des idéaux premiers de A est totalement ordonné par inclusion, alors A est une algèbre de Fréchet.

Démonstration. A est évidemment un anneau local. D'après le lemme précédent l'application $x \to x^{-1}$ est continue sur G. Le fait que A est une algèbre de Fréchet résulte alors du théorème 2.

Rappelons qu'un anneau commutatif unitaire et intègre A est appelé un anneau de valuation si l'ensemble des idéaux de A est totalement ordonné par inclusion.

COROLLAIRE. *Toute B_0-algèbre qui est un anneau de valuation est une algèbre de Fréchet.*

On a montré dans [1] que toute algèbre de Fréchet de dimension infinie qui est un anneau de valuation est une algèbre de séries formelles, c'est-à-dire qu'il existe un homomorphisme injectif de A dans $C[[X]]$, continu pour la topologie naturelle d'algèbre de Fréchet de $C[[X]]$ et dont l'image contient les polynômes. On voit donc que toute B_0-algèbre de dimension infinie qui est un anneau de valuation est une algèbre de Fréchet de séries formelles. Il résulte également du corollaire 1 et d'un résultat de [1] que si une telle B_0-algèbre A ne possède aucune norme continue, A est isomorphe à $C[[X]]$.

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Operators of Bochner-Riesz type for the helix

by

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Abstract. We consider in R^3 a family of multipliers $m_\delta(\vec{\xi})$ depending upon a parameter $\delta > 0$. Their singularities lie along a cylindrical helix. The boundedness on $L_p(R^3)$ of the operators $T_\delta f(\vec{x})$ defined by $\widehat{T_\delta f}(\vec{\xi}) = m_\delta(\vec{\xi})\hat{f}(\vec{\xi})$ is studied after establishing a sharp estimate on the corresponding maximal function. The operators T_δ are the analogue for the helix in R^3 of the Bochner–Riesz spherical summation operators for the circle in R^2.

Introduction. Bochner–Riesz spherical summation operators U_δ are defined in $R^n (n \geqslant 2)$ by the formula $\widehat{U_\delta f}(\vec{\xi}) = n_\delta(\vec{\xi})\hat{f}(\vec{\xi})$, $\delta > 0$, where $n_\delta(\vec{\xi}) = (1 - \|\vec{\xi}\|^2)^\delta$ if $\|\vec{\xi}\| < 1$ and $n_\delta(\vec{\xi}) = 0$ otherwise. They have been studied extensively (see [1], [3], [5], [7], [8], [10], [17]). In R^2 the results on the boundedness of these operators acting on L_p functions are sharp (they actually hold not only for the unit circle but for more general curves with the function n_δ replaced by any function with compact support which is smooth except near the curve where it is the distance to the curve raised to the power δ; see [11], [15]). The studies on the subject have shown that these operators are tight up with restriction theorems of the Fourier transform to the unit sphere of R^n. In [13] a restriction theorem to smooth curves in R^3 with nonvanishing curvature and torsion is proved, so it is natural to look for the related multipliers. We are going to define them in R^3 for the helix Γ of equation $\gamma(t) = (\cos t/\sqrt{2},\ \sin t/\sqrt{2},\ t/\sqrt{2})$, $-1/10 \leqslant t \leqslant 1/10$. (The helix being the only curve with constant curvature and torsion might be thought as the analogue in R^3 of the unit circle in R^2.) This choice makes the geometry slightly easier, at the same time it captures the nature of the problem which has to do with the local behavior of the curve.

Let $(\xi_1^t, \xi_2^t, \xi_3^t)$ be the coordinates of $\vec{\xi} = (\xi_1, \xi_2, \xi_3) \in R^3$, with respect to the Frenet frame of Γ at $\gamma(t)$. Denote by N_t the normal plane at $\gamma(t)$. If $\vec{\xi}$ is a point lying in the angle determined by the planes N_t for $t = \pm 1/10$, which contains Γ, and moreover $\vec{\xi}$ is close enough to Γ, say $\operatorname{dist}(\vec{\xi}, \Gamma) < 1/100$ (denote by $U(\Gamma)$ this set) then there exists one and only one normal plane $N_{t(\vec{\xi})}$ through $\vec{\xi}$ ($t(\vec{\xi})$ is the solution of the equation $-\xi_1 \sin t + \xi_2 \cos t + \xi_3 = 0$, unique under our assumptions). Now we define the following multipliers:

$$m_\delta(\vec{\xi}) = (|\xi_2^{t(\vec{\xi})}|^6 + |\xi_3^{t(\vec{\xi})}|^4)^{\delta/4} G(\vec{\xi}), \qquad \delta > 0 \tag{1}$$

where $G(\vec{\xi})$ is a C_0^∞ function identically one on the ball centered at (1, 0, 0) of radius 1/200 and zero outside the double of it. We study the boundedness of the operators $T_\delta f(\vec{x})$, $\vec{x}=(x, y, z)$, defined by

$$\widehat{T_\delta f}(\vec{\xi}) = m_\delta(\vec{\xi})\hat{f}(\vec{\xi}). \tag{2}$$

Associated to T_δ, for $N \geqslant 1$ and $d > 0$, we consider the maximal function M_N^d defined on locally integrable functions by the formula

$$M_N^d f(\vec{x}) = \operatorname*{Sup}_{\vec{x}\in R\in B_N^d} |R|^{-1}\int_R |f(\vec{y})|\,d\vec{y} \tag{3}$$

where B_N^d is the set of boxes R centered at the origin of dimensions (dN, dN^2, dN^3) along the tangent, normal and binormal directions at $\gamma(2\sqrt{2}\pi j/N)$, $j=1, \ldots, N$. In Section 1 we prove the following theorems:

THEOREM 1. *Suppose* $1 \leqslant p \leqslant 2$. *Then if* $M_N^d f(\vec{x})$ *is defined as in* (3) *we have that for any* $d > 0$ *and* $N \geqslant 1$

(a) *there exist positive constants* $C(p)$ *and* $\alpha(p)$ *independent of* N, d *and* f *such that*

$$\|M_N^d f\|_p \leqslant C(p)\,N^{(2-p)/p}(\lg N)^{\alpha(p)}\|f\|_p;$$

(b) *there exists a constant* $c(p)$ *independent of* N *and* d *such that*

$$\|M_N^d\|_{(p,p)} \geqslant c(p)\,N^{(2-p)/p}.$$

THEOREM 2. *The operator* T_δ *defined in* (2) *is a bounded operator from* $L_p(R^2)$ *to itself in each one of the following cases*:

(A) $1 \leqslant p \leqslant 6/5$ *and* $\delta > 5/3p-4/3$;

(B) $6/5 \leqslant p \leqslant 4/3$ *and* $\delta > 2/3p-1/2$;

(C) $4/3 \leqslant p \leqslant 2$ *and* $\delta > 0$.

The range $2 < p \leqslant \infty$ *is covered by duality.*

In Section 2 we study the behavior at infinity of the kernel K_δ of the convolution operator T_δ and we show that K_δ is integrable for $\delta > 1/3$. Then we prove that if μ denotes the uniform measure on the curve (t, t^2, t^3), $0 \leqslant t \leqslant 1/10$, then $\hat{\mu}$ belongs to $L_p(R^3)$ if and only if $p > 7$.

We wish to thank A. Cordoba, C. Fefferman and P. Sjölin for valuable conversations.

Section 1. In what follows by C we denote a constant not necessarily the same in all instances. Before proving Theorem 1 let us recall that the spherical indicatrix of the tangents and binormals of the helix $\gamma(t)$, $0 \leqslant t \leqslant 2\sqrt{2}\pi$, is the parallel $z=1/\sqrt{2}$ in the (x, y, z)-coordinates and the spherical indicatrix of the normals is the equator $z=0$.

Proof of Theorem 1. Let us consider for $d > 0$ and $N \geqslant 1$ the following maximal function:

$$\tilde{M}_N^d f(\vec{x}) = \operatorname*{Sup}_{d,j}(2d)^{-1}\int_{-d}^{d} |f(\vec{x}+t\vec{\omega}_j)|\,dt$$

where $\vec{\omega}_j$ is the point of the unit sphere of coordinates $(1/\sqrt{2}\cos 2\pi j/N,\ 1/\sqrt{2}\sin 2\pi j/N,\ 1/\sqrt{2})$, $j=1, \ldots, N$. In [6] it has been shown that $\|\tilde{M}_N^d f\|_2 \leqslant C(\lg N)^\alpha \|f\|_2$ where C and α are constants independent of N, d and f. This implies $\|M_N^d f\|_2 \leqslant C(\lg N)^{3\alpha}\|f\|_2$. Moreover since the number of boxes R belonging to B_N^d is N, we have trivially that $\|M_N^d f\|_1 \leqslant N\|f\|_1$. Then (a) follows by interpolation. Statement (b) is proved for $d=1$ by applying M_N^1 to the characteristic function χ of the ball of radius N^2 centered at the origin. It is easy to check that $M_N^1\chi(\vec{x}) \geqslant (100N)^{-1}$ on the set

$$\{(r, \varphi, \theta)\colon\ N^3/2 \leqslant r \leqslant N^3,\ |\varphi-\pi/4| \leqslant (2N)^{-1},\ 0 \leqslant \theta \leqslant 2\pi\}$$

where (r, φ, θ) are the usual spherical coordinates of R^3. This implies (b) in the case $d=1$. By a suitable dilation the counterexample can be made to work for any $d > 0$.

We shall now consider the multiplier problem. We are going to break up $m_\delta(\vec{\xi})$ by means of a smooth partition of unity. There exists a C_0^∞ function in R^2 $\tau(u, v)$ supported on $\{(u, v)\colon 1/4 \leqslant |u|, |v| \leqslant 2\}$ such that if we denote $\tau_k(u, v) = \tau(2^{2k/3}u, 2^k v)$ then

$$\sum_{k\geqslant 0}\tau_k(u, v) = 1 \quad \text{on} \quad \{(u, v)\colon |u|, |v| \leqslant 1\}\setminus\{0, 0)\}.$$

Now we consider a smooth partition of unity on Γ. Precisely there exists a C_0^∞ function $\psi(t) = \varphi(\gamma(t))$ defined for $-1/5 \leqslant t \leqslant 1/5$ with bounds independent of k such that if we denote

$$\psi_{kj}(t) = \varphi\big(\gamma(2^{k/3}(t-j/2^{k/3}))\big), \qquad k > 90, \qquad j = [-2^{k/3}/5], \ldots, [2^{k/3}/5]$$

then $\sum_j \psi_{kj}(t) = 1$ for $-1/10 \leqslant t \leqslant 1/10$ and for every $k > 90$ ($[\cdot]$ is the greatest integer function). Now as we already pointed out for every $\vec{\xi}$ in $U(\Gamma)$ there exists a unique value $t = t(\vec{\xi})$ such that the normal plane $N_{t(\vec{\xi})}$ goes through $\vec{\xi}$ and the correspondence $\vec{\xi} \to t(\vec{\xi})$ is C^∞. Hence the function

$$\psi_{kj}(\vec{\xi}) = \tau_k(\xi_2^{t(\vec{\xi})}, \xi_3^{t(\vec{\xi})})\,\psi_{kj}(t(\vec{\xi}))$$

is well defined in $U(\Gamma)$ and C^∞. Moreover if we denote $\psi_k(\vec{\xi}) = \sum_j \psi_{kj}(\vec{\xi})$ we have $\sum_k \psi_k(\vec{\xi}) = 1$ on support$(G)\setminus\Gamma$. Let us denote

$$\widehat{T_k^\delta f}(\vec{\xi}) = m_\delta(\vec{\xi})\psi_k(\vec{\xi})\hat{f}(\vec{\xi}), \qquad \widehat{T_{kj}^\delta f}(\vec{\xi}) = m_\delta(\vec{\xi})\psi_{kj}(\vec{\xi})\hat{f}(\vec{\xi}). \tag{4}$$

Obviously,

(5) $$T_\delta f = \sum_{\substack{k>90 \\ j}} T_{kj}^\delta f + T_0^\delta f$$

where $T_0^\delta f$ is defined by (5). Clearly T_0^δ is a bounded operator on L_p, $p \geqslant 1$, $\delta > 0$. To handle T_{kj}^δ we need to know the behavior at infinity of the corresponding kernel. Obviously

$$|\widehat{m_\delta \psi_{kj}}(\tilde{x}, \tilde{y}, \tilde{z})| \leqslant C 2^{-k(2+\delta)}$$

on

$$R_{kj} = \{(\tilde{x}, \tilde{y}, \tilde{z}) \in R^3 \colon |\tilde{x}| \leqslant 2^{k/3},\ |\tilde{y}| \leqslant 2^{2k/3},\ |\tilde{z}| \leqslant 2^k\}$$

where $(\tilde{x}, \tilde{y}, \tilde{z})$ are coordinates with respect to the Frenet frame at $P_j = \gamma(j2^{-k/3})$ translated at the origin of the (x, y, z) coordinates. Outside R_{kj} an integration by parts shows that for any integer $M, M', M'' \geqslant 0$ there exists a constant $C_{M,M',M''}$ such that

(6) $$|\widehat{m_\delta \psi_{kj}}(\tilde{x}, \tilde{y}, \tilde{z})| \leqslant C_{M,M',M''} 2^{-k(2+\delta)} (2^{k/3}/|\tilde{x}|)^M (2^{2k/3}/|\tilde{y}|)^{M'} (2^k/|\tilde{z}|)^{M''}.$$

Therefore the operator T_{kj}^δ is dominated by the convolution operator whose kernel is given by

(7) $$C 2^{-k\delta} \sum_{h \geqslant 0} 2^{-h} |R_{kj}^h|^{-1} \chi_{R_{kj}^h}(x, y, z)$$

where R_{kj}^h denotes the box obtained from R_{kj} by a dilation of 2^h, namely

$$R_{kj}^h = \{(\tilde{x}, \tilde{y}, \tilde{z}) \in R^3 \colon |\tilde{x}| \leqslant 2^{h+k/3},\ |\tilde{y}| \leqslant 2^{h+2k/3},\ |\tilde{z}| \leqslant 2^{h+k}\}.$$

For every $k > 90$ we are going to estimate the norm of T_k^δ as an operator from L_6 to itself. We shall prove the following

LEMMA 1. *If T_k^δ is the operator defined in* (4) *then for every $k > 90$ there exists a constant ν such that*

(8) $$\|T_k^\delta f\|_6 \leqslant C 2^{-k(\delta - 1/18)} k^\nu \|f\|_6.$$

Proof. Let us write T_k instead of T_k^δ. By Plancherel theorem we have

$$\int |T_k f(\vec{x})|^6 \, d\vec{x} = \int |\sum_j T_{kj} f(\vec{x})|^6 \, d\vec{x} = \int |\sum_{j_1, j_2, j_3} T_{kj_1} f(\vec{x}) \, T_{kj_2} f(\vec{x}) \, T_{kj_3} f(\vec{x})|^2 \, d\vec{x}$$

$$= \int |\sum_{j_1, j_2, j_3} \widehat{T_{kj_1} f}(\vec{\xi}) * \widehat{T_{kj_2} f}(\vec{\xi}) * \widehat{T_{kj_3} f}(\vec{\xi})|^2 \, d\vec{\xi}$$

$$\leqslant C \sum_{j_1, j_2, j_3} \int |\widehat{T_{kj_1} f}(\vec{\xi}) * \widehat{T_{kj_2} f}(\vec{\xi}) * \widehat{T_{kj_3} f}(\vec{\xi})|^2 \, d\vec{\xi}$$

with C a constant independent of k. The last inequality follows from the fact that every point in R^3 belongs to at most a hundred of the following sets:

$$\text{support}\,(\widehat{T_{kj_1} f} * \widehat{T_{kj_2} f} * \widehat{T_{kj_3} f}) = \text{support}\,(\psi_{kj_1}) + \text{support}\,(\psi_{kj_2}) + \text{support}\,(\psi_{kj_3}),$$

$j_1, j_2, j_3 = [-2^{k/3}/5], \ldots, [2^{k/3}/5]$ (this is the connection with the restriction theorem [13]). Now we proceed as follows:

$$\int |T_k f(\vec{x})|^6 \, d\vec{x} \leqslant C \sum_{j_1, j_2, j_3} \int |T_{kj_1} f(\vec{x}) T_{kj_2} f(\vec{x}) T_{kj_3} f(\vec{x})|^2 \, d\vec{x}$$

$$= C \int \Big(\sum_j |T_{kj} f(\vec{x})|^2\Big)^3 d\vec{x} = C \Big\|\sum_j T_{kj} f(\vec{x})|^2\Big\|_3^3$$

$$\leqslant \Big(\sup_{\|g\|_{3/2} = 1} \int \sum_j |T_{kj} f(\vec{x})|^2 g(\vec{x}) \, d\vec{x}\Big)^3.$$

We split the sum over j into the sum over j odd and j even. The two terms can be treated in the same way. So let j be even. We can find a configuration of congruent disjoint squares Q_{kj} in the plane $z = y$ (notice that it is parallel to $\gamma'(0)$) lined up in the direction of the vector $(0, 1, 1)$ and equally spaced such that if E_{kj} denotes the strip orthogonal to the plane $z = y$ whose projection on it is Q_{kj} then

$$\chi_{E_{kj'}}(\vec{\xi}) \psi_{kj}(\vec{\xi}) = \psi_{kj}(\vec{\xi}) \quad \text{if} \quad j = j',$$
$$\chi_{E_{kj'}}(\vec{\xi}) \psi_{kj}(\vec{\xi}) = 0 \quad \text{if} \quad j \neq j'.$$

We denote by S_{kj} the operator defined on C_0^∞ functions by $\widehat{S_{kj} f}(\vec{\xi}) = \chi_{E_{kj}}(\vec{\xi}) \hat{f}(\vec{\xi})$. Evidently $S_{kj} T_{kj} = T_{kj}$. Now observe that it is enough to deal with the convolution operator whose kernel is $\sum_j 2^{-k\delta} |R_{kj}|^{-1} \chi_{R_{kj}}(\vec{x})$ instead of $\widehat{m_\delta \psi_k}(\vec{x})$ because of the exponential decay 2^{-h} in (7). If we do so and $g \geqslant 0$, by Lemma 1 of [9] we have

$$\int \sum_j |T_{kj} f(\vec{x})|^2 g(\vec{x}) \, d\vec{x} = \sum_j \int |S_{kj} T_{kj} f(\vec{x})|^2 g(\vec{x}) \, d\vec{x}$$

$$\leqslant C \sum_j \int |2^{-k\delta} |R_{kj}|^{-1} \chi_{R_{kj}} * S_{kj} f(\vec{x})|^2 g(\vec{x}) \, d\vec{x}$$

$$\leqslant C 2^{-2k\delta} \Big\|\Big(\sum_j |S_{kj} f(\vec{x})|^2\Big)^{1/2}\Big\|_6^2 \, \|M^1_{2k/3} g(\vec{x})\|_{3/2}.$$

Therefore by Theorem 1 and the following Lemma 2 we obtain

(9) $$\int |T_k f(\vec{x})|^6 \, d\vec{x} \leqslant C \Big\|\sum |T_{kj} f|^2\Big\|_3^3$$

$$\leqslant C 2^{-6k\delta} \Big\|\Big(\sum_j |S_{kj} f|^2\Big)^{1/2}\Big\|_6^6 \sup_{\substack{\|g\|_{3/2} = 1 \\ g \geqslant 0}} \|M^1_{2k/3} g\|_{3/2}^3$$

$$\leqslant C 2^{-6k(\delta - 1/18)} k^{3\alpha(3/2)} \|f\|_6^6.$$

This proves (8) once we have the following lemma.

LEMMA 2. *Let the operators S_{kj} be defined as above. Then for $p \geqslant 2$ the following inequality holds*:

$$\|(\sum_j |S_{kj} f|^2)^{1/2}\|_p \leqslant C_p \|f\|_p.$$

Proof. The proof follows the same lines as the proof of Theorem 18 of [5] to which we refer the reader.

The following lemma is due to A. Cordoba.

LEMMA 3. *For every $k > 90$ there exists a positive constant ϱ such that*

(10) $$\|T_k^\delta f\|_4 \leqslant C 2^{-k\delta} k^\varrho \|f\|_4.$$

Proof. In (9) we proved that $\|T_k^\delta f\|_6 \leqslant C \|(\sum_j |T_{kj}^\delta f|^2)^{1/2}\|_6$. On the other hand, it is obvious that

$$\|T_k^\delta f\|_2 \leqslant C \|(\sum_j |T_{kj}^\delta f|^2)^{1/2}\|_2.$$

Hence by interpolation

$$\|T_k^\delta f\|_4 \leqslant C \|(\sum_j |T_{kj}^\delta f|^2)^{1/2}\|_4 \leqslant C 2^{-k\delta} \|(\sum_j |S_{kj} f|^2)^{1/2}\|_4 \{\operatorname*{Sup}_{\substack{\|g\|_2 = 1 \\ g \geqslant 0}} \|M^1_{2k/3} g\|_2\}^{1/2}.$$

Now Lemma 3 follows by applying Theorem 1 and Lemma 2.

We shall give now the

Proof of Theorem 2. (A), (B), (C) follow respectively by interpolation between the trivial estimate $\|T_k^\delta f\|_1 \leqslant C 2^{-k(\delta - 1/3)} \|f\|_1$ and the dual estimate of (8), between the dual estimate of (8) and the dual estimate of (10), between the dual estimate of (10), and the trivial estimate in L_2 and by adding a geometric series.

Because of Theorem 1 we expect Theorem 2 to be sharp in the range $6/5 \leqslant p \leqslant 6$. If so, in the $(1/p, \delta)$-plane, the shape of the region in which Theorem 2 holds is strikingly different from the analogous one for Bochner–Riesz operators.

Section 2. We are going to study the behavior at infinity of $K_\delta(\vec{x})$, the kernel of the convolution operator T_δ. As a consequence we will have that $K_\delta(\vec{x})$ is integrable for $\delta > 1/3$. To the preceding notations we add the following ones. Let $D_k = \bigcup_j R_{kj}$, $D_{kh} = \bigcup_j R_{k+h,j} \setminus \bigcup_j R_{k+h-1,j}$ $(h \geqslant 1)$ and let $A_k = \{\vec{x} \in R^3 \colon 2^{k-1} \leqslant \|\vec{x}\| \leqslant 2^k\}$. Write $K(\vec{x}) = K_\delta(\vec{x})$. Then we have the following

PROPOSITION 1. *There exists an integer k_0 such that for $k \geqslant k_0$ and any integer $h \geqslant 1$ the following estimate is true*:

(11) $$|K(\vec{x})| \leqslant \begin{cases} C 2^{-k(\delta+2)} & \text{if } \vec{x} \in D_k \cap A_k, \\ C 2^{-(k+h)(\delta+2)} 2^h & \text{if } \vec{x} \in D_{kh} \cap A_k. \end{cases}$$

Therefore $K(\vec{x})$ belongs to $L_1(R^3)$ for $\delta > 1/3$.

Proof. As in (4) we split $K(\vec{x}) = \sum_{\substack{k > 90 \\ j}} K_{kj}(\vec{x}) + K_0(\vec{x})$. If k_0 is big enough then in $A_{\bar{k}}$, $\bar{k} \geqslant k_0$, we have $|K_0(\vec{x})| \leqslant |\sum_{k,j} K_{kj}(\vec{x})|$. Let us estimate the size of $\sum_{k,j} K_{kj}(\vec{x})$ in $A_{\bar{k}}$. To do so, let us recall that for $\vec{x} \in R_{kj}$ we have $|K_{kj}(\vec{x})| \leqslant C 2^{-k(\delta+2)}$ and that outside R_{kj} estimate (6) holds. Now since the $R_{\bar{k}j}$'s are essentially disjoint in $A_{\bar{k}}$, we have that the main contribution to the size of $K(\vec{x})$ in $D_{\bar{k}} \cap A_{\bar{k}}$ is due to $\sum_j K_{\bar{k}j}(\vec{x})$. By this we mean that

$$|\sum_{\substack{k \neq \bar{k} \\ j}} K_{kj}(\vec{x})| \leqslant C |\sum_j K_{\bar{k}j}(\vec{x})|, \quad \vec{x} \in D_{\bar{k}} \cap A_{\bar{k}}.$$

Hence the first part of (11) is proved. Similarly in $D_{\bar{k}h} \cap A_{\bar{k}}$ the main contribution is due to $\sum_j K_{\bar{k}+h,j}(\vec{x})$. This time, though the $R_{\bar{k}+h,j}$'s are not disjoint in $A_{\bar{k}}$, the estimate has to be modified by a factor of 2^h which takes into account the overlapping. Therefore (11) is proved. Now it is not difficult to see that $\|K(\vec{x})\|_{L_1(A_{\bar{k}})} \leqslant C 2^{-\bar{k}(\delta+2)} 2^{7\bar{k}/3}$ which implies that $K(\vec{x})$ is integrable for $\delta > 1/3$.

Next we are going to study the behavior at infinity of the Fourier transform of the uniform measure on the curve (t, t^2, t^3), $0 \leqslant t \leqslant 1/10$. We shall need Van der Corput's Lemma which sounds as follows (see [12], p. 220):

(a) *Let φ be real valued on $[a, b]$ and assume that it has a monotone derivative $\varphi'(t) > \varrho > 0$ on $[a, b]$. Then $|\int_a^b e^{i\varphi(t)}\, dt| < 2/\varrho$;*

(b) *Instead of assuming $\varphi' > \varrho$ on $[a, b]$, assume that φ is twice differentiable and that $\varphi''(t) > k > 0$ on $[a, b]$. Then $|\int_a^b e^{i\varphi(t)}\, dt| < 6/k^{1/2}$.*

Now let us prove the following

LEMMA 4. *Let $\hat{v}(x, y) = \int_0^{1/10} e^{i(xt + yt^3)}\, dt$. Then using polar coordinates (r, θ), if $r > 10^{10}$ the following estimate holds*:

(12)

$$|\hat{v}(r, \theta)| \leqslant \begin{cases} C r^{-1/3} & \text{if } |\theta - \pi/2| < (3r^{2/3})^{-1} \text{ or } |\theta - 3\pi/2| < (3r^{2/3})^{-1}, \\ C r^{-1/2} |\cos\theta|^{-1/4} & \text{otherwise.} \end{cases}$$

Proof. We are going to prove the lemma for $0 \leqslant \theta \leqslant \pi$. In a similar way one can take care of the range $\pi \leqslant \theta \leqslant 2\pi$. Write

$$\int_0^{1/10} e^{i(xt + yt^3)}\, dt = \int_0^{(3r^{1/3})^{-1}} e^{i(xt + yt^3)}\, dt + \int_{(3r^{1/3})^{-1}}^{1/10} e^{i(xt + yt^3)}\, dt = I_1 + I_2.$$

Write $\varphi(t) = xt + yt^3 = r(t\cos\theta + t^3 \sin\theta)$ and observe that $\varphi'(t)$ is an increasing function of t for every θ fixed. First we study I_1. If $0 \leqslant \theta < \pi/2 - \pi/10^4$ then $\varphi'(t) > r/10^5$; hence by (a) $|I_1| \leqslant 20^5/r$ in this range of θ. If

$\pi/2+\pi/10^4 \leqslant \theta \leqslant \pi$ then $\varphi'(t) < -r/20^5$; hence by (a) $|I_1| \leqslant 40^5/r$ in this range of θ. If instead $|\theta-\pi/2| \leqslant r^{-2/3}$ then obviously $|I_1| \leqslant (3r^{1/3})^{-1}$. Finally if $2^n r^{-2/3} \leqslant |\theta-\pi/2| \leqslant 2^{n+1} r^{-2/3}$ with $0 \leqslant n \leqslant \lg(r^{2/3}\pi/10)$ then $\varphi'(t) > 10^{-1} 2^n r^{1/3}$ for $\theta \leqslant \pi/2$ and $\varphi'(t) < -10^{-1} 2^n r^{1/3}$ for $\theta \geqslant \pi/2$. In any case $|I_1| \leqslant 20\, 2^{-n} r^{-1/3}$ in the range of θ under consideration.

Now we are going to study I_2. In the region $|\pi/2-\theta| < r^{-2/3}$, I_2 satisfies the same estimate as I_1. To show this, it is convenient to break up I_2 in the following way:

$$I_2 = \int_{(3r^{1/3})^{-1}}^{10r^{-1/3}} e^{i\varphi(t)}\,dt + \int_{10r^{-1/3}}^{1/10} e^{i\varphi(t)}\,dt = I_3 + I_4.$$

Evidently $|I_3| \leqslant 10r^{-1/3}$. Moreover $|I_4| \leqslant 10r^{-1/3}$ since $\varphi'(t) > r^{1/3}$. This takes care of the range of θ we were studying. In the range $0 < \theta \leqslant \pi/4$, $3\pi/4 \leqslant \theta < \pi$, respectively, $\varphi'(t) > r\sqrt{2}/2$ and $\varphi'(t) < -r/100$. In any case $|I_2| \leqslant 200/r$. To study I_2 in the region $\pi/4 \leqslant \theta \leqslant \pi/2 - r^{-2/3}$ it is convenient to further subdivide it as we did before into $\pi/2 - 2^{n+1} r^{-2/3} < \theta \leqslant \pi/2 - 2^n r^{-2/3}$ for $n = 0, \ldots, [\lg(r^{2/3}\pi/10^4)]$. Here $\varphi'(t) > 2^n r^{1/3}$. Hence $|I_2| \leqslant 2^{-n+1} r^{-1/3}$. It remains to study I_2 in the region $\pi/2 + r^{2/3} \leqslant \theta \leqslant 3\pi/4$. Observe that $\varphi'(t) = 0$ for $t = t_\theta = (-\cos\theta/3\sin\theta)^{1/2}$. If t_θ belongs to $[(3r^{1/3})^{-1}, 1/10]$, then we split I_2 in the following way

$$I_2 = \int_{(3r^{1/3})^{-1}}^{99t_\theta/100} e^{i\varphi(t)}\,dt + \int_{99t_\theta/100}^{101t_\theta/101} e^{i\varphi(t)}\,dt + \int_{101t_\theta/100}^{1/10} e^{i\varphi(t)}\,dt = I_5 + I_6 + I_7$$

(the case $t_\theta < (3r^{1/3})^{-1}$ is easier to handle and it can be treated similarly). We start with I_6. Observe that $\varphi''(t) = 6rt\sin\theta > 100^{-1}(-\cos\theta)^{1/2} r$. By (b) we have $|I_6| \leqslant 600 r^{-1/2} |\cos\theta|^{-1/4}$. For I_5 observe that $\varphi'(t) < 10^{-4} r\cos\theta$. Hence $|I_5| \leqslant 20^4 r^{-1} |\cos\theta|^{-1}$. For I_7 observe that $\varphi'(t) > -10^{-4} r\cos\theta$ which implies that I_7 satisfies the same estimate as I_5. Hence (12) has been proved.

Let us denote $\hat{\mu}(x, y, z) = \int_0^{1/10} e^{i(xt+yt^2+zt^3)}\,dt$, that is, $\hat{\mu}$ is the Fourier transform of the uniform measure on the curve (t, t^2, t^3), $0 \leqslant t \leqslant 1/10$. We are going to prove the following

PROPOSITION 2. *$\hat{\mu}$ belongs to $L_p(R^3)$ for $p > 7$.*

Proof. By the change of variables $t = u - y/3z$, $\hat{\mu}$ can be written as follows:

$$\hat{\mu}(x, y, z) = \sigma \int_{y/3z}^{1/10+y/3z} e^{i((x-y^2/3z)u + zu^3)}\,du$$

where σ is a number with module one. (Notice that $\hat{\mu}(x, y, 0)$ is the Fourier transform of the uniform measure on the curve (t, t^2), $0 \leqslant t \leqslant 1/10$ and it is well known – see e.g. [11] – that $|\hat{\mu}(r\cos\theta, r\sin\theta)| \leqslant Cr^{-1/2}$.) Hence we are led to study the following integral:

$$I(u, z) = \int_a^{a+1/10} e^{i(wu+zu^3)}\,du, \qquad a \in R.$$

If $a \geqslant 10^{-4}$ or $a \leqslant -10^{-4} - 10^{-1}$ then $I(w, z)$ is the Fourier transform of the

uniform measure on (t, t^3), $a \leqslant t \leqslant a + 1/10$. Such a curve has curvature bounded away from zero; hence $|I(r\cos\theta, r\sin\theta)| \leqslant Cr^{-1/2}$ (see e.g. [11]). If $-10^{-1} - 10^{-4} \leqslant a \leqslant 10^{-4}$ then (12) holds as the preceding lemma or a slight modification of it shows. Let us observe that Lemma 4 says that the decrease at infinity of $\hat{\mu}$ becomes worse as we approach the cone $3xz - y^2 = 0$. Observe that the binormals of (t, t^2, t^3), $0 \leqslant t \leqslant 1/10$ have the directions of the vectors $(6t^2, -6t, 2)$ and that if we set $x_0 = 6t^2$, $y_0 = -6t$, $z_0 = 2$ then $x_0 - y_0/3z_0 = 0$. Now we are going to estimate the L_p-norm of $\hat{\mu}$ restricted to the set $\{\vec{x} \in R^3\colon 2^{n-1} \leqslant \|\vec{x}\| \leqslant 2^n\}$, $n \geqslant 2^{100}$. If φ_0 is any direction of the cone $3xz - y^2 = 0$ then on the set

$$\bigcup_{\varphi_0} \{(r, \varphi, \theta)\colon 2^{n-1} \leqslant r \leqslant 2^n,\ |\varphi - \varphi_0| \leqslant 2^{-2n/3}\},$$

we have $|\mu| \leqslant C2^{-n/3}$. On the sets

$$\bigcup_{\varphi_0} \{(r, \varphi, \theta)\colon 2^{n-1} \leqslant r \leqslant 2^n,\ 2^{i-2n/3} \leqslant |\varphi - \varphi_0| \leqslant 2^{i+1-2n/3}\}$$

we know that $|\hat{\mu}| \leqslant C2^{-n/2}(2^{2n/3} 2^{-i})^{1/4}$ for $i = 0, \ldots, [\lg(2^{2n/3}\pi/10^4)]$. Hence

$$\|\hat{\mu}\|^p_{L_p(2^{n-1} \leqslant \|\vec{x}\| \leqslant 2^n)} \leqslant C2^{-n(p-7)/3}.$$

Therefore $\hat{\mu}$ belongs to L_p for $p > 7$ and the proposition is proved.

Remark. P. Sjölin recently informed us that in [16] he proved a stronger estimate than (12) using different methods. In the range $\pi/2 - \delta < \theta < \pi/2 - r^{-2/3}$, $3\pi/2 - \delta < \theta < 3\pi/2 - r^{-2/3}$ (for some $\delta > 0$) his estimate reads

$$|\hat{v}(r, \theta)| = c_0 \psi(\cos^{1/2}\theta)/r^{1/2}(\cos\theta)^{1/4} + \mathcal{O}((r\cos\theta)^{-1})$$

where c_0 is a fixed constant, $\psi(t)$ is $C_0^\infty(-\delta, \delta)$ and bounded away from zero in $(-\delta/2, \delta/2)$. This implies that $\hat{\mu}$ does not belong to L_p for $p \leqslant 7$.

Lemma 4 and the preceding Remark allow us to conjecture, by analogy with the R^2 case, that the sharp restriction theorem for smooth curves in R^3, with never vanishing curvature and torsion, involves an $(L_{7/6-\varepsilon}, L_{7/6})$ estimate, $\varepsilon > 0$ a small number.

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Banach S-algebras and conditional basic sequences in non-Montel Fréchet spaces

by

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Abstract. A non-associative multiplication is defined on Banach spaces with symmetric basis. This multiplication is continuous exactly when each basic sequence generated by one vector is equivalent to the original basis. Upper and lower l_p-estimates are proved for such algebra norms. As an application, these results are combined with the techniques of Figiel, Lindenstrauss and Milman to produce conditional basic sequences in a large class of non-Montel Fréchet spaces. This class includes subspaces of l_p-Köthe sequence spaces and subspaces of products of superreflective spaces. This partially answers a question of Pełczyński.

Altshuler, in [1], studied the class of Banach spaces X with a symmetric basis $\{x_n\}$ which have the further property that each basic sequence generated by one vector is equivalent to $\{x_n\}$. We show (Proposition 3.1) that such spaces X are exactly those which can be re-normed into a Banach S-algebra; that is, there is a (non-associative) multiplication which singles out this class of Banach spaces with a symmetric basis. The algebra norm of a Banach S-algebra must satisfy some l_p-estimates (Theorem 3.2). In fact, for each such X there is a p with $1 \leqslant p \leqslant \infty$, so that for each $q > p$ there is a constant C_q so that

$$C_q\left(\sum|\alpha_n|^q\right)^{1/q} \geqslant \left\|\sum\alpha_n x_n\right\| \geqslant \left(\sum|\alpha_n|^p\right)^{1/p}$$

for any scalar sequence $\{\alpha_n\}$. (The lower estimate is essentially in Altshuler [1].)

Thus a Banach S-algebra can replace some l_q and still preserve the ordering (as sets of sequence spaces) of the l_p-spaces. In Section 2, such spaces are defined to have index q. The techniques of Figiel, Lindenstrauss and Milman [8] applied to spaces of index $q < \infty$, yield "nearly" the same Dvoretzky-type results as obtained for l_q in Example 3.1 of [8] (Proposition 2.7).

In Section 4, these results are combined to affirmatively answer the following question of Pełczyński [13] for "most" non-Montel spaces:

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