[2^k, 2^{k+1}]$ is greater than $[\varkappa(2^k)]^{-1/2}$ for $k > k_0$.

THEOREM 2. *Let* $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$, *where* $m(\xi_1, \xi_2) = \varphi(\xi_2 - r(\xi_1))$. *Assume r is regular in the sense of Ruiz, and that φ is in $L^q \cap L^\infty$ for some $q > 0$. If φ is not identically zero, then T is bounded on $L^p(R^2)$ if and only if $p = 2$.*

We need a slightly different Kakeya set than that used for Theorem 1.

LEMMA 3 (Kakeya's set). *For positive numbers r_1', $\varkappa_1$, ζ, there exist a set E and a family of pairwise disjoint rectangles $\{R_j\}$ with dimensions $\frac{(r_1'\varkappa_1)^{1/2}}{L} \times r_1'/8\zeta$, whose directions are within an angle $8\zeta k^{1/2}$, such that*

$$|E| \leqslant \frac{M \log\log r_1'}{\log r_1'} \sum |R_j| \qquad (M \text{ constant}),$$

and $|E \cap \tilde{R}_j| \geqslant \frac{1}{20}|R_j|$, where $\tilde{R}_j$ is the usual rectangle adjacent to R_j.

Proof of Theorem 2. Without loss of generality, we can suppose $\varphi \in L^1 \cap L^\infty(R)$ and $p > 2$. We shall follow the proof of Ruiz [3]. Take a positive integer k such that $\zeta \cos\alpha < 1$, where $\alpha > \pi/4$ is the inclination of tangent at 2^k, ζ is a number so that

$$\int_0^\zeta \varphi > \frac{9}{10}\int_0^\infty \varphi \quad \text{and} \quad \int_{-\zeta}^0 \varphi > \frac{9}{10}\int_{-\infty}^0 \varphi.$$

We choose a, b satisfying $2^k = a < b < 2^{k+1}$ and the length of the arc of curve $\xi_2 = r(\xi_1)$ from $(a, r(a))$ to $(b, r(b))$ is $\varkappa_1^{1/2}$ $(\varkappa_1 = \varkappa(a))$. Let $I = [a, b] \times [r(a), r(b)]$ and $\widehat{Sf} = m\chi_I\hat{f}$. If $\|Tf\|_p \leqslant C_p\|f\|_p$, then $\|Sf\|_p \leqslant A\|f\|_p$.

We arrange the strips $\{B_j\}$, in Lemma 2, with width $\lambda = 1/(r_1'\varkappa_1)^{1/2}$ $(r_1' = r'(a))$ and directions the same as that of the normal to the curve at $(a+b)/2$, centered at the points on the curve at which the normal directions coincide with those of the suitably rotated Kakeya rectangles R_j (in Lemma 3) (Figure 8).

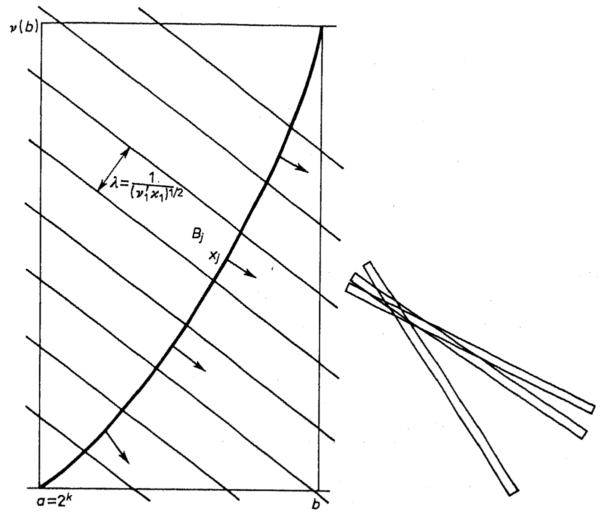

Fig. 8

We take $f_j(x) = \chi_{R_j}(x)e^{ix_j\cdot x}$, $\widehat{T_j^\lambda f} = \chi_{B_j}\widehat{Sf}$. Then, by Lemma 2,

$$\Big\|\Big(\sum |T_j^\lambda f_j|^2\Big)^{1/2}\Big\|_p \leqslant C_p'\Big\|\Big(\sum |f_j|^2\Big)^{1/2}\Big\|_p.$$

But we claim that C_p' cannot be bounded as $k \to \infty$ if $p > 2$. It is sufficient to show that $|T_j^\lambda f_j(x)| \geqslant \text{const}(r, p)$ for every $x \in \tilde{R}_j$. We have

$$|T_j^\lambda f_j(x)| = \big|\{[\varphi(\xi_2 - r(\xi_1))\chi_{B_j}(\xi)\chi_I(\xi)]\check{} * \chi_{R_j}(\xi)e^{ix_j\cdot\xi}\}(x)\big|$$
$$= \Big|\int_{\bar{B}_j} e^{-i\xi y}\varphi(\xi_2 - r(\xi_1)) \int_{x-R_j} e^{ix_j(x-y)}\,d\xi\,dy\Big| \qquad (\bar{B}_j = B_j \cap I)$$
$$\geqslant \int_{\bar{B}_j} \varphi(\xi_2 - r(\xi_1)) \int_{x-R_j} \cos((x_j - \xi)\cdot y)\,dy\,d\xi.$$

The curve $\xi_2 - r(\xi_1) = 0$ separates the strip $\bar{B}_j$ into two parts B_j^+ and B_j^- (Figure 9). It is enough to show

$$\int_{B_j^+}\int_{x-R_j} \varphi(\xi_2 - r(\xi_1))\cos((x_j - \xi)\cdot y)\,dy\,d\xi$$
$$= \Big(\int_{B_j^0} + \int_{B_j^c}\Big)\int_{x-R_j} \varphi(\xi_2 - r(\xi_1))\cos((x_j - \xi)\cdot y)\,dy\,d\xi$$
$$\geqslant \text{const},$$

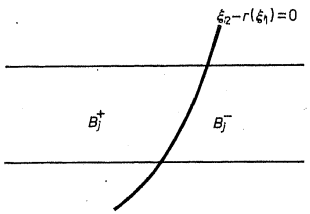

Fig. 9

where B_j^0 is a subset of the strip $\bar{B}_j$ between curves $\xi_2-r(\xi_1)=0$ and $\xi_2-r(\xi_1)=\zeta$, $B_j^c=B_j^+\setminus B_j^0$ (Figure 10).

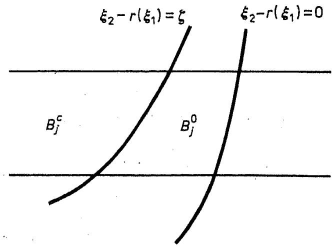

Fig. 10

As in Section one, when $x\in\tilde{R}_j$, $y\in x-R_j$,

$$\cos\big((x_j-\xi)\cdot y\big)\begin{cases}\geqslant \frac{1}{4} & \text{if } \xi\in B_j^0,\\ <1 & \text{otherwise}.\end{cases}$$

Hence

$$\Big(\int\limits_{B_j^0}+\int\limits_{B_j^c}\Big)\int\limits_{x-R_j}\varphi(\xi_2-r(\xi_1))\cos\big((x_j-\xi)\cdot y\big)\,dy\,d\xi$$
$$\geqslant \tfrac{1}{4}\int\limits_{B_j^0}\int\limits_{x-R_j}\varphi(\xi_2-r(\xi_1))\,dy\,d\xi-\int\limits_{B_j^c}\int\limits_{x-R_j}\varphi(\xi_2-r(\xi_1))\,dy\,d\xi_1$$
$$=|R_j|\Big(\tfrac{1}{4}\int\limits_{B_j^0}-\int\limits_{B_j^c}\Big)\varphi(\xi_2-r(\xi_1))\,d\xi.$$

Changing variables

$$\begin{cases}\xi_2-r(\xi_1)=u, & \text{Jacobian}=1,\\ \xi_1=v\end{cases}$$

sides l_1 and l_2 of strips B_j will be

$$(*)\qquad u+r(v)-\eta v-b_i=0.$$

η is the slope of l_1, l_2, b_i the ξ_2 intercept of l_i; $i=1, 2$. Seeing b_i as a parameter, $(*)$ as an implicit function of variable u, we obtain a function $v=v(u, b)$, and

$$\frac{1}{4}\int\limits_{B_j^0}\varphi(\xi_2-r(\xi_1))\,d\xi=\frac{1}{4}\int\limits_0^{\zeta}\varphi(u)\int\limits_{v(u,b_1)}^{v(u,b_2)}dv\,du$$
$$=\frac{1}{4}\int\limits_0^{\zeta}\varphi(u)\,[v(u, b_2)-v(u, b_1)]\,du$$
$$=\frac{1}{4}\,\frac{b_2-b_1}{r'(v_0)-\eta}\int\limits_0^{\zeta}\varphi(u)\,du,\qquad a\leqslant v_0\leqslant b$$

and

$$\int\limits_{B_j^c}\varphi(\xi_2-r(\xi_1))\,d\xi\leqslant\int\limits_{\zeta}^{\infty}\varphi(u)\int\limits_{v(u,b_1)}^{v(u,b_2)}dv\,du$$
$$=\int\limits_{\zeta}^{\infty}\varphi(u)\,[v(u, b_2)-v(u, b_1)]\,du$$
$$=\frac{b_2-b_1}{r'(v_0')-\eta}\int\limits_{\zeta}^{\infty}\varphi(u)\,du,\qquad a\leqslant v_0'\leqslant b.$$

So

$$\int\limits_{B_j^+}\int\limits_{x-R_j}\varphi(\xi_2-r(\xi_1))\cos\big((x_j-\xi)\cdot y\big)\,dy\,d\xi$$
$$\geqslant|R_j|\Big(\frac{1}{4}\int\limits_{B_j^0}-\int\limits_{B_j^c}\Big)\varphi(\xi_2-r(\xi_1))\,d\xi$$
$$\geqslant\frac{(r_1'^3\varkappa_1)^{1/2}}{8L\zeta}\,\frac{b_2-b_1}{L(r_1'-\eta)}\Big(\frac{1}{4}\int\limits_0^{\zeta}\varphi-\int\limits_{\zeta}^{\infty}\varphi\Big)$$

$$\geqslant \frac{(r_1'^3 \varkappa_k)^{1/2}\sqrt{2}\,\lambda}{8L^2\zeta r_1'}\left(\frac{5}{4}\int_0^{\zeta} \varphi - \int_0^{\infty} \varphi\right), \qquad \lambda = \frac{1}{(r_1'\varkappa_1)^{1/2}},$$

$$\geqslant \frac{\sqrt{2}}{64L^2\zeta}\int_0^{\infty} \varphi .$$

For the same reason

$$\int_{B_j^-}\int_{x-R_j} \varphi(\xi_2 - r(\xi_1)) \cos\left((x_j-\xi)\cdot y\right) dy\, d\xi \geqslant \frac{\sqrt{2}}{64L^2\zeta}\int_{-\infty}^{0} \varphi .$$

Therefore

$$|T_j^\lambda f_j(x)| \geqslant \frac{\sqrt{2}}{64L^2\zeta}\, \|\varphi\|_1 .$$

Then T is $L^p \to L^p$ bounded operator if and only if $p = 2$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TEXAS AT AUSTIN
PEKING NORMAL UNIVERSITY
and
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TEXAS AT AUSTIN

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Invertibility of some second order differential operators

by

YANG-CHUN CHANG (Peking) and P. A. TOMAS (Austin, Tex.)

Abstract. The authors examine L^p invertibility of second order linear partial differential operators with constant coefficients. Invertibility of such operators is shown to depend upon the geometric structure of the level surfaces associated to the symbol of the operator.

The purpose of this paper is to examine the invertibility of second order linear partial differential operators with constant coefficients. In general dimension we shall treat operators with level hyper-surfaces; in two dimensions we shall give a complete classification. Following the remarks of Kenig–Tomas [2], we shall see that the invertibility of such operators depends upon the geometric structure of the level surfaces. The main tools in our approach are the Kakeya counterexample of C. Fefferman, the classical multiplier theorems of Marcinkiewicz and Hörmander, and a multiplier theorem of Littman, McCarthy and Rivière.

Let $d(x)$, $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$, denote a second order polynomial on $\boldsymbol{C}$. $d(x)$ can be expressed as

$$d(x) = P(x) + iQ(x),$$
$$P(x) = X^T AX + \alpha\cdot X + t,$$
$$Q(x) = X^T BX + \beta\cdot X + s,$$

where A, B are real symmetric matrices, α, β are real vectors, t and s are real constants.

THEOREM 1. *Assume $d(x)$, $\frac{1}{d(x)} \in L^\infty(\boldsymbol{R}^n)$, has $(n-1)$-dimensional level hyper-surface. Then $\frac{1}{d(x)}$ is a Fourier multiplier of L^p, $1 < p < \infty$, except in the following two cases*:

(1) *α is not an eigenvector of A.*

(2) *The rank of A is at least three and the restriction of $X^T AX$ to the eigenspace is not positive definite.*