des fonctions entières de deux variables (voir [2], Theorem 2.15, ou [10], pp. 329–331).

Le théorème 2.6 permet d'améliorer les résultats de [5] de la façon suivante:

COROLLAIRE 2.7. *Soit $\lambda \to K(\lambda)$ une fonction analytique multiforme sur* **C**. *Supposons que pour tout $\lambda \in C$, $K(\lambda)$ ait au plus 0 comme point limite (resp. $K(\lambda)$ soit fini ou dénombrable). Alors ou bien $K(\lambda)$ est constant ou bien il existe un ensemble fermé F de capacité nulle tel que pour $z \notin F$ l'ensemble des λ tels que $z \in K(\lambda)$ soit discret (resp. dénombrable) et non vide.*

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Computing norms and critical exponents of some operators in L^p-spaces

by

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Abstract. Among other results we prove the following:

1° Let $(f_j)_{1\leqslant j\leqslant n}$ be a sequence in L^p such that $\|f_j\|_p = 1$ for $j = 1, 2, \ldots, n$ and

$$\|\sum c_j f_j\|_p = \|\sum c_j \cdot \alpha_j f_{\pi(j)}\|_p$$

for every sequence of scalars (c_j), for every sequence of unimodular scalars (α_j), and for every permutation π of the indices. Then

$$\|\sum_{j=1}^n c_j f_j\|_p^p \geqslant n^{-1} \|\sum_{j=1}^n f_j\|_p^p \sum_{j=1}^n |c_j|^p \quad \text{for} \quad 1 < p < 2,$$

$$\|\sum_{j=1}^n c_j f_j\|_p^p \leqslant n^{-1} \|\sum_{j=1}^n f_j\|_p^p \sum_{j=1}^n |c_j|^p \quad \text{for} \quad 2 < p < \infty.$$

2° Let E be a finite dimensional subspace of L^∞. Let P denote either P_E – the orthogonal projection onto E or P^E – the orthogonal projection onto the orthogonal complement of E. Then $\sup\{p: \|P\|^{\infty,p} = 1\} > 2$ where $\|P\|^{\infty,p}$ denotes the norm of P regarded as an operator from L^∞ into L^p.

In particular, if $E = \operatorname{span}\{1, e^{it}\}$, then $\sup\{p: \|P_E\|^{\infty,p} = 1\} = 4$.

Introduction. Evaluation of norms of linear operators in Banach spaces requires various techniques like the Lagrange multipliers, variational methods, combinatorial calculations, etc. It is often related to finding the best constants in classical inequalities.

The present paper consists of three loosely connected parts.

The first one has a general character. Roughly speaking, we consider there the following problem: Given an operator acting between real L^p-spaces. Under what condition the complexification of the operator has the same norm as the original operator? The norms are the same in the case where the operator acts from L^p-space into L^q-space for $p \leqslant q$. Our general result is stated in terms of "mixed norms". For $p = q$ this fact has been established by F. W. Levi [5].

A consequence of the main result of Section 2 is the following inequality:

Given n Rademacher functions $r_1, r_2, \ldots, r_n$ ($n = 1, 2, \ldots$); then for arbitrary scalars $c_1, c_2, \ldots, c_n$,

$$\int_0^1 \Big|\sum_{j=1}^n c_j r_j(t)\Big|^p dt \geqslant n^{-1} \int_0^1 \Big|\sum_{j=1}^n r_j(t)\Big|^p dt \sum_{j=1}^n |c_j|^p \quad (1 < p < 2),$$

$$\int_0^1 \Big|\sum_{j=1}^n c_j r_j(t)\Big|^p dt \leqslant n^{-1} \int_0^1 \Big|\sum_{j=1}^n r_j(t)\Big|^p dt \sum_{j=1}^n |c_j|^p \quad (2 < p < \infty).$$

Our Theorem 2.1 asserts that in the above inequalities the Rademacher functions can be replaced by any 1-symmetric basic sequence normalized in L^p (in particular, by a sequence of equally distributed independent random variables). Moreover, the inequality becomes the equality either for all sequences of scalars or only for sequences of scalars whose absolute values are equal.

In the third section we study the following phenomenon. Let E be a finite dimensional subspace of L^∞. Let P_E denote the orthogonal (in L^2) projection onto E. Let $\|P_E\|^{\infty,p}$ denote the norm of P_E regarded as an operator from L^∞ into L^p. Then there is a $p > 2$ such that $\|P_E\|^{\infty,p} = 1$. Hence if E contains a unimodular function then the function $p \to \|P_E\|^{\infty,p}$ is constant in some interval $2 \leqslant p \leqslant p_0$ and is strictly increasing for $p > p_0$. We call the lower upper bound of p such that $\|P_E\|^{\infty,p} = 1$ the *critical exponent* of P_E. We prove that the critical exponent always exists for P_E as for P^E = "the orthogonal projection on the complement of E". We discuss some special cases; in particular, for the projection

$$f \to \frac{1}{2\pi} \int_0^{2\pi} f(t)\,dt + \frac{e^{is}}{2\pi} \int_0^{2\pi} f(t) e^{-it}\,dt$$

the critical exponent equals 4. This fact requires some delicate computation involving Gauss' hypergeometric series.

Finally, in the Appendix we consider the problem of maximizing the functional

$$\int_0^1 g(t)\,dt \cdot \int_0^1 [g(t)]^p\,dt \cdot \Big(\int_0^1 [g(t)]^2\,dt\Big)^{-1}$$

for non-increasing g satisfying the conditions $1 \geqslant g \geqslant 0$ and $\int_0^1 g(t)\,dt = M^{-1}$ ($M \geqslant 1$ is fixed). It is shown that the problem has a "bang-bang" solution.

1. The norms of complexification and tensor products of operators in L^p. Let (S, Ω, μ) be a measure space. Denote by $L^0(\mu)$ (resp. $L^0_R(\mu)$) the space of all μ-equivalence classes of μ-measurable complex-valued (resp. real-valued) functions on S. For $f \in L^0(\mu)$ we put

$$\|f\|_{L^p(\mu)} = \|f\|_p = \Big(\int |f(s)|^p \mu(ds)\Big)^{1/p} = \Big(\int |f|^p\,d\mu\Big)^{1/p} \quad (0 < p < \infty),$$
$$\|f\|_{L^\infty(\mu)} = \|f\|_\infty = \operatorname{ess\,sup}_{s \in S} |f(s)|.$$

Given measure spaces (S_j, Ω_j, μ_j) ($j = 1, 2$) we denote by $(S_1 \times S_2, \Omega_1 \times \Omega_2, \mu_1 \times \mu_2)$ the product measure space. For $f \in L^0(\mu_1 \times \mu_2)$ and for $p_j \in (0, \infty]$ ($j = 1, 2$) we put

$$\|f\|_{p_1,p_2} = \Big(\int\Big(\int |f(s_1, s_2)|^{p_1} \mu_1(ds_1)\Big)^{p_2/p_1} \mu_2(ds_2)\Big)^{1/p_2} \quad (\max_{j=1,2} p_j < \infty),$$
$$\|f\|_{\infty,p_2} = \Big\|\operatorname{ess\,sup}_{s_1 \in S_1} |f(s_1, s_2)|\Big\|_{L^{p_2}(\mu_2)} \quad (0 < p_2 \leqslant \infty).$$

We put

$$L^p(\mu) = \{f \in L^0(\mu)\colon \|f\|_p < \infty\},$$
$$L^p_R(\mu) = L^p(\mu) \cap L^0_R(\mu),$$
$$L^{p_1,p_2}(\mu_1 \times \mu_2) = \{f \in L^0(\mu_1 \times \mu_2)\colon \|f\|_{p_1,p_2} < \infty\},$$
$$L^{p_1,p_2}_R(\mu_1 \times \mu_2) = L^{p_1,p_2}(\mu_1 \times \mu_2) \cap L^0(\mu_1 \times \mu_2).$$

Clearly, $L^{p,p}(\mu_1 \times \mu_2) = L^p(\mu_1 \times \mu_2)$.

Recall that a non-negative functional $\|\cdot\|$ on a linear space X is a *quasi-norm* provided that $\|x\| = 0$ implies $x = 0$, $\|\alpha x\| = |\alpha|\,\|x\|$ for all $x \in X$ and all scalars α, and there is a $C > 0$ such that $\|x+y\| \leqslant C(\|x\| + \|y\|)$ for all $x, y \in X$.

A *quasi-normed space* $(X, \|\cdot\|)$ is a linear space X with a quasi-norm $\|\cdot\|$. A quasi-normed space is a linear metric space with the topology given by a basis of neighbourhoods of zero,

$$\{\{x \in X\colon \|x\| < c\}\colon c > 0\}.$$

It is well known that the spaces $L^p(\mu)$ and $L^{p_1,p_2}(\mu_1 \times \mu_2)$ and their linear subspaces are quasi-normed spaces; they are Banach spaces iff $p \geqslant 1$, resp. $\min(p_1, p_2) \geqslant 1$.

If $u\colon X \to Y$ is a linear operator acting between quasi-normed spaces then

$$\|u\| = \sup\{\|ux\|\colon \|x\| \leqslant 1\}.$$

If $\|u\| < \infty$ then u is said to be *bounded*.

Let E_j be a linear subspace of $L^0(\mu_j)$ ($j = 1, 2$). By $E_1 \otimes E_2$ we denote the linear subspace of $L^0(\mu_1 \times \mu_2)$ spanned by the set $\{e_1 \otimes e_2\colon e_1 \in E_1,\ e_2 \in E_2\}$ where $e_1 \otimes e_2(s_1, s_2) = e_1(s_1) e_2(s_2)$. If $E_1 \otimes E_2$ is contained in $L^{p_1,p_2}(\mu_1 \times \mu_2)$ then by $E_1 \underset{p_1,p_2}{\otimes} E_2$ we denote the closure of $E_1 \otimes E_2$ in $L^{p_1,p_2}(\mu_1, \mu_2)$. If $p_1 = p_2$ we write $E_1 \underset{p}{\otimes} E_2$ instead of $E_1 \underset{p,p}{\otimes} E_2$.

Let E_j and F_j be linear subspaces of $L^0(\mu_j)$ and $L^0(\nu_j)$, respectively, and let $u_j\colon E_j \to F_j$ be linear operators $(j = 1, 2)$. By $u_1 \otimes u_2\colon E_1 \otimes E_2 \to F_1 \otimes F_2$ we denote the unique linear operator such that $(u_1 \otimes u_2)(e_1 \otimes e_2) = u_1(e_1) \otimes u_2(e_2)$ for $e_1 \in E_1$ and $e_2 \in E_2$. Finally, by 1_X we denote the identity operator on a linear space X, and by $u_{|X_1}$ we denote the restriction of an operator $u\colon X \to Y$ to a subspace X_1 of X.

Now we are ready to state the main result of this section. (For a slightly weaker result cf. [7], Lemma 2.)

THEOREM 1.1. *Let $0 < p \leqslant q$. Let (S, Σ, μ) and (T, Ω, ν) be measure spaces. Let E be a linear subspace of $L^p(\mu)$ and let $u\colon E \to L^q(\nu)$ be a bounded linear operator. Then for every measure space (Z, Λ, σ) the operator*

$$1_{L^p(\sigma)} \otimes u\colon L^p(\sigma) \otimes E \to L^p(\sigma) \otimes L^q(\nu)$$

has the unique extension to the bounded linear operator

$$^{L^p(\sigma)}u\colon L^p(\sigma) \underset{p}{\otimes} E \to L^{p,q}(\sigma \times \nu).$$

Moreover, $\|^{L^p(\sigma)}u\| = \|u\|$.

Proof. First observe that without loss of generality one may assume 1° E is finite dimensional, 2° all measures in question are sigma-finite. Indeed, given a quasi-normed space X, denote by $\mathscr{F}(X)$ the family of all finite dimensional subspaces of X. Now for 1° observe that for every subspace E of $L^p(\mu)$,

$$\|u\| = \sup\{\|u_{|F}\|\colon F \in \mathscr{F}(E)\},$$

$$\|1_{L^p(\sigma)} \otimes u\| = \sup\{\|1_{L^p(\sigma)} \otimes u_{|F}\|\colon F \in \mathscr{F}(E)\}.$$

For 2° note that for every measure τ any separable, in particular, any finite dimensional subspace of $L^p(\tau)$, is isometrically isomorphic to a subspace of $L^p(\tau_0)$ for a sigma-finite measure τ_0; furthermore,

$$\|1_{L^p(\tau)} \otimes u\| = \sup\{\|1_F \otimes u\|\colon F \in \mathscr{F}(L^p(\tau))\}.$$

Assuming 1° and 2° we begin with the case $0 < p \leqslant q < \infty$. Let $f_1, f_2, \ldots, f_n$ be a basis for E, let $g_j = u(f_j)$ for $j = 1, 2, \ldots, n$. Then for every scalars $c_1, c_2, \ldots, c_n$ one has

$$\Big\|\sum_{j=1}^n c_j g_j\Big\|_{L^q(\nu)} \leqslant \|u\| \Big\|\sum_{j=1}^n c_j f_j\Big\|_{L^p(\mu)}.$$

Hence for arbitrary functions $c_1(\cdot), \ldots, c_n(\cdot)$ in $L^p(\sigma)$,

$$\Big(\int_T \Big|\sum_{j=1}^n c_j(z) g_j(t)\Big|^q \nu(dt)\Big)^{p/q} \leqslant \|u\|^p \int_S \Big|\sum_{j=1}^n c_j(z) f_j(s)\Big|^p \mu(ds) \quad \text{for } z \ \sigma\text{-a.e.}$$

Integrating against $d\sigma$ we obtain

$$\int_Z \Big(\int_T \Big|\sum_{j=1}^n c_j(z) g_j(t)\Big|^q \nu(dt)\Big)^{p/q} \sigma(dz) \leqslant \|u\|^p \int_Z \int_S \Big|\sum_{j=1}^n c_j(z) f_j(s)\Big|^p \mu(ds)\,\sigma(dz). \tag{1}$$

Using the Fubini Theorem to the right-hand side of (1) (this is rigorous because of assumption 2°) we get

$$\int_Z \int_S \Big|\sum_{j=1}^n c_j(z) f_j(s)\Big|^p \mu(ds)\,\sigma(dz) = \int_S \Big\|\sum_{j=1}^n c_j(\cdot) f_j(s)\Big\|_{L^p(\sigma)}^p \mu(ds). \tag{2}$$

Next we estimate from below the left-hand side of (1). Put $\alpha = q/p > 1$ and define $\Psi\colon Z \to L^\alpha(\nu)$ by

$$\Psi(z) = \Big|\sum_{j=1}^n c_j(z) g_j(\cdot)\Big|^p \quad \text{for} \quad z \in Z.$$

Then for the left-hand side of (1) we have the identity

$$\int_Z \Big(\int_T \Big|\sum_{j=1}^n c_j(z) g_j(t)\Big|^q \nu(dt)\Big)^{p/q} \sigma(dz) = \int_Z \|\Psi(z)\|_{L^\alpha(\nu)} \sigma(dz).$$

Using the integral version of the Minkowski inequality we get

$$\int_Z \|\Psi(z)\|_{L^\alpha(\nu)} \sigma(dz) \geqslant \Big\|\int_Z \Psi(z)\,\sigma(dz)\Big\|_{L^\alpha(\nu)}. \tag{3}$$

(Here we use that $\alpha = q/p > 1$.) Rewriting the right-hand side of (3), we get

$$\begin{aligned} \Big\|\int_Z \Psi(z)\,\sigma(dz)\Big\|_{L^\alpha(\nu)} &= \Big(\int_T \Big(\int_Z \Big|\sum_{j=1}^n c_j(z) g_j(t)\Big|^p \sigma(dz)\Big)^{q/p} \nu(dt)\Big)^{p/q} \\ &= \Big(\int_T \Big\|\sum_{j=1}^n c_j(\cdot) g_j(t)\Big\|_{L^p(\sigma)}^q \nu(dt)\Big)^{p/q}. \end{aligned} \tag{4}$$

Combining (1)–(4), we obtain

$$\Big(\int_T \Big\|\sum_{j=1}^n c_j(\cdot) g_j(t)\Big\|_{L^p(\sigma)}^q \nu(dt)\Big)^{1/q} \leqslant \|u\| \Big(\int_S \Big\|\sum_{j=1}^n c_j(\cdot) f_j(s)\Big\|_{L^p(\sigma)}^p \mu(ds)\Big)^{1/p};$$

equivalently,

$$\Big\|\sum_{j=1}^n c_j \otimes g_j\Big\|_{p,q} \leqslant \|u\| \Big\|\sum_{j=1}^n c_j \otimes f_j\Big\|_p.$$

The last inequality, in view of arbitrariness of $c_1(\cdot), \ldots, c_n(\cdot)$ in $L^p(\sigma)$ and the definition of the g_j's, yields

$$\|1_{L^p(\sigma)} \otimes u\| = \sup\Big\{\Big\|\sum_{j=1}^n c_j \otimes g_j\Big\|_{p,q}\colon \Big\|\sum_{j=1}^n c_j \otimes f_j\Big\|_p = 1\Big\} \leqslant \|u\|.$$

Since the reverse inequality is trivial and since the space $L^p(\sigma) \otimes E$ is dense in $L^p(\sigma) \underset{p}{\otimes} E$, we get the desired conclusion for $0 < p \leqslant q < \infty$. The proof in the case $q = \infty$ is similar.

The following well-known fact completes Theorem 1.1.

THEOREM 1.2. *Let* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $0 < r \leqslant \infty$. *Let* (S, Σ, μ), (T, Ω, ν), (Z, Λ, σ) *be measure spaces. Let* E *be a linear subspace of* $L^p(\mu)$, *let* $u\colon E \to L^q(\nu)$ *be a bounded linear operator. Then the operator*

$$u \otimes 1_{L^r(\sigma)}\colon E \otimes L^r(\sigma) \to L^q(\nu) \otimes L^r(\sigma)$$

has the unique extension to the bounded linear operator

$$u^{L^r(\sigma)}\colon E \underset{p,r}{\otimes} L^r(\sigma) \to L^{q,r}(\nu \times \sigma).$$

Moreover, $\|u^{L^r(\sigma)}\| = \|u\|$.

Proof. Pick $f_1, f_2, \ldots, f_n$ in E and $c_1, c_2, \ldots, c_n$ in $L^r(\sigma)$. Put $g_j = u(f_j)$ for $j = 1, 2, \ldots, n$. By the assumption,

$$\Big\|\sum_{j=1}^{n} c_j(z) g_j\Big\|_{L^q(\nu)} \leqslant \|u\| \Big\|\sum_{j=1}^{n} c_j(z) f_j\Big\|_{L^p(\mu)} \quad \text{for } z\ \sigma\text{-a.e.}$$

Hence for $r < \infty$,

$$\Big(\int_Z \Big\|\sum_{j=1}^{n} c_j(z) g_j\Big\|_{L^q(\nu)}^r \sigma(dz)\Big)^{1/r} \leqslant \|u\| \Big(\int_Z \Big\|\sum_{j=1}^{n} c_j(z) f_j\Big\|_{L^p(\mu)}^r\Big)^{1/r}$$

and for $r = \infty$

$$\operatorname*{ess\,sup}_{z\in Z} \Big\|\sum_{j=1}^{n} c_j(z) g_j\Big\| \leqslant \|u\| \operatorname*{ess\,sup}_{z\in Z} \Big\|\sum_{j=1}^{n} c_j(z) f_j\Big\|.$$

The last inequalities in view of arbitrariness of $c_1, \ldots, c_n$ in $L^r(\sigma)$ yield $\|u \otimes 1_{L^r(\sigma)}\| \leqslant \|u\|$. Since the reverse inequality is trivial, we get the desired conclusion.

COROLLARY 1.1. *Let* $0 < p \leqslant q \leqslant \infty$. *Let* $E_j \subset L^p(\mu_j)$ *be linear spaces and let* $u_j\colon E_j \to L^q(\nu_j)$ *be bounded linear operators* ($j = 1, 2$). *Then the operator*

$$u_1 \otimes u_2\colon E_1 \otimes E_2 \to L^q(\nu_1) \otimes L^q(\nu_2)$$

has the unique extension to a bounded linear operator

$$v\colon E_1 \underset{p}{\otimes} E_2 \to L^q(\nu_1 \times \nu_2).$$

Moreover, $\|v\| = \|u_1\|\,\|u_2\|$.

Proof. Let us put $v = u_1^{L^q(\nu_2)} \circ {}^{L^p(\mu_1)}u_{2|E_1 \underset{p}{\otimes} E_2}$. By Theorems 1.1 and 1.2, $\|v\| \leqslant \|u_1\|\,\|u_2\|$. Clearly, $v_{|E_1 \otimes E_2} = u_1 \otimes u_2$. Thus $\|v\| \geqslant \|u_1\|\,\|u_2\|$.

The next corollary generalizes a result of F. W. Levi [5], cf. also [6], p. 175.

COROLLARY 1.2. *Let* $0 < p \leqslant q \leqslant \infty$. *Let* $E \subset L^p(\mu)$ *be a linear space and let* $u\colon E \to L^q(\nu)$ *be a bounded linear operator. Then for every Hilbert space* $L^2(\tau)$ *the operator*

$$1_{L^2(\tau)} \otimes u\colon L^2(\tau) \otimes E \to L^2(\tau) \otimes L^q(\nu)$$

has the unique extension to a linear operator

$${}^{L^2(\tau)}u\colon L^2(\tau) \underset{2,p}{\otimes} E \to L^{2,q}(\tau \times \nu).$$

Moreover, $\|{}^{L^2(\tau)}u\| = \|u\|$.

Proof. For every $p \in (0, \infty]$ the space $L^2(\tau)$ is isometrically isomorphic to a subspace, say G, of $L^p(\sigma)$ for sufficiently rich measure space (Z, Λ, σ). Identifying $L^2(\tau)$ with G we put ${}^{L^2(\tau)}u = {}^{L^p(\sigma)}u_{|G \underset{p}{\otimes} E}$ where the operator ${}^{L^p(\sigma)}u$ is that of Theorem 1.1.

Remark. Theorems 1.1 and 1.2 and Corollaries 1.1 and 1.2 remain valid if complex spaces are replaced by appropriate real spaces.

To formulate the last corollary we need some notation. In the sequel we shall identify the space $L^p_{\mathbf{R}}(\mu)$ with the subset of $L^p(\mu)$ consisting of real-valued functions. If $E \subset L^p_{\mathbf{R}}(\mu)$ is a real linear space then by $\tilde{E}$ we denote the linear subspace of $L^p(\mu)$ generated by E. If $u\colon E \to L^q_{\mathbf{R}}(\nu)$ is a real linear operator then $\tilde{u}\colon \tilde{E} \to L^q(\nu)$ denotes the unique linear operator over the complex scalars which extends u. (Every element of $\tilde{E}$ is of the form $e_1 + ie_2$ for some $e_1, e_2 \in E$; we put $\tilde{u}(e_1 + ie_2) = u(e_1) + iu(e_2)$.) The space $\tilde{E}$ is called the *complexification of* E *in* $L^p(\mu)$ and the operator $\tilde{u}$ the *complexification of* u.

COROLLARY 1.3. *Let* $0 < p \leqslant q \leqslant \infty$. *Let* E *be a linear subspace of* $L^p_{\mathbf{R}}(\mu)$ *and let* $u\colon E \to L^q_{\mathbf{R}}(\nu)$ *be a real bounded linear operator. Then* $\|\tilde{u}\| = \|u\|$.

Proof. Let l_2^2 denote the 2-dimensional real Hilbert space. Then the spaces $\tilde{E}$ and $L^q(\nu)$ regarded as real spaces can be identified with $l_2^2 \underset{2,p}{\otimes} E$ and $l_2^2 \underset{2,q}{\otimes} L^q_{\mathbf{R}}(\nu)$, respectively; the operator $\tilde{u}$ regarded as an operator over the real scalars can be identified with ${}^{l_2^2}u$. The desired conclusion follows from the real counterpart of Corollary 1.2.

2. An inequality for 1-symmetric basic sequences in L^p. In this section we deal with diagonal operators acting from finite dimensional subspaces of L^p spaces spanned by independent symmetric random variables (like the Rademacher functions and the Steinhaus variables) into l_n^p. We prove some results which allow to compute the norm of such operators and their inverses.

We begin with recalling some concepts.

Let X be a Banach space over a scalar field K and let n be a positive integer. A sequence $x_1, x_2, \ldots, x_n$ in X is *normalized* if $\|x_j\| = 1$ for $j = 1$, 2, ..., n; is 1-*symmetric* if for every sequence of scalars $t_1, t_2, \ldots, t_n$ and for every permutation of the indices π: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$,

$$\|\sum t_j x_j\| = \|\sum t_j x_{\pi(j)}\|;$$

is 1-*unconditional* if

$$\|\sum t_j x_j\| = \|\sum \varepsilon_j t_j x_j\|$$

for every sequence $(t_j) \subset K$ and every sequence $(\varepsilon_j) \subset K$ with $|\varepsilon_j| = 1$ for $j = 1, 2, \ldots, n$.

It is well-known that a normalized 1-unconditional sequence is linearly independent.

By $[(f_j)_{j=1}^n]_p$, or briefly by $[(f_j)]_p$, we denote the linear subspace of $L^p(\mu)$ (resp. $L^p_{\mathbf{R}}(\mu)$) generated by a sequence $(f_j)_{j=1}^n \subset L^p(\mu)$ (resp. $(f_j)_{j=1}^n \subset L^p_{\mathbf{R}}(\mu)$). By Δ we denote the diagonal operator from $[(f_j)_{j=1}^n]_p$ into l^p_n which is defined by

$$\Delta\Big(\sum_{j=1}^{n} t_j f_j\Big) = (t_j)_{j=1}^n.$$

Now we are ready to formulate the main result of this section.

THEOREM 2.1. *Let $(f_j)_{j=1}^n$ be a normalized 1-symmetric and 1-unconditional sequence in $L^p(\mu)$ (resp. in $L^p_{\mathbf{R}}(\mu)$). Assume*:

(1) $1 < p < 2$. *Then*

(1a) $$\|\Delta\| = n^{1/p}\Big\|\sum_{j=1}^{n} f_j\Big\|_p^{-1},$$

(1b) *either the only elements in the unit ball of $[(f_j)]_p$ at which Δ attains its norm are of the form*

$$\Big(\sum_{j=1}^{n} \varepsilon_j f_j\Big)\Big\|\sum_{j=1}^{n} f_j\Big\|_p^{-1} \quad \textit{with} \quad |\varepsilon_j| = 1 \textit{ for } j = 1, 2, \ldots, n,$$

or the functions $f_1, f_2, \ldots, f_n$ have mutually disjoint supports.

(2) $2 < p < \infty$. *Then*

(2a) $$\|\Delta^{-1}\| = \Big\|\sum_{j=1}^{n} f_j\Big\|_p (n)^{-1/p},$$

(2b) *either the only elements in the unit ball of l^p_n at which Δ^{-1} attains its norm are of the form $(\varepsilon_j n^{-1/p})_{j=1}^n$ with $|\varepsilon_j| = 1$ for $j = 1, 2, \ldots, n$ or the functions $f_1, f_2, \ldots, f_n$ have mutually disjoint supports.*

Our proof of Theorem 2.1 involves several steps. An important role in the argument is played by the concept of exchangeable random variables. Recall that a sequence $(g_j)_{j=1}^n$ of real μ-measurable functions is said to be a sequence of *exchangeable and symmetric random variables* if for every open set $\Omega \subset \mathbf{R}^n$,

$$\mu\{t\colon (g_j(t))_{j=1}^n \in \Omega\} = \mu\{t\colon (\varepsilon_j g_{\pi(j)}(t))_{j=1}^n \in \Omega\}$$

for every sequence $\varepsilon = (\varepsilon_j)$ with $\varepsilon_j = \pm 1$ ($j = 1, 2, \ldots, n$) and for every permutation π: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

The next proposition (cf. [4], Lemma 1.2) goes back to P. Lévy.

PROPOSITION 2.1. *Given a normalized 1-symmetric and 1-unconditional sequence $(f_j)_{j=1}^n$ in $L^p_{\mathbf{R}}(\mu)$ there exist a probability measure space (S, Ω, ν) and a sequence $(g_j)_{j=1}^n \subset L^p_{\mathbf{R}}(\nu)$ of exchangeable and symmetric random variables such that the map $f_j \to g_j$ for $j = 1, 2, \ldots, n$ extends to the isometric isomorphism from $[(f_j)]_p$ onto $[(g_j)]_p$.*

Moreover if the f_j's have mutually disjoint supports then the g_j's have the same property.

Sketch of the proof. It is well-known that every finite dimensional subspace of $L^p_{\mathbf{R}}(\mu)$ is isometrically isomorphic to a finite dimensional subspace of $L^p_{\mathbf{R}}(\mu_0)$ for some probability measure μ_0. Moreover, this isometric isomorphism preserves disjointness of supports of the functions.

Let (T, Σ, μ_0) be a probability measure space. Let W denote the set of all $N = 2^n n!$ pairs (ε, π) where $\varepsilon = (\varepsilon_j)_{j=1}^n$ is a sequence with $\varepsilon_j = \pm 1$ ($j = 1, 2, \ldots, n$) and π: $\{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation. Let $S = \bigcup_{(\varepsilon,\pi)\in W} T_{(\varepsilon,\pi)}$ be the union of N different copies of T, $\Omega = \bigcup_{(\varepsilon,\pi)\in W} \Sigma_{(\varepsilon,\pi)}$. The measure ν on Ω is defined as follows: if $A \in \Omega$ then $A = A_{(\varepsilon,\pi)} \in \Sigma_{(\varepsilon,\pi)}$ for some $(\varepsilon, \pi) \in W$. We put $\nu(A) = N^{-1}\mu_{(\varepsilon,\pi)}(A_{(\varepsilon,\pi)})$ where $\mu_{(\varepsilon,\pi)}$ denotes the measure corresponding to μ_0 on $\Sigma_{(\varepsilon,\pi)}$. Finally, to $f_j \in L^p_{\mathbf{R}}(\mu_0)$ ($j = 1, 2, \ldots, n$) we assign the function $g_j \in L^p_{\mathbf{R}}(\nu)$ as follows:

$$g_j(s) = \varepsilon_j f_{\pi(j)}(t_{(\varepsilon,\pi)})$$

where for given $s \in S$, (ε, π) is the unique pair in W such that $s = t_{(\varepsilon,\pi)} \in T_{(\varepsilon,\pi)}$.

To simplify several formulae it is convenient to introduce the following notation. For $t \in R$ and $p > 0$ we put

(5) $$(t)^p = |t|^p \operatorname{sign} t.$$

With this notation we have:

If $\varphi(t) = |t|^p$, $p > 1$, $t \in R$, then

(6) $$\varphi'(t) = p(t)^{p-1}.$$

First we shall prove Theorem 2.1 for $1 < p < 2$ in the case of real functions and real scalars. To this end in view of Proposition 2.1 it is enough to prove

PROPOSITION 2.2. *Let* $1 < p < 2$. *Let* $(f_j)_{j=1}^n$ *be a normalized sequence of exchangeable and symmetric random variables in* $L^p_R(\mu)$ (μ — *a probability measure*). *Let*

$$m = \inf\{\|\sum_{j=1}^n t_j f_j\|_p\colon t_j \in R\ (j = 1, 2, \ldots, n), \sum |t_j|^p = 1\}.$$

Then $m = \|\sum_{j=1}^n f_j\|_p \cdot (n)^{-1/p}$.

Moreover, if there exists a real sequence $(a_j)_{j=1}^n$ *with* $\sum_{j=1}^n |a_j|^p = 1$, $\|\sum_{j=1}^n a_j f_j\|_p = m$ *and* $|a_k| \neq |a_l|$ *for some indices* k *and* l *then the functions* (f_j) *have mutually disjoint supports.*

Proof. Clearly $m \leqslant n^{-1/p}\|\sum_{j=1}^n f_j\|_p$. Pick a sequence $\boldsymbol{a} = (a_j)_{j=1}^n$ so that $1 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$, $\sum_{j=1}^n a_j^p = 1$, $\|\sum_{j=1}^n a_j f_j\|_p = m$. The existence of $\boldsymbol{a}$ follows by a standard compactness argument and from the symmetry conditions imposed on the f_j's. By the Lagrange Multiplier Theorem there exists a $\lambda_0 \in R$ such that if

$$\varphi(t_1, t_2, \ldots, t_n, \lambda) = \int |\sum_{j=1}^n t_j f_j|^p d\mu - \lambda \cdot (\sum_{j=1}^n |t_j|^p - 1)$$

then

$$\frac{\partial \varphi}{\partial t_j}(\boldsymbol{a}, \lambda_0) = 0 \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{7}$$

Using differentiation rules and (6) we infer that (7) is equivalent to

$$\int f_j (\sum a_j f_j)^{p-1} d\mu = \lambda_0 (a_j)^{p-1} \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{8}$$

Multiplying the jth identity of (8) by a_j and adding all of them together "by sides" we get

$$\int |\sum_{j=1}^n a_j f_j|^p d\mu = \lambda_0 \sum_{j=1}^n |a_j|^p = \lambda_0.$$

(Here we have used an obvious consequence of (5) that $t(t)^q = |t|^{q+1}$ for $q > 0$ and $t \in R$.) Hence

$$\lambda_0 = m^p \leqslant n^{-1} \|\sum_{j=1}^n f_j\|_p^p.$$

Next observe that $a_1 > 0$, because $n a_1^p \geqslant \sum_{j=1}^n a_j^p = 1$. Let us put

$$b_j = a_j a_1^{-1} \quad (j = 1, 2, \ldots, n).$$

Clearly $1 = b_1 \geqslant b_2 \geqslant \ldots \geqslant b_n \geqslant 0$. Dividing the first identity of (8) (for $j = 1$) by $(a_1)^{p-1}$ we get

$$\lambda_0 = \int f_1 (\sum_{j=1}^n b_j f_j)^{p-1} d\mu.$$

Thus

$$\int f_1 (\sum_{j=1}^n b_j f_j)^{p-1} d\mu \leqslant n^{-1} \int |\sum_{j=1}^n f_j|^p d\mu.$$

Let us put

$$H(x_1, x_2, \ldots, x_n) = \int f_1 (\sum_{j=1}^n x_j f_j)^{p-1} d\mu,$$

$$\varrho = \inf\{H(x_1, \ldots, x_n)\colon 1 = x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0\}.$$

By the Hölder inequality H is well defined. Using the exchangeability of the f_j's we get

$$H(1, 1, \ldots, 1) = \int f_k (\sum_{j=1}^n f_j)^{p-1} d\mu \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Summing over k and using the formula $t(t)^{p-1} = |t|^p$ we get

$$H(1, 1, \ldots, 1) = n^{-1} \int |\sum_{j=1}^n f_j|^p d\mu.$$

Now to complete the proof of Proposition 2.2 it is enough to prove

PROPOSITION 2.3. *Under the assumption of Proposition* 2.2,

$$\varrho \geqslant n^{-1} \int |\sum_{j=1}^n f_j|^p d\mu = H(1, 1, \ldots, 1).$$

Moreover, if there is a $\boldsymbol{b} = (b_j)_{j=1}^n$ *such that*

$$1 = b_1 \geqslant b_2 \geqslant \ldots \geqslant b_n \geqslant 0, \quad b_n < 1, \qquad H(b_1, b_2, \ldots, b_n) = \varrho \tag{9}$$

then the functions $f_1, f_2, \ldots, f_n$ *have mutually disjoint supports.*

Our proof of Proposition 2.3 is based on the following

LEMMA 2.1. *Let* $1 < p < 2$.

(i) *If* $u, v \in R$ *and*

$$h(t) = u[(u+tv)^{p-1} + (u-tv)^{p-1}] \quad \textit{for} \quad t \geqslant 0$$

then either $uv = 0$ *and then* $h(t) = \text{const} = 2|u|^p$, *or* $h(t)$ *is a strictly decreasing function for* $t \geqslant 0$.

(ii) *If* $u, v \in L_{\mathbf{R}}^p(\mu)$ *and*

$$h(t) = \int u[(u+tv)^{p-1} + (u-tv)^{p-1}]\,d\mu$$

then either $uv = 0$ *μ-almost everywhere and then* $h(t) = \text{const} = 2\|u\|_p^p$, *or* $h(t)$ *is a strictly decreasing function for* $t \geq 0$.

Proof. Part (i) uses a routine argument of examining the sign of the derivative of $h(\cdot)$. Part (ii) follows from part (i) by pointwise integration.

Proof of Proposition 2.3. If $\boldsymbol{b} = (b_j)_{j=1}^n$ satisfies (9) then there is a k with $1 \leq k < n$ such that $b_k = 1$ and $b_{k+1} < 1$. First we show that

$$(\sum_{j=1}^k f_j)(\sum_{j=k+1}^n b_j f_j) = 0 \quad \mu\text{-a.e.} \tag{10}$$

This is trivial if $b_{k+1} = 0$ because in that case $b_j = 0$ for $k+1 \leq j \leq n$ (by (9)). Assume that $b_{k+1} \neq 0$. Let us put

$$u = \sum_{j=1}^k b_j f_j = \sum_{j=1}^k f_j, \qquad v = \sum_{j=k+1}^n b_j f_j,$$
$$h(t) = \int u[(u+tv)^{p-1} + (u-tv)^{p-1}]\,d\mu.$$

Since $0 < b_{k+1} < 1$, Lemma 2.1 yields $h(1) \geq h(b_{k+1}^{-1})$. Observe that $h(t) = 2\int u(u+tv)^{p-1}\,d\mu$ because of the exchangeability and symmetry of the random variables $f_1, \ldots, f_n$. Moreover, since $b_1 = b_2 = \ldots = b_k = 1$, the exchangeability of the variables f_j yields

$$H(b_1, b_2, \ldots, b_n) = \int f_i (\sum_{j=1}^k f_j + \sum_{j=k+1}^n b_j f_j)^{p-1}\,d\mu \quad \text{for} \quad 1 \leq i \leq k.$$

Hence

$$H(b_1, b_2, \ldots, b_n) = \frac{1}{k}\int (\sum_{j=1}^k f_j)(\sum_{j=1}^k f_j + \sum_{j=k+1}^n b_j f_j)^{p-1}\,d\mu = \frac{1}{2k} h(1).$$

Let us put $b_j^* = b_j$ for $1 \leq j \leq k$, $b_j^* = b_{k+1}^{-1} b_j$ for $k+1 \leq j \leq n$. Clearly $1 = b_1^* = b_2^* \ldots = b_{k+1}^* \geq b_{k+2}^* \geq \ldots \geq b_n^* \geq 0$. The same argument as before shows that

$$H(b_1^*, b_2^*, \ldots, b_n^*) = \frac{1}{2k} h\left(\frac{1}{b_{k+1}}\right).$$

It follows from (9) that $H(b_1, b_2, \ldots, b_n) = \varrho \leq H(b_1^*, b_2^*, \ldots, b_n^*)$. Thus $h(1) \leq h(1/b_{k+1})$. Hence $h(1) = h(1/b_{k+1})$ and, by Lemma 2.1, $u \cdot v = 0$ μ-a.e. This completes the proof of (10).

Next we show that

$$(\sum_{j=1}^k f_j) f_{k+1} = 0 \quad \mu\text{-a.e.} \tag{11}$$

Case 1°. $b_{k+1} \neq 0$. Then (10) and exchangeability and symmetry of the f_j's yield

$$(\sum_{j=1}^k f_j)(-b_{k+1} f_{k+1} + b_{k+2} f_{k+2} + \ldots + b_n f_n) = 0 \quad \mu\text{-a.e.}$$

Substracting the last identity from (10) we get

$$2b_{k+1} f_{k+1} (\sum_{j=1}^k f_j) = 0 \quad \mu\text{-a.e.}$$

Dividing by $b_{k+1} \neq 0$ we get (11).

Case 2°. $b_{k+1} = 0$. Then $b_j = 0$ for $j > k$ and obviously $b_j = 1$ for $j \leq k$. Put $b_j^* = 1$ for $j \leq k+1$ and $b_j^* = 0$ for $j > k+1$. Let us consider the function

$$h(t) = (\sum_{j=1}^k f_j)[(\sum_{j=1}^k f_j + tf_{k+1})^{p-1} + (\sum_{j=1}^k f_j - tf_{k+1})^{p-1}].$$

Using the exchangeability and symmetry of the random variables $(f_j)_{j=1}^n$ as in the proof of (10), we infer that

$$h(0) = 2kH(b_1, b_2, \ldots, b_n), \qquad h(1) = 2kH(b_1^*, b_2^*, \ldots, b_n^*).$$

Since $H(b_1, b_2, \ldots, b_n) = \varrho \leq H(b_1^*, b_2^*, \ldots, b_n^*)$, we get $h(0) \leq h(1)$. Thus, by Lemma 2.1, we get (11).

Clearly, (11) yields $(-f_1 + f_2 + \ldots + f_k) f_{k+1} = 0$ μ-a.e. (by the exchangeability and symmetry of the f_j's). Substracting the last identity from (11) and dividing by 2 we get $f_1 \cdot f_{k+1} = 0$ μ-a.e. Hence the exchangeability condition yields $f_l \cdot f_k = 0$ μ-a.e. for $k \neq l$. Thus the f_j's have mutually disjoint supports.

The proof of Theorem 2.1 for $2 < p < \infty$ in the real case is similar to that for $1 < p < 2$. Instead of Proposition 2.2 we use

Proposition 2.2′. *Let* $2 < p < \infty$. *Let* $(f_j)_{j=1}^n$ *be a normalized sequence of exchangeable and symmetric random variables in* $L_{\mathbf{R}}^p(\mu)$ (μ — *a probability measure*). *Let*

$$M = \sup\{\|\sum_{j=1}^n t_j f_j\|_p : t_j \in R \ (j = 1, 2, \ldots, n), \ \sum_{j=1}^n |t_j|^p = 1\}.$$

Then $M = \|\sum_{j=1}^n f_j\|_p n^{-1/p}$.

Moreover, if there exists a real sequence $(a_j)_{j=1}^n$ *with* $\sum_{j=1}^n |a_j|^p = 1$, $\|\sum_{j=1}^n a_j f_j\|_p = M$ *and* $|a_k| \neq |a_l|$ *for some* k *and* l *then the functions* $(f_j)_{j=1}^n$ *have mutually disjoint supports.*

Proof of Proposition 2.2′. The proof reduces via the Lagrange multipliers method to an analogue of Proposition 2.3 with ϱ replaced by

$$\varrho^* = \sup\{H(x_1, x_2, \ldots, x_n) : 1 = x_1 \geq x_2 \geq \ldots \geq x_n \geq 0\}$$

and the inequality involving ϱ replaced by the reverse inequality involving ϱ^*. The proof of this analogue uses an analogue of Lemma 2.1 with "strictly decreasing function" replaced by "strictly increasing function".

The next proposition enables us to extend Theorem 2.1 to the case of complex-valued functions.

PROPOSITION 2.4. *The space $L^p(\mu)$ regarded as a linear metric space over the real scalars is isometrically isomorphic to a subspace of $L^p_R(\nu)$ for some other measure ν.*

Proof. Let $\nu = \mu\times\frac{1}{2\pi}\,d\theta$ where $d\theta$ is the Lebesgue measure on $[0, 2\pi]$. To each $f\in L^p(\mu)$ we assign $F\in L^p_R\left(\mu\times\frac{1}{2\pi}\,d\theta\right)$ defined by

$$F(s,\theta) = \operatorname{Re} f(s)\cos\theta + \operatorname{Im} f(s)\sin\theta = |f|(s)\left[\cos\left(w(s)-\theta\right)\right]$$

where $w(s) = \arg f(s)\in[0, 2\pi)$. The desired isometrically isomorphic embedding is given by the map

$$f\to c_p^{-1}F \quad \text{where} \quad c_p = \left(\frac{1}{2\pi}\int_0^{2\pi}|\cos\theta|^p\,d\theta\right)^{1/p}.$$

This is clearly a real linear map. Since for every $\alpha\in[0, 2\pi)$ we have

$$c_p^p = \frac{1}{2\pi}\int_0^{2\pi}|\cos(\alpha-\theta)|^p\,d\theta,$$

we get

$$\|c_p^{-1}F\|_p^p = c_p^{-p}\int_S\left(\frac{1}{2\pi}\int_0^{2\pi}|f|^p(s)\left|\cos\left(w(s)-\theta\right)\right|^p d\theta\right)\mu(ds) = \|f\|_p^p.$$

Remark. Clearly, the map $f\to c_p^{-1}F$ preserves the property of the disjointness of supports of functions.

Now we are ready for the

Proof of Theorem 2.1 for complex-valued functions. Assume that $f_1, f_2, \ldots, f_n\in L^p(\mu)$ is a 1-(complex) unconditional and 1-symmetric sequence. Then the f_j's as elements of $L^p(\mu)$ regarded as a space over reals are 1-(real) unconditional and 1-symmetric. Therefore the corresponding (via Proposition 2.4) real functions $c_p^{-1}F_1, c_p^{-1}F_2, \ldots, c_p^{-1}F_n$ in $L^p_R(\nu)$ have the same property. Applying Theorem 2.2 for $L^p_R(\nu)$ which has already been established, we infer that if

$$W = \{t_j\in R,\ j = 1, 2, \ldots, n\colon \textstyle\sum|t_j|^p = 1\}$$

then

$$\inf_W\|\sum t_j c_p^{-1}F_j\|_p = \inf_W\|\sum t_j f_j\|_p = \|\sum_{j=1}^n f_j\|_p n^{-1/p} \quad \text{for} \quad 1<p<2,$$

$$\sup_W\|\sum t_j c_p^{-1}F_j\|_p = \sup_W\|\sum t_j f_j\|_p = \|\sum_{j=1}^n f_j\|_p n^{-1/p} \quad \text{for} \quad 2<p<\infty.$$

Also the "moreover" part, where the inf and the sup are attained, holds. To complete the proof observe that the 1-(complex) unconditionality implies

$$\|\sum_{j=1}^n z_j f_j\|_p = \|\sum_{j=1}^n |z_j| f_j\|_p$$

for every complex sequence $(z_j)_{j=1}^n$.

Now we derive some corollaries from Theorem 2.1.

COROLLARY 2.1. *Let $(f_j)_{j=1}^n$ be a normalized 1-unconditional and 1-symmetric sequence in $L^p(\mu)$ (resp. $L^p_R(\mu)$). Let H be a Hilbert space over complex or real scalars, respectively.*

Then if $1<p<2$

$$\inf\{(\int\|\sum x_j f_j\|_H^p\,d\mu)^{1/p}\colon x_j\in H\,(j = 1, 2, \ldots, n), \sum\|x_j\|_H^p = 1\} = n^{-1/p}\|\sum_{j=1}^n f_j\|_p;$$

if $2<p<\infty$,

$$\sup\{(\int\|\sum x_j f_j\|_H^p\,d\mu)^{1/p}\colon x_j\in H\,(j = 1, 2, \ldots, n), \sum\|x_j\|_H^p = 1\} = n^{-1/p}\|\sum_{j=1}^n f_j\|_p.$$

Moreover, either the inf *(respectively the* sup*) is attained only for sequences $(x_j)_{j=1}^n\subset H$ such that $\sum\|x_j\|_H^p = 1$ and there is an $x_0\in H$ and a sequence of scalars $(\varepsilon_j)_{j=1}^n$ with $|\varepsilon_j| = 1$ for $j = 1, 2, \ldots, n$ such that $x_j = \varepsilon_j x_0$ for $j = 1, 2, \ldots, n$ or the f_j's have mutually disjoint supports.*

Proof. The first part of Corollary 2.1 is an immediate consequence of Theorem 2.1 and Corollary 1.2. We give here an alternative proof which allows to obtain the "moreover" part, too.

Since we deal with sequences $(x_j)_{j=1}^n\subset H$ of length n, without loss of generality we may assume that $H = l_n^2$ (either real or complex). Let $\langle\,,\rangle$ denote the scalar product in l_n^2, let S_{n-1} be the unit sphere of l_n^2 and let λ be the normalized Haar measure on S_{n-1}. We embed l_n^2 isometrically into $L^p(\lambda)$ (resp. $L^p_R(\lambda)$) via the map $x\to g_x$ for $x\in l_n^2$ where $g_x(y) = a_p^{-1}\langle x, y\rangle$ for $y\in S_{n-1}$ and $a_p = (\int_{S_{n-1}}|\langle e_1, y\rangle|^p\lambda(dy))^{1/p}$, $e_1 = (1, 0, 0, \ldots, 0)$. Clearly, for every $x\in l_n^2$,

$$\int_{S_{n-1}}|\langle x, y\rangle|^p\lambda(dy) = a_p^p\|x\|_H^p.$$

We restrict ourselves to the case $1 < p < 2$. The proof for $2 < p < \infty$ is analogous. Fix a sequence $(x_j)_{j=1}^n \subset H$. By Theorem 2.1, for every $y \in S_{n-1}$ we have

$$\int \Big|\sum_{j=1}^n \langle x_j, y\rangle f_j(s)\Big|^p \mu(ds) \geq \Big\|\sum_{j=1}^n f_j\Big\|_p^p n^{-1} \sum_{j=1}^n |\langle x_j, y\rangle|^p. \tag{12}$$

Integrating against λ over S_{n-1} we get

$$\int\int \Big|\sum_{j=1}^n \langle x_j, y\rangle f_j(s)\Big|^p \mu(ds)\,\lambda(dy) \geq a_p^p \Big\|\sum_{j=1}^n f_j\Big\|_p^p \cdot n^{-1}. \tag{13}$$

Applying the Fubini Theorem to the left-hand side of (13) we get

$$\int\int \Big|\sum_{j=1}^n \langle x_j, y\rangle f_j(s)\Big|^p \mu(ds)\,\lambda(dy) = \int\int \Big|\Big\langle \sum_{j=1}^n f_j(s)\,x_j, y\Big\rangle\Big|^p \lambda(dy)\,\mu(ds)$$
$$= \int a_p^p \Big\|\sum_{j=1}^n f_j(s)\,x_j\Big\|_H^p \mu(ds).$$

Thus

$$\int \Big\|\sum_{j=1}^n f_j(s)\,x_j\Big\|_H^p \mu(ds) \geq \Big\|\sum_{j=1}^n f_j\Big\|_p^p n^{-1}. \tag{14}$$

If in (14) we have equality then (12) is also an equality for y λ-a.e. Thus, by continuity of the functions g_{x_j}, we infer that we have equality in (12) for all $y \in S_{n-1}$. Thus by the "moreover" part of Theorem 2.1 if the f_j's do not have mutually disjoint supports then

$$|\langle x_j, y\rangle| = |\langle x_k, y\rangle|$$

for all $y \in S_{n-1}$ and all pairs (j, k) of the indices. The last property easily implies that x_j and x_k are linearly dependent and $\|x_j\| = \|x_k\|$ which yields the desired conclusion.

Corollary 2.2. *Let $(f_j)_{j=1}^n$ be a normalized 1-(real) unconditional and 1-symmetric sequence in $L_{\mathbf{R}}^p(\mu)$. Then for every sequence $(z_j)_{j=1}^n$ of complex numbers satisfying the condition $\sum_{j=1}^n |z_j|^p = 1$ we have*

$$\int \Big|\sum_{j=1}^n z_j f_j\Big|^p d\mu \leq n^{-1} \int \Big|\sum_{j=1}^n f_j\Big|^p d\mu \quad \text{for} \quad 1 < p < 2,$$

$$\int \Big|\sum_{j=1}^n z_j f_j\Big|^p d\mu \geq n^{-1} \int \Big|\sum_{j=1}^n f_j\Big|^p d\mu \quad \text{for} \quad 2 < p < \infty.$$

Moreover, in both cases the inequality becomes the equality iff $z_j = z_0 \varepsilon_j$ $(j = 1, 2, \ldots, n)$ where $\varepsilon_j = \pm 1$ $(j = 1, 2, \ldots, n)$ and z_0 is a complex number with $|z_0| = n^{-1/p}$.

Proof. Apply Corollary 2.1 regarding the complex plane as the 2-dimensional real Euclidean space.

The first part of Theorem 2.1 (as well as the first parts of Corollaries 2.1 and 2.2) extends to the "limit cases" $p = 1$ and $p = \infty$. However, a more general fact is well-known:

Proposition 2.5. *Let X be an n-dimensional Banach space with a 1-symmetric and 1-unconditional normalized basis $(x_j)_{j=1}^n$. Let $\Delta\colon X \to l_n^1$ and $\nabla\colon l_n^\infty \to X$ be diagonal maps. Then*

$$\|\Delta\| = \frac{n}{\|\sum_{j=1}^n x_j\|}, \qquad \|\nabla\| = \Big\|\sum x_j\Big\|.$$

The first formula follows by a simple averaging procedure using the fact that the norm of l_n^1 restricted to the positive cone of l_n^1 is an additive functional; the second uses only 1-unconditionality of the basis $(x_j)_{j=1}^n$ and the form of the extreme points of the unit ball of l_n^∞.

Recall that the Rademacher functions (r_j) are defined by $r_j(t) = \operatorname{sign} \sin 2^j t\pi$ for $t \in [0, 1]$ $(j = 1, 2, \ldots)$. They are independent random variables essentially ± 1-valued with mean zero.

The Steinhaus variables (γ_j) are independent random variables each distributed as the function $t \to e^{2\pi i t}$ for $t \in [0, 1]$. A model for the sequence $(\gamma_j)_{j=1}^n$ of n Steinhaus variables are the functions $\gamma_j(t_1, t_2, \ldots, t_n) = e^{2\pi i t_j}$ defined on the cube $\{0 \leq t_j \leq 1\colon j = 1, 2, \ldots, n\}$.

The Rademacher functions can be identified as the coordinate functions of the Cantor group $\boldsymbol{Z}_2^\infty$ while the Steinhaus functions – as the coordinate functions of the infinite torus group $\boldsymbol{T}^\infty$.

Applying Theorem 2.1 and Corollaries 2.1 and 2.2 for the Rademacher functions we obtain

Corollary 2.3. *Let $n = 1, 2, \ldots$, let $(r_j)_{j=1}^n$ and $(\gamma_j)_{j=1}^n$ be the Rademacher and the Steinhaus functions, respectively, let*

$$\alpha_{p,n} = \Big(\int_0^1 \Big|\sum_{j=1}^n r_j(t)\Big|^p dt\Big)^{1/p}, \qquad \beta_{p,n} = \Big(\int_0^1 \Big|\sum_{j=1}^n \gamma_j\Big|^p d\mu\Big)^{1/p}$$

$(1 \leq p \leq \infty)$. Then for every sequence $(x_j)_{j=1}^n$ of vectors in a Hilbert space, in particular of complex numbers, we have:

If $1 \leq p \leq 2$ then

$$\Big(\int_0^1 \Big\|\sum_{j=1}^n x_j r_j(t)\Big\|^p dt\Big)^{1/p} \geq \alpha_{p,n} n^{-1/p} \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p},$$

$$\Big(\int \Big\|\sum_{j=1}^n x_j \gamma_j\Big\|^p d\mu\Big)^{1/p} \geq \beta_{p,n} n^{-1/p} \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p};$$

if $2 \leqslant p \leqslant \infty$ *then*

$$\Big(\int_0^1 \Big\|\sum_{j=1}^{n} x_j r_j(t)\Big\|^p dt\Big)^{1/p} \leqslant \alpha_{p,n} n^{-1/p}\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p},$$

$$\Big(\int \Big\|\sum_{j=1}^{n} x_j \gamma_j\Big\|^p d\mu\Big)^{1/p} \leqslant \beta_{p,n} n^{-1/p}\Big(\sum_{j=1}^{n} \|x_j\|^p\Big)^{1/p}.$$

Moreover, for $1 < p \neq 2 < \infty$ *these inequalities become equalities iff* $x_j = \varepsilon_j x_0$, $\varepsilon_j = \pm 1$ *for the Rademacher and* $|\varepsilon_j| = 1$ *for the Steinhaus variables* $(j = 1, 2, \ldots, n)$ *and* x_0 *is an arbitrary vector.*

Remark. By the Central Limit Theorem the asymptotic behaviour of the sequences $(\alpha_{p,n})$ and $(\beta_{p,n})$ is determined by the formulae

$$\lim_{n=\infty} n^{-1/2}\alpha_{p,n} = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |t|^p e^{-t^2/2}\, dt\right)^{1/p},$$

$$\lim_{n=\infty} n^{-1/2}\beta_{p,n} = \left(\frac{1}{\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (s^2+t^2)^{p/2} e^{-s^2-t^2}\, ds\, dt\right)^{1/p} \qquad (1 \leqslant p < \infty).$$

We end this section by discussing some relationship of Corollary 2.3 to the Hlawka inequality (cf. [6], pp. 171–172).

Observe that $\int_0^1 \Big|\sum_{j=1}^{3} r_j(t)\Big|^p dt = 4^{-1}(3^p+3)$. Now specifying in Corollary 2.3 $n = 3$ and using the Hölder inequality

$$\|x_1\|+\|x_2\|+\|x_3\| \leqslant 3^{(p-1)/p}(\|x_1\|^p+\|x_2\|^p+\|x_3\|^p)^{1/p}$$

we get

Corollary 2.4. *If* $1 \leqslant p \leqslant 2$ *then for every vectors* x_1, x_2, x_3 *in a Hilbert space, in particular, for every complex numbers,*

$$\text{(15)} \qquad \left(\int_0^1 \Big\|\sum_{j=1}^{3} x_j r_j\Big\|^p\right)^{1/p} \geqslant \left(\frac{3^{1-p}+1}{4}\right)^{1/p}(\|x_1\|+\|x_2\|+\|x_3\|).$$

Moreover, if $1 < p < 2$ *then* (15) *becomes the equality iff* $x_j = \varepsilon_j x_0$, $\varepsilon_j = \pm 1$ $(j = 1, 2, 3)$ *and* x_0 *is an arbitrary vector.*

Putting in (15)

$$x_1 = \frac{u+v}{2}, \quad x_2 = \frac{u+w}{2}, \quad x_3 = \frac{v+w}{2}$$

and evaluating the integral in the left-hand side of (15) we get

Corollary 2.5. *For* $1 \leqslant p \leqslant 2$ *and for every vectors* u, v, w *in a Hilbert space*

$$\text{(16)} \qquad \|u+v+w\|^p+\|u\|^p+\|v\|^p+\|w\|^p \geqslant \frac{3^{1-p}+1}{2^p}(\|u+v\|+\|w+u\|+\|v+w\|)^p.$$

Moreover, if $1 < p < 2$, *then the inequality becomes the equality if either* $u = v = w$, *or two of the vectors* u, v, w *equal zero, or one of the vectors* u, v, w *equals zero and two others are equal.*

Remark. For $p = 1$ (16) is the well-known Hlawka inequality (cf. [6], p. 171). For $p = 1$ the "moreover" part of Corollary 2.5 fails; for instance, for $u+v+w = 0$ we have always the equality.

It would be desirable to extend Theorem 2.1 to the case of complex-valued functions $f_1, \ldots, f_n$ satisfying the following "mixed" condition:

For arbitrary complex $(z_j)_{j=1}^n$, arbitrary permutation π of the indices and arbitrary real $(\varepsilon_j)_{j=1}^n$ with $\varepsilon_j^2 = 1$ $(j = 1, 2, \ldots, n)$

$$\Big\|\sum_j \varepsilon_j z_j f_{\pi(j)}\Big\|_p = \Big\|\sum_j z_j f_j\Big\|_p.$$

In particular, we do not know whether the assertion of Theorem 2.1 remains valid for the sequence (ξ_n) of independent equally distributed random variables each of which is distributed according to the law

$$\mu\{\xi = 1\} = \mu\{\xi = -1\} = \mu\{\xi = i\} = \mu\{\xi = -i\} = 4^{-1}.$$

3. Critical exponents. This section is devoted to the study of the behaviour of norms of orthogonal projections onto finite dimensional subspaces of L^∞ as well as onto orthogonal complements of finite dimensional subspaces of L^∞ regarded as operators from L^∞ into L^p $(p > 2)$.

To formulate the results we introduce some notation.

In this section we consider only probability measures. Let E be a finite dimensional subspace of $L^\infty(\mu)$, μ – a probability measure. By P_E (resp. P^E) we denote the orthogonal projection (in $L^2(\mu)$) onto E (respectively in the direction of E, i.e., onto the orthogonal complement of E). By $\|P\|^{\infty,p}$ we denote the norm of a projection P regarded as an operator from $L^\infty(\mu)$ into $L^p(\mu)$. If P is either P_E or P^E then clearly $\|P\|^{\infty,\infty} \geqslant 1$ while $\|P\|^{\infty,2} \leqslant 1$. We put

$$\operatorname{cr}(P) = \sup\{p\colon \|P\|^{\infty,p} \leqslant 1\}.$$

The value $\operatorname{cr}(P)$ is called the *critical exponent* of P.

Principal results of this section are

Theorem 3.1. *For every probability measure* μ *and every finite dimensional subspace* E *of* $L^\infty(\mu)$ *one has* $\operatorname{cr}(P_E) > 2$.

Theorem 3.2. *For every probability measure* μ *and every finite dimensional subspace* E *of* $L^\infty(\mu)$ *one has* $\operatorname{cr}(P^E) > 2$.

We begin with the proof of Theorem 3.1. It is convenient to introduce more notation. Let us put

$$M_E = \sup\{\|x\|_\infty:\ x \in E,\ \|x\|_1 \leqslant 1\}.$$

Clearly $1 \leqslant M_E < \infty$.

Given $0 \neq g \in L^\infty(\mu)$ we denote by P_g the orthogonal projection onto the one dimensional space generated by g. Clearly

$$P_g(f) = \int f \cdot \bar{g}\, d\mu \cdot \frac{g}{\|g\|_2^2} \quad \text{for} \quad f \in L^2(\mu).$$

Thus

$$\|P_g\|^{\infty,p} = \frac{\|g\|_1 \|g\|_p}{\|g\|_2^2}.$$

Given M with $1 \leqslant M < \infty$ we put

$$Z_M(\mu) = \{g \in L^\infty(\mu):\ \|g\|_1 = M^{-1} \text{ and } \|g\|_\infty \leqslant 1\},$$

$$a^\mu_{M,p} = \sup\{\|P_g\|^{\infty,p}:\ g \in Z_M(\mu)\} \qquad (2 \leqslant p < \infty).$$

Theorem 3.1 is an obvious consequence of the next two propositions.

Proposition 3.1. *For every finite dimensional subspace E of $L^\infty(\mu)$ we have*

$$\|P_E\|^{\infty,p} \leqslant a^\mu_{M_E,p}.$$

Proposition 3.2. *For each M with $1 < M < \infty$ there is a $p_0(M) > 2$ such that $a^\mu_{M,p} = 1$ for $2 \leqslant p \leqslant p_0(M)$.*

For the precise description of $p_0(M)$ see Appendix.

Proof of Proposition 3.1. Fix $p \geqslant 2$. Let $g_1, g_2, \ldots, g_n$ be an orthonormal basis in E. Then $P_E(f) = \sum_{j=1}^n \left(\int f \bar{g}_j\, d\mu\right) g_j$ for $f \in L^\infty(\mu)$. Clearly P_E is continuous in the w^*-topology of $L^\infty(\mu)$ (induced by $L^1(\mu)$) and the norm topology of $(E, \|\cdot\|_p)$. Since E is finite dimensional and the unit ball of $L^\infty(\mu)$ is compact in the w^*-topology, there is an f_0 with $\|f_0\|_\infty = 1$ such that $\|P_E(f_0)\|_p = \|P_E\|^{\infty,p}$. Put $g = P_E(f_0)$. Then

$$P_E(f_0) = \|g\|_2^{-2}\left(\int f_0 \bar{g}\, d\mu\right) g.$$

(This is clear if one chooses the orthonormal basis $g_1, g_2, \ldots, g_n$ so that $g_1 = \|g\|_2^{-1} g$.) Hence $\|P_E\|^{\infty,p} = \|P_g(f_0)\|_p \leqslant \|P_g\|^{\infty,p}$. Since $g \in E$, $g_0 = (\|g\|_1 M_E)^{-1} \cdot g \in Z_{M_E}(\mu)$. Clearly $P_g = P_{g_0}$. Thus $\|P_g\|^{\infty,p} \leqslant a^\mu_{M_E,p}$.

Proof of Proposition 3.2. The general case can be reduced to the case of the Lebesgue measure λ on the unit interval $[0, 1]$ as follows. First we reduce the case of an arbitrary probability measure space (S, Ω, μ) to the case of a separable probability measure space. Let $g \in L^\infty(\mu)$. Let Ω_0 be the smallest sigma-subfield of Ω such that g is Ω_0 measurable. Let $\mu_0 = \mu_{|\Omega}$.

Then, by the Stone Representation Theorem, there is a probability measure space $(S_0^*, \Omega_0^*, \mu_0^*)$ and a measure preserving homomorphism of Boolean algebras φ: $(\Omega_0^*, \mu_0^*) \to (\Omega_0, \mu_0)$. Clearly μ_0, and therefore μ_0^*, are separable probability measures. This homomorphism induces the linear isomorphism of $L^0(\mu_0^*)$ into $L^0(\mu)$, say u, such that $u(L^0(\mu_0^*))$ consists of all Ω_0 measurable functions. Furthermore, there exists the operator P: $L^1(\mu) \to u(L^1(\mu_0^*))$ of conditional expectation which regarded as an operator from $L^p(\mu)$ into $u(L^p(\mu_0^*))$ is a contractive projection onto $L^p(\mu_0^*)$ $(1 \leqslant p \leqslant \infty)$. Clearly $P_g = P_{g|u(L^2(\mu_0^*))} \circ P$. One can easily see that

$$\|P_g\|^{\infty,p} = \|P_{g|u(L^2(\mu_0^*))}\|^{\infty,p} = \|P_{u^{-1}(g)}\|^{\infty,p}.$$

Now let (S, Ω, μ) be a separable probability measure space. Let $s_1, s_2, \ldots$ be atoms of μ and let $S_0 = S \setminus \bigcup \{s_j\}$. Let $\Delta_0, \Delta_1, \ldots$ be the decomposition of $[0, 1]$ into mutually disjoint intervals such that $\lambda(\Delta_i) = \mu(\{s_i\})$ for $i = 1, 2, \ldots$ Then $\mu(S_0) = 1 - \sum_i \mu(\{s_i\}) = \lambda(\Delta_0)$. Let $\mu_c = \mu_{|\Omega \cap S_0}$. Then μ_c is an atomless separable measure. Hence there exists a measure preserving one-to-one map φ from Δ_0 onto S_0 (cf. [3]). Next we consider the map u: $L^0(\mu) \to L^0(\lambda)$ defined by

$$u(f) = f \cdot \chi_{\Delta_0} + \sum_i f(s_i)\, \chi_{\Delta_i}.$$

Let P: $L^2(\lambda) \to u(L^2(\mu))$ be the orthogonal projection. Hence

$$Ph = h \cdot \chi_{\Delta_0} + \sum_i \int_{\Delta_i} h\, d\lambda.$$

One can easily see that P regarded as a map from $L^p[0, 1]$ into $L^p[0, 1]$ is a contractive projection. Clearly, if $g \in Z_M(\mu)$ then $u(g) \in Z_M(\lambda)$ because $\|u(g)\|_p \leqslant \|g\|_p$ for $1 \leqslant p \leqslant \infty$. Furthermore

$$\|P_g\|^{\infty,p} = \|P_{u(g)}:\ u(L^\infty(\mu)) \to L^p(\lambda)\| \leqslant \|P_{u(g)}\|^{\infty,p}.$$

Thus $a^\mu_{M,p} \leqslant a^\lambda_{M,p}$.

Finally, in the case of the Lebesgue measure λ on the interval $[0, 1]$, the assertion of Proposition 3.2 is an immediate consequence of Theorem A of the Appendix which says that $a^\lambda_{M,p}$ equals the unique positive root of the equation $M^p - 1 = p(M^2 - M)$.

Proof of Theorem 3.2. The proof consists of several steps. Given $0 \neq h \in L^\infty$ we denote by P^h the orthogonal projection of codimension 1 onto the orthogonal complement of h.

1° *Reduction to projections of codimension* 1. Fix $p > 2$. Since $P^E = I - P_E$ (I: $L^\infty \to L^p$ denotes the natural injection) is as an operator from $L^\infty(\mu)$ into $L^p(\mu)$ weak*-weak continuous, it attains its norm at some extreme point of the unit ball of $L^\infty(\mu)$. Hence there exists a unimodular $f_0 \in L^\infty(\mu)$ such that

$\|P^E(f_0)\|_p = \|P^E\|^{\infty,p}$. Put $g = P^E(f_0)$. Clearly $0 \neq g \perp E$. If $g = f_0$ then $\|P^E\|^{\infty,p} = 1$; this case is trivial. If $g \neq f_0$ then obviously $f_0 - g \perp g$ (because $P^E(f_0-g) = 0$ and therefore $f_0 - g \in E$). Put $h_0 = \dfrac{f_0-g}{\|f_0-g\|_2}$. Then $h_0 \in E$ and $P^{h_0}(f_0-g) = 0$. Thus $P^{h_0}(f_0) = g$. Therefore

$$\|P^E\|^{p,\infty} = \|g\|_p = \|P^{h_0}(f_0)\|_p \leqslant \|P^{h_0}\|^{p,\infty}.$$

Hence

$$\operatorname{cr}(P^E) \geqslant \inf\{\operatorname{cr}(P^h):\ h \in E\}.$$

2° *Reduction to the case of atomless measures.* Similarly as in the proof of Proposition 3.2 we show that if ν is an arbitrary probability measure then there exists an atomless probability measure μ and a map u: $L^1(\nu) \to L^1(\mu)$ such that $u(L^p(\nu)) \subset L^p(\mu)$ for $1 \leqslant p \leqslant \infty$ and u regarded as a map from $L^p(\nu)$ into $L^p(\mu)$ is an isometry. Moreover, the orthogonal projection from $L^2(\mu)$ onto $u(L^2(\nu))$ regarded as a map from $L^p(\mu)$ onto $u(L^p(\nu))$ has norm one. Thus $\operatorname{cr}(P^E) \geqslant \operatorname{cr}(P^{u(E)})$ for every finite dimensional $E \subset L^\infty(\nu)$.

From now until the end of the proof we assume that μ is an atomless probability measure.

3° *Reduction to the case of P^h with auxiliary real h.* Let α be a unimodular measurable function. Denote by m_α the operator of multiplication by α. Observe that m_α regarded as an operator from $L^p(\mu)$ into $L^p(\mu)$ is for every $1 \leqslant p \leqslant \infty$ an isometric isomorphism; moreover, $m_\alpha P^{h_0} m_{\bar\alpha} = P^{\alpha h_0}$. Hence $\|P^{h_0}\|^{\infty,p} = \|P^{\alpha h_0}\|^{\infty,p}$ and therefore $\operatorname{cr}(P^{\alpha h_0}) = \operatorname{cr}(P^{h_0})$. Now given $h_0 \in L^\infty$ with $\|h_0\|_2 = 1$ we specify α so that

$$\alpha(s) = \begin{cases} e^{-i\arg h_0(s)} & \text{for} \quad s \in A, \\ -e^{-i\arg h_0(s)} & \text{for} \quad s \notin A \end{cases}$$

where the measurable set A is chosen so that

$$\int_A |h_0(s)|\,\mu(ds) = \int_{S\setminus A} |h_0(s)|\,\mu(ds)$$

(S denotes the whole space). A set A with the above property exists because μ is an atomless probability measure and $h_0 \in L^\infty(\mu) \subset L^1(\mu)$. Let us put $h = \alpha h_0$. Then

(i) $h \in L^\infty_R(\mu)$,

(ii) $\|h\|_2 = 1$,

(iii) $\int h\,d\mu = 0$.

Clearly, $\|h\|_p = \|h_0\|_p$ and $\|P^h\|^{\infty,p} = \|P^{h_0}\|^{\infty,p}$ for $2 \leqslant p \leqslant \infty$.

Observe that if $h \in L^\infty_R$ then $P^h(L^\infty_R) \subset L^\infty_R$. We denote by P^h_R the restriction of P^h to L^∞_R regarded as a linear operator acting between real spaces. In particular, $\|P^h_R\|^{\infty,p}$ denotes the norm of P^h_R regarded as an operator from L^∞_R into L^p_R.

4° $\|P^h_R\|^{\infty,p} = \|P^h\|^{\infty,p}$ *for every* $2 \leqslant p \leqslant \infty$ *and for every* $h \in L^\infty_R(\mu)$ (μ — atomless!). Clearly $\|P^h_R\|^{\infty,p} \leqslant \|P^h\|^{\infty,p}$. To prove the reverse inequality pick arbitrary unimodular $f \in L^\infty(\mu)$. Let $f = x + iy$ with $x, y \in L^\infty_R(\mu)$ and $x^2(s) + y^2(s) = 1$ for $s \in S$. Replacing f if necessary by $c \cdot f$ for appropriate complex c with $|c| = 1$ one may assume without loss of generality that $\int f h\,d\mu$ is a real number; equivalently $\int y h\,d\mu = 0$ (because $h \in L^\infty_R(\mu)$). We then have

$$P^h(f) = f - \left(\int f h\,d\mu\right)h = x - \left(\int x h\,d\mu\right)h + iy.$$

Thus for $p < \infty$

$$\|P^h f\|_p = \left(\int \left[\left(x - \left(\int x h\,d\mu\right)h\right)^2 + y^2\right]^{p/2} d\mu\right)^{1/p} = \left(\int \psi(x)^{p/2}\,d\mu\right)^{1/p}$$

where for $g \in L^\infty_R(\mu)$ with $\|g\|_\infty \leqslant 1$ we have

$$\psi(g) = 1 - 2\left(\int g h\,d\mu\right)g h + \left(\int g h\,d\mu\right)^2 h^2.$$

Now if $p \geqslant 2$ there is a $\varphi \in L^{(p/2)^*}_R(\mu)$ (where $(p/2)^* = \dfrac{p/2}{p/2-1}$ for $p > 2$ and $(p/2)^* = \infty$ for $p = 2$) such that

$$\left(\int \psi(x)^{p/2}\,d\mu\right)^{2/p} = \int \psi(x(s))\,\varphi(s)\,\mu(ds) \quad \text{and} \quad \|\varphi\|_{(p/2)^*} = 1.$$

Next we use the following well-known fact (cf. e.g. [2]).

If μ is an atomless measure then for every $x \in L^\infty_R(\mu)$ with $\|x\|_\infty \leqslant 1$ there exists in $L^\infty_R(\mu)$ a sequence (f_n) of unimodular functions (i.e., $f_n(s) = \pm 1$ for $s \in S$ and $n = 1, 2, \ldots$) such that for every r with $1 \leqslant r < \infty$ $\lim\limits_n f_n = x$ weakly in L^r_R.

Since h belongs to $L^\infty_R(\mu)$, the functions $h\varphi$ and $h^2\varphi$ belong to $L^{(p/2)^*}_R(\mu)$. Thus the weak convergence in $L^{p/2}_R(\mu)$ of the sequence f_n to x easily implies that

$$\int \psi(x)\,\varphi\,d\mu = \lim_n \int \psi(f_n)\,\varphi\,d\mu.$$

Next observe that the property $f_n^2 = 1$ yields

$$\psi(f_n) = \left[f_n - \left(\int f_n h\,d\mu\right)h\right]^2 = [P^h_R(f_n)]^2.$$

Thus, by the Hölder inequality,

$$\|P^h(f)\|_p = \left[\lim_n \int \psi(f_n)\,\varphi\,d\mu\right]^{1/2}$$
$$\leqslant \left[\overline{\lim_n}\left(\int \psi(f_n)^{p/2}\right)^{2/p} \|\varphi\|_{(p/2)^*}\right]^{1/2}$$
$$= \overline{\lim_n}\, \|P^h_R(f_n)\|_p \leqslant \|P^h_R\|^{\infty,p}.$$

Since the latter inequality holds for arbitrary unimodular f in $L^\infty(\mu)$, we infer that $\|P^h\|^{\infty,p} \leqslant \|P^h_R\|^{\infty,p}$. This completes the proof of 4° in the case $2 \leqslant p < \infty$.

If $p = \infty$ then for $h \in L_R^\infty(\mu)$ with $\|h\|_2 = 1$ we have the explicit formula

$$(*) \qquad \|P_R^h\|^{\infty,\infty} = 1 + \|h\|_1 \|h\|_\infty.$$

Since for every $f \in L^\infty(\mu)$ with $\|f\|_\infty = 1$ one can easily check that

$$\|P^h(f)\|_\infty \leqslant \|f\|_\infty + \|h\|_1 \|f\|_\infty \|h\|_\infty = 1 + \|h\|_1 \|h\|_\infty,$$

we infer that $\|P^h\|^{\infty,\infty} \leqslant \|P_R^h\|^{\infty,\infty}$.

To establish $(*)$ fix $\varepsilon > 0$ and pick unimodular $f \in L_R^\infty(\mu)$ so that $\int f h\, d\mu > \|h\|_1 - \varepsilon$. Next pick a measurable set $A \subset S$ so that $\mu(A) < 2^{-1}\|h\|_\infty^{-2}\varepsilon$ and $\|h\|_\infty = \|h\chi_A\|_\infty$. Now define f_A by

$$f_A(s) = \begin{cases} f(s) & \text{for } s \notin A, \\ -\operatorname{sign}(\int f h\, d\mu)\, h(s) & \text{for } s \in A. \end{cases}$$

Clearly, $f_A \in L_R^\infty$ is unimodular. Moreover, using the inequality

$$\left|\int f h\, d\mu - \int f_A h\, d\mu\right| \leqslant 2\mu(A)\|h\|_\infty$$

one can easily check that

$$\|P_R^h f_A\|_\infty \geqslant 1 + \|h\|_1 \|h\|_\infty - \varepsilon\|h\|_\infty - \varepsilon.$$

Letting ε tend to zero we get $\|P_R^h\| \geqslant 1 + \|h\|_1 \|h\|_\infty$.

5° *If h satisfies* (i)–(iii) *then* $\operatorname{cr}(P^h) > 2$. Assume to the contrary that $\operatorname{cr}(P^h) = 2$. Then there would exist a sequence $p_n \searrow 2$ and a sequence (f_n) in $L^\infty(\mu)$ of unimodular real functions such that

$$(17) \qquad \|P^h f_n\|_{p_n} > 1 \quad \text{for} \quad n = 1, 2, \ldots$$

Remembering that $\|h\|_2 = \|f_n\|_2 = 1$ and using the Pithagoras Theorem we get

$$\left(\int f_n h\, d\mu\right)^2 = \|f_n - P^h f_n\|_2^2 = 1 - \|P^h f_n\|_2^2.$$

Observe that (17) implies that $P^h f_n \neq f_n$, hence $\int f_n h\, d\mu \neq 0$. Let $\|h\|_\infty = K$. Clearly, by (ii), $K \geqslant 1$. Then $\|P^h(f_n)\|_\infty \leqslant K+1$; hence

$$\|P^h f_n\|_2^2 \geqslant (K+1)^{2-p_n} \|P^h f_n\|_{p_n}^{p_n} \geqslant (K+1)^{2-p_n}$$

(the latter inequality by (17)). Thus

$$\left(\int f_n h\, d\mu\right)^2 \leqslant 1 - (K+1)^{2-p_n} \quad \text{for} \quad n = 1, 2, \ldots$$

Hence

$$\lim_n \int_S f_n h\, d\mu = 0.$$

Let $A_n = \{s \in S\colon f_n(s) = -1\}$. Then, by (iii),

$$\int_S f_n h\, d\mu = -2\int_{A_n} h\, d\mu, \qquad \int_{S \setminus A_n} h\, d\mu = -\int_{A_n} h\, d\mu.$$

Let n be any index such that $K\left|\int_S f_n h\, d\mu\right| < 1$. Then

$$\|P^h f_n\|_{p_n}^{p_n} = \int_{S \setminus A_n} \left[1 - \left(\int_S f_n h\, d\mu\right) h\right]^{p_n} d\mu + \int_{A_n} \left[1 + \left(\int_S f_n h\, d\mu\right) h\right]^{p_n} d\mu$$
$$= \int_{S \setminus A_n} \left[1 + 2\left(\int_{A_n} h\, d\mu\right) h\right]^{p_n} d\mu + \int_{A_n} \left[1 - 2\left(\int_{A_n} h\, d\mu\right) h\right]^{p_n} d\mu.$$

Expanding each term into power series and integrating we get

$$\int_{S \setminus A_n} \left[1 + 2\left(\int_{A_n} h\, d\mu\right) h\right]^{p_n} d\mu = \int_{S \setminus A_n} \Big[1 + 2p_n\left(\int_{A_n} h\, d\mu\right) h +$$
$$+ 2p_n(p_n - 1)\left(\int_{A_n} h\, d\mu\right)^2 h^2 + O\Big(\Big(\int_{A_n} h\, d\mu\Big)^3\Big)\Big] d\mu$$
$$= \mu(S \setminus A_n) - 2p_n\left(\int_{A_n} h\, d\mu\right)^2 +$$
$$+ 2p_n(p_n - 1)\left(\int_{A_n} h\, d\mu\right)^2 \int_{S \setminus A_n} h^2\, d\mu + O\Big(\Big(\int_{A_n} h\, d\mu\Big)^2\Big),$$

$$\int_{A_n} \left[1 - 2\left(\int_{A_n} h\, d\mu\right) h\right]^{p_n} d\mu = \int_{A_n} \Big[1 - 2p_n\left(\int_{A_n} h\, d\mu\right) h +$$
$$+ 2p_n(p_n - 1)\left(\int_{A_n} h\, d\mu\right)^2 h^2 + O\Big(\Big(\int_{A_n} h\, d\mu\Big)^3\Big)\Big] d\mu$$
$$= \mu(A_n) - 2p_n\left(\int_{A_n} h\, d\mu\right)^2 +$$
$$+ 2p_n(p_n - 1)\left(\int_{A_n} h\, d\mu\right)^2 \int_{A_n} h^2\, d\mu + O\Big(\Big(\int_{A_n} h\, d\mu\Big)^2\Big).$$

Adding both expressions together and taking into account that $\mu(S) = 1$ and $\int_S h^2\, d\mu = 1$ we get

$$\|P^h f_n\|_{p_n}^{p_n} = 1 + 2p_n(p_n - 3)\left(\int_{A_n} h\, d\mu\right)^2 + O\Big(\Big(\int_{A_n} h\, d\mu\Big)^2\Big)$$

and $\int_{A_n} h\, d\mu \neq 0$ for all n. Since $\lim_n p_n = 2$ and $\lim_n \left(\int_{A_n} h\, d\mu\right)^2 = 0$, we infer that for large n, $\|P^h f_n\|_{p_n}^{p_n} < 1$ which contradicts (17).

This completes the proof of 5° and thus of Theorem 3.2.

Remark. Given a finite dimensional subspace $E \subset L^\infty(\mu)$ we put

$$K_E = \sup\{\|x\|_\infty\colon x \in E,\ \|x\|_2 \leqslant 1\}.$$

A careful analysis of the proof of Theorem 3.2 shows that similarly like in the case of Theorem 3.1 there exists a function $K \to \varphi(K)$ such that $\operatorname{cr}(P^E) > \varphi(K_E)$. However we do not know the explicit formula for $\varphi(K)$.

In the next two examples μ is an atomless probability measure; (S, Ω, μ) is a measure space.

EXAMPLE 3.1. Let $A \subset \Omega$, $\mu(A) > 0$. Pick $h \in L^\infty(\mu)$ so that $|h| = \chi_A$. Then $\mathrm{cr}(P^h) = 3$.

Proof. Put $P^A = P^{\chi_A}$. By step 3° of the proof of Theorem 3.2, $\mathrm{cr}(P^A) = \mathrm{cr}(P^h)$. Now fix $p > 2$. By step 4° of the proof of Theorem 3.2, $\|P^A\|^{\infty,p} = \|P^A_{\mathbf{R}}\|^{\infty,p}$.

To evaluate $\|P^A_{\mathbf{R}}\|^{\infty,p}$ put $a = \mu(A)$ and fix $\varepsilon \in [-a, a]$. Since the measure μ is atomless, there is a set $A_\varepsilon \in \Omega$ with $A_\varepsilon \subset A$ and $\mu(A_\varepsilon) = (a+\varepsilon)/2$. Define the unimodular f_ε by $f_\varepsilon = \chi_{A_\varepsilon} - \chi_{S\setminus A_\varepsilon}$. Then

$$P^A f_\varepsilon = f_\varepsilon - \int_A f_\varepsilon \, d\mu \, \chi_A a^{-1} = \chi_{S\setminus A} + \left(1-\frac{\varepsilon}{a}\right)\chi_{A_\varepsilon} - \left(1+\frac{\varepsilon}{a}\right)\chi_{A\setminus A_\varepsilon}.$$

Thus

$$\|P^A f_\varepsilon\|_3^3 = 1 - \frac{\varepsilon^4}{a^3} \tag{18}$$

and for all $2 < p < \infty$

$$\|P^A f_\varepsilon\|_p^p = (1-a) + \left(1-\frac{\varepsilon}{a}\right)^p \frac{a+\varepsilon}{2} + \left(1+\frac{\varepsilon}{a}\right)^p \frac{a-\varepsilon}{2}$$
$$= (1-a) + \frac{a}{2}\left(1-\left(\frac{\varepsilon}{a}\right)^2\right)\left[\left(1-\frac{\varepsilon}{a}\right)^{p-1} + \left(1+\frac{\varepsilon}{a}\right)^{p-1}\right]$$
$$= (1-a) + \frac{a}{2}\left[2 + p(p-3)\left(\frac{\varepsilon}{a}\right)^2 + O\left(\left(\frac{\varepsilon}{a}\right)^4\right)\right]$$
$$= 1 + \frac{p(p-3)}{2a}\varepsilon^2 + O\left(\left(\frac{\varepsilon}{a}\right)^4\right).$$

The latter formula implies that for every $p > 3$, for sufficiently small positive ε, $\|P^A f_\varepsilon\|_p^p > 1$. Thus $\mathrm{cr}(P^A) \leqslant 3$. On the other hand, for every unimodular real f one has

$$\|P^A f\|_p^p = \|P^A f_\varepsilon\|_p^p$$

for $\varepsilon = \int_A f \, d\mu = \mu(A \cap \{f = 1\}) - \mu(A \cap \{f = -1\})$. Therefore, by (18), $\|P^A f\|_3^3 \leqslant 1$. Hence $\|P^A_{\mathbf{R}}\|^{\infty,3} = 1$. Thus $\mathrm{cr}(P^A) \geqslant 3$. Hence $\mathrm{cr}(P^A) = \mathrm{cr}(P^h) = 3$.

COROLLARY 3.1. *If μ is an atomless measure, A a measurable set with $\mu(A) > 0$, then for every $h \in L^\infty(\mu)$ with $|h| = \chi_A$ and every $p > 3$ there exists a $\varphi_p \in L^\infty(\mu)$ with $|\varphi_p| = \chi_A$ such that*

$$\int |\varphi_p - (\int_A h\varphi_p \, d\mu) h\mu(A)^{-1}|^p \, d\mu > \mu(A)$$

while for every $\varphi \in L^\infty(\mu)$ with $|\varphi| = \chi_A$

$$\int |\varphi - (\int_A h\varphi \, d\mu) h\mu(A)^{-1}|^3 \, d\mu \leqslant \mu(A).$$

Proof. Apply the argument of Example 3.1 to the measure space $(A, \Omega_A, \mu(A)^{-1}\mu)$ where $\Omega_A = \{B \in \Omega \colon B \subset A\}$.

EXAMPLE 3.2. Let $(A_j)_{j=1}^n$ be a finite family of mutually disjoint measurable sets. Pick $h_1, h_2, \ldots, h_n$ so that $|h_j| = \chi_{A_j}$ for $j = 1, 2, \ldots, n$. Let $E = \mathrm{span}(h_1, h_2, \ldots, h_n)$. Then $\mathrm{cr}(P^E) = 3$.

Proof. One has

$$P^E(f) = f - \sum_{j=1}^n (\int_{A_j} f h_j \, d\mu) h_j \mu(A_j)^{-1}.$$

Thus for every unimodular f,

$$\|P^E(f)\|_p^p = \int_{S\setminus\bigcup A_j} |f|^p \, d\mu - \int_{\bigcup_j A_j} |f - \sum_{k=1}^n (\int_{A_k} f h_k \, d\mu) h_k \mu(A_k)^{-1}|^p \, d\mu$$
$$= \mu(S\setminus\bigcup A_j) + \sum_{j=1}^n \int_{A_j} |f - (\int_{A_j} f h_j \, d\mu) h_j \mu(A_j)^{-1}|^p \, d\mu.$$

Using Corollary 3.1 for every fixed $p > 3$ there exist $\varphi_{p,j}$ such that $|\varphi_{p,j}| = \chi_{A_j}$ and

$$\int_{A_j} |\varphi_{p,j} - (\int_A h_j \varphi_{p,j} \, d\mu) h_j \mu(A_j)^{-1}|^p \, d\mu > \mu(A_j) \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Let us put

$$f_p = \chi_{S\setminus\bigcup_{j=1}^n A_j} + \sum_{j=1}^n \varphi_{p,j} \cdot \chi_{A_j}.$$

Then f_p is unimodular and

$$\|P^E(f_p)\|_p^p > \mu(S\setminus\bigcup_{j=1}^n A_j) + \sum_{j=1}^n \mu(A_j) = \mu(S) = 1.$$

Thus, for $p > 3$, $\|P^E\|^{\infty,p} > 1$. Hence $\mathrm{cr}(P^E) \leqslant 3$.

On the other hand, by the second part of Corollary 3.1, for every unimodular $f \in L^\infty(\mu)$,

$$\int_{A_j} |f - (\int_{A_j} h_j f \, d\mu) h_j \mu(A_j)^{-1}|^3 \, d\mu \leqslant \mu(A_j) \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Thus

$$\|P^E(f)\|_3^3 \leqslant \mu(S\setminus\bigcup_{j=1}^n A_j) + \sum_{j=1}^n \mu(A_j) = 1.$$

Therefore $\|P^E\|^{\infty,3} \leqslant 1$ (in fact $\|P^E\|^{\infty,3} = 1$). Hence $\mathrm{cr}(P^E) = 3$.

Remark. A similar technique allows to evaluate critical exponents of complementary projections to some infinite dimensional expectations. We mention here the following fact.

Let (S, Ω, μ) be a probability space with μ atomless. Let Ω_0 be a subfield of Ω. Assume for simplicity that $S \in \Omega_0$. Suppose for each $\eta \in (0, 1)$ there exists an $A_\eta \in \Omega$ such that $P_{\Omega_0}(\chi_{A_\eta}) = \eta\chi_S$. Then $\mathrm{cr}(P^{\Omega_0}) = 3$. Here P_{Ω_0} denotes the orthogonal projection from $L^2(\mu)$ onto the subspace consisting of all Ω_0 measurable functions, and $P^{\Omega_0} = I - P_{\Omega_0}$.

Next we show that the fact that $\mathrm{cr}(P^E) = 3$ at least for real finite dimensional E holds only in the situation described in Example 3.2.

PROPOSITION 3.3. *Let (S, Ω, μ) be a probability space with μ atomless. Let $E \subset L_R^\infty(\mu)$ be a finite dimensional space. Assume that $\|P_R^E\|^{\infty,3} \leqslant 1$ (in particular, that $\|P^E\|^{\infty,3} = 1$).*

Then there are mutually disjoint sets $A_1, \ldots, A_n$ in Ω with $\mu(A_j) > 0$ $(j = 1, 2, \ldots, n)$ and $h_1, h_2, \ldots, h_n$ in $L_R^\infty(\mu)$ such that $|h_j| = \chi_{A_j}$ for $j = 1, 2, \ldots, n$ and $\mathrm{span}(h_1, \ldots, h_n) = E$.

Proof. Fix a unimodular $f \in L_R^\infty(\mu)$. Clearly, $f^2 = 1$ and $f^3 = f$. Also $\int P_E f(f - P_E f)\, d\mu = 0$ because for real E and f, $P_E f$ is real. Thus, by our hypothesis,

$$1 \geqslant \|P^E f\|_3^3 = \int |f - P_E(f)|^3\, d\mu = \int |1 - fP_E(f)|^3\, d\mu$$
$$\geqslant \int (1 - fP_E(f))^3\, d\mu = 1 - \int f[P_E(f)]^3\, d\mu.$$

Hence

$$\int f[P_E(f)]^3\, d\mu \geqslant 0 \quad \text{for every unimodular} \quad f \in L_R^\infty(\mu).$$

The latter inequality implies

$$\int x(P_E(x))^3\, d\mu \geqslant 0 \quad \text{for every} \quad x \in L_R^\infty(\mu). \tag{19}$$

Indeed, it is enough to check (19) for $x \in L_R^\infty(\mu)$ with $\|x\|_\infty \leqslant 1$. Pick as in step 4° of the proof of Theorem 3.2 a sequence (f_n) of real unimodulars which tends to x weakly in $L^2(\mu)$. Then, because of finite dimensionality of E,

$$\lim_n \|P_E(f_n) - P_E(x)\|_\infty = 0.$$

Thus

$$\lim_n \|[P_E(f_n)]^3 - [P_E(x)]^3\|_\infty = 0$$

and finally

$$\lim_n \int f_n[P_E(f_n)]^3\, d\mu = \int x[P_E(x)]^3\, d\mu$$

which yields (19).

Next we show

$$y \in E \quad \text{implies} \quad y^3 \in E. \tag{20}$$

If (20) were not true then there would exists a $y \in E$ and a $z \in L_R^\infty(\mu)$ such that (i) $\int zy^3\, d\mu \neq 0$ and (ii) $\int zx\, d\mu = 0$ for all $x \in E$. (We regard E as a closed linear subspace of $L^1(\mu)$ and apply the Hahn–Banach Theorem to find a $z \in L_R^\infty(\mu) = [L^1(\mu)]^*$ which separates y^3 from E.) The orthogonality of P_E combined with (ii) yields $P_E z = 0$. Now using (19) for all $t \in R$ we get

$$0 \leqslant \int (z+ty)[P_E(z+ty)]^3\, d\mu = \int (z+ty)t^3y^3\, d\mu = t^3 \int zy^3\, d\mu + t^4 \int y^4\, d\mu;$$

this is impossible for $|t|$ small enough and $\mathrm{sign}\, t = -\mathrm{sign} \int zy^3\, d\mu$ unless $\int zy^3\, d\mu = 0$ which contradicts (i).

Finally, observe that for every finite dimensional subspace E of $L_R^\infty(\mu)$ there is a set $S_E \subset S$ of full measure such that evaluations at points of S_E are linear and multiplicative functionals on the smallest closed subalgebra of $L_R^\infty(\mu)$ generated by E. Thus for $n = \dim E$ one can find points $s_1, s_2, \ldots, s_n$ of S_E and functions $h_1, h_2, \ldots, h_n$ in E so that $h_j(s_k) = \delta_{j,k}$ for $j, k = 1, 2, \ldots, n$. Hence $x = \sum_{k=1}^n x(s_k)h_k$ for $x \in E$. By (20), $h_j^3 \in E$. Thus $h_j^3 = \sum_{k=1}^n h_j^3(s_k)h_k = h_j$. We put $A_j = \{s \in S\colon h_j^2(s) = 1\}$ $(j = 1, 2, \ldots, n)$. One can easily check that the functions $h_1, h_2, \ldots, h_n$ together with the sets $A_1, A_2, \ldots, A_n$ fulfil the assertion of the proposition.

We do not know whether Proposition 3.3 remains valid for arbitrary finite dimensional subspaces of $L^\infty(\mu)$.

However, we have

COROLLARY 3.2. *Let $h \in L^\infty(\mu)$, $\|h\|_\infty = 1$, μ – an atomless probability measure. Then $\mathrm{cr}(P^h) = 3$ iff $|h|$ is a characteristic function of a set of positive measure.*

Proof. Combine Proposition 3.1 and Example 3.2 with the argument of step 3° of the proof of Theorem 3.2.

Our next result provides an additional information about functions at which the norm $\|P_E\|^{\infty,p}$ is attained. It seems to be useful in computing critical exponents.

PROPOSITION 3.4. *Let μ be a probability measure not necessarily atomless. Let E be a finite dimensional subspace of $L^\infty(\mu)$. Let $p \in (1, \infty)$ be fixed. If $f \in L^\infty(\mu)$, with $\|f\|_\infty = 1$, satisfies the condition $\|P_E(f)\|_p = \|P_E\|^{\infty,p}$, then*

$$P_E(P_E(f)|P_E(f)|^{p-2}) = cf$$

for some non-negative $c \in L^\infty(\mu)$.

Moreover, if $A = \{t\colon |f(t)| < 1\}$, and $e \in E$ then $e = 0$ a.e. on A.

In particular, if no non-zero element of E vanishes on a set of positive measure *then one has*

$$f(t) = \frac{e(t)}{|e(t)|} \quad \mu\text{-a.e.}$$

for some $e \in E$, *viz* $e = P_E(P_E(f)|P_E(f)|^{p-2})$.

Proof. We may assume that $E \neq \{0\}$ and hence $\|P_E(f)\|_p \neq 0$. Recall that if $g \in L^\infty(\mu)$ then the function

$$\varphi(s) = \|P_E(f+sg)\|_p$$

is differentiable at zero; if $h = P_E(f)|P_E(f)|^{p-2}$ then

$$\varphi'(0) = \operatorname{Re} \int P_E(g)\bar{h}\,d\mu = \operatorname{Re} \int g\overline{P_E(h)}\,d\mu.$$

(Here we make use of the assumption that P_E is an orthogonal projection.) Next observe that if $v \in L_R^\infty(\mu)$ and $g = ivf$ then the assumption $\|f\|_\infty \leqslant 1$ yields

$$\|f+sg\|_\infty \leqslant (1+s^2\|v\|_\infty^2)^{1/2} \quad \text{for} \quad s \in \boldsymbol{R}.$$

Hence taking into account that

$$\|P_E\|^{\infty,p} = \|P_E(f)\|_p \geqslant \left\| P_E\left(\frac{f+sg}{\|f+sg\|_\infty}\right)\right\|_p$$

we obtain

$$\|P_E(f)\|_p(1+s^2\|v\|_\infty^2)^{1/2} \geqslant \|P_E(f+sg)\|_p$$
$$= \|P_E(f)\|_p + s\varphi'(0) + O(|s|).$$

This implies that $\varphi'(0) = 0$, i.e., for all $v \in L_R^\infty(\mu)$

$$0 = \varphi'(0) = \operatorname{Re} \int ifv\overline{P_E(h)}\,d\mu$$
$$= \int v \operatorname{Im}(\bar{f}P_E(h))\,d\mu.$$

Therefore $c = \bar{f}P_E(h)$ must be a real function.

Now to prove that $c \geqslant 0$, take $g = vf$ for some $v \in L_R^\infty(\mu)$ with $v \leqslant 0$. Then for $0 \leqslant s \leqslant \|v\|_\infty^{-1}$ one has $\|f+sg\|_\infty \leqslant \|f\|_\infty = 1$. Hence $\varphi(s) \leqslant \varphi(0)$ for $0 \leqslant s \leqslant \|v\|_\infty^{-1}$. This shows that

$$0 \geqslant \varphi'(0) = \operatorname{Re} \int vf\overline{P_E(h)}\,d\mu = \operatorname{Re} \overline{\int v\bar{f}P_E(h)\,d\mu}$$
$$= \operatorname{Re} \int v\bar{f}P_E(h)\,d\mu = \int vc\,d\mu.$$

Since $v \leqslant 0$ is arbitrary, we infer that $c \geqslant 0$. Clearly on the set where $|f| = 1$ one has $c = |P_E(h)|$.

Suppose that $A = \{t\colon |f(t)| < 1\}$ has positive measure. Let χ_n denote the characteristic function of the set $\{t\colon |f(t)| < 1-n^{-1}\}$. Observe that

$$P_E(\chi_n \cdot g) = 0 \quad \text{for} \quad 0 \neq g \in L^\infty(\mu).$$

For, let $g_0 = (2n\|g\|_\infty)^{-1} g\chi_n$. Then $\|f \pm g_0\|_\infty \leqslant 1$, and hence $\|P_E(f \pm g_0)\|_p \leqslant \|P_E\|^{\infty,p}$. On the other hand,

$$\|P_E\|^{\infty,p} = \|P_E(f)\| \leqslant \tfrac{1}{2}(\|P_E(f+g_0)\| + \|P_E(f-g_0)\|).$$

Since $L^p(\mu)$ is strictly convex, it follows that

$$P_E(f+g_0) = P_E(f-g_0).$$

This proves that $P_E(g_0) = 0$. Hence $P_E(\chi_n \cdot g) = 0$.

Now if $g \in L^\infty(\mu)$ vanishes outside A, then $\chi_n g \to g$ as $n \to \infty$ in the weak*-topology of $L^\infty(\mu)$. Thus for P_E being weak*-norm continuous we have $P_E(g) = \lim_n P_E(\chi_n g) = 0$. Therefore for arbitrary $e \in E$ we have

$$\int e\bar{g}\,d\mu = \int P_E(e)\bar{g}\,d\mu = \int e\overline{P_E(g)}\,d\mu = 0.$$

Consequently all elements of e vanish μ-a.e. on A. Thus $c(t) = 0$ for $t \in A$.

Remark. Proposition 3.4 can be extended to the case $p = 1$ and $p = \infty$ as well as for the orthogonal projection P_E regarded as an operator from $L^q(\mu)$ into $L^p(\mu)$ for $\infty > q > p$.

Example 3.3. Here L^p denotes $L^p[0, 1]$ with respect to the Lebesgue measure. We put

$$e(t) = \exp(2\pi it).$$

Let $\boldsymbol{E}$ denote the two dimensional space spanned by $\boldsymbol{e}$ and the constant functions, $\boldsymbol{P}$ the orthogonal projection onto $\boldsymbol{E}$. Clearly,

$$\boldsymbol{P}f = \hat{f}(0) + \hat{f}(1)\,e \quad \text{for} \quad f \in L^1$$

where $\hat{f}(n) = \int_0^1 f(t)\,e(-nt)\,dt$ $(n = 0, \pm 1, \ldots)$.

Theorem 3.3. *One has* $\operatorname{cr}(\boldsymbol{P}) = 4$.

Theorem 3.3 is an immediate consequence of the four lemmas stated below. The following one parameter family of unimodular functions plays an important role in the proof:

$$h_r = (1+re)|1+re|^{-1}, \quad 0 \leqslant r \leqslant 1.$$

The argument culminates in establishing that the function of r,

$$\Phi(r) = \|\boldsymbol{P}(h_r)\|_4^4, \quad 0 \leqslant r \leqslant 1,$$

is strictly decreasing.

LEMMA 3.1. *For every* $p\in[2,\infty]$ *there is an* $r(p)$ *such that*

$$\|\boldsymbol{P}(h_{r(p)})\|_p = \|\boldsymbol{P}\|^{\infty,p};$$

in other words the norm $\|\boldsymbol{P}\|^{\infty,p}$ *is attained on the set* $\{h_r\}_{0\leqslant r\leqslant 1}$.

LEMMA 3.2. *One has* $\boldsymbol{P}(h_r) = A(r)+B(r)e$ *where*

$$A(r) = \sum_{j=0}^{\infty}\binom{1/2}{j}\binom{-1/2}{j} r^{2j} = 1-\frac{r^2}{4}-\ldots,$$

$$B(r) = \sum_{j=0}^{\infty}\binom{1/2}{j+1}\binom{-1/2}{j} r^{2j+1} = \frac{r}{2}+\frac{r^3}{16}+\ldots$$

LEMMA 3.3. *One has*

$$\|\boldsymbol{P}(h_r)\|_p^p = 1+\frac{p(p-4)}{16}r^2+O(r^3)\quad (2<p<\infty).$$

Hence if $p>4$ *then for* r *positive and small enough* $\|\boldsymbol{P}(h_r)\|_p>1$; *thus* $\|\boldsymbol{P}\|^{\infty,p}>1$.

LEMMA 3.4. *The function* $r\to\Phi(r)$ $(0\leqslant r\leqslant 1)$ *is strictly decreasing. Thus*

$$\|\boldsymbol{P}\|^{\infty,4} = \|\boldsymbol{P}(h_0)\|_4 = [\Phi(0)]^{1/4} = 1.$$

Proof of Lemma 3.1. Put

$$\varphi_{a,b} = \frac{a+be}{|a+be|}\quad\text{with}\quad |a|+|b|\geqslant 1.$$

Clearly $\varphi_{1,r} = h_r$. Observe that $f(t)\neq 0$ for t a.e. for every $f\in\boldsymbol{E}$. Thus, by Proposition 3.4, for every $p\in[2,\infty)$ there are $a(p)$ and $b(p)$ such that $\|\boldsymbol{P}(\varphi_{a(p),b(p)})\|_p = \|\boldsymbol{P}\|^{\infty,p}$. Next note that $\|\boldsymbol{P}(\varphi_{a,b})\|_p = \|\boldsymbol{P}(\varphi_{b,a})\|_p$. Thus without loss of generality one may assume that $|a(p)|>0$ and $|a(p)|\geqslant|b(p)|$. Clearly, if $a\neq 0$ then $\varphi_{a,b} = \varphi_{1,b/a} = \varphi^{\theta}_{1,|b/a|}$ for $\theta = -\arg\dfrac{b}{a}$ where $\varphi^{\theta}_{a,b}(t) = \varphi_{a,b}(t+\theta)$. Thus $\|\boldsymbol{P}(\varphi_{a,b})\|_p = \|\boldsymbol{P}(\varphi_{1,|b/a|})\|_p$. Thus

$$\|\boldsymbol{P}\|^{\infty,p} = \|\boldsymbol{P}(h_{r(p)})\|_p\quad\text{for}\quad r(p) = \frac{|b(p)|}{|a(p)|}\in(0,1].$$

Finally note that

$$\|\boldsymbol{P}\|^{\infty,\infty} = \frac{1}{2\pi}\int_0^{2\pi}|1+e(t)|\,dt = \|\boldsymbol{P}(h_1)\|_\infty.$$

Proof of Lemma 3.2. Let $0\leqslant r<1$. Then expanding into Taylor series we get

$$A(r) = \int_0^1 h_r(t)\,dt = \int_0^1\frac{1+re(t)}{|1+re(t)|}\,dt$$

$$= \int_0^1 (1+re(t))^{1/2}(1+re(-t))^{1/2}\,dt$$

$$= \int_0^1\Big(\sum_{k,l=0}^{\infty}\binom{1/2}{k}\binom{-1/2}{l}r^{k+l}e(kt-lt)\Big)dt$$

$$= \sum_{j=0}^{\infty}\binom{1/2}{j}\binom{-1/2}{j}r^{2j}.$$

Similarly

$$B(r) = \int_0^1 e(-t)\frac{1+re(t)}{|1+re(t)|}\,dt = \sum_{j=0}^{\infty}\binom{1/2}{j+1}\binom{-1/2}{j}r^{2j+1}.$$

Since the power series also converges for $r=1$, the Abel Theorem yields that the formulae for $A(r)$ and $B(r)$ hold also for $r=1$.

Proof of Lemma 3.3. By Lemma 3.2 one has

$$A(r) = 1-\frac{r^2}{4}+o(r^2);\qquad B(r) = \frac{r}{2}+o(r^2).$$

Fix $p\in(2,\infty)$. Then for small $r>0$,

$$\|\boldsymbol{P}(h_r)\|_p^p = \int_0^1|A(r)+B(r)e(t)|^p\,dt$$

$$= \int_0^1\left|\Big(1-\frac{r^2}{4}+\frac{r}{2}\cos 2\pi t\Big)^2+\frac{r^2}{4}\sin^2 2\pi t+o(r^2)\right|^{p/2}dt$$

$$= \int_0^1\Big(1-\frac{r^2}{4}+r\cos 2\pi t+o(r^2)\Big)^{p/2}dt$$

$$= 1+\frac{p(p-4)}{16}r^2+o(r^2).$$

To prove Lemma 3.4 we recall some facts about Gauss hypergeometric series,

$$F(a, b; c, x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \quad c \neq 0, -1, -2, \ldots$$

where $(a)_0 = 1$; $(a)_n = a(a+1)\ldots(a+n-1)$.

Note that $(1)_n = n!$ and $(2)_n = (n+1)!$. Thus

$$\binom{1/2}{j}\binom{-1/2}{j} = \frac{(1/2)_j(-1/2)_j}{(1)_j j!}; \quad \binom{1/2}{j+1}\binom{-1/2}{j} = \frac{1}{2}\frac{(1/2)_j(1/2)_j}{(2)_j j!}.$$

Therefore Lemma 3.2 yields

$$A(r) = F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, r^2), \qquad B(r) = \tfrac{1}{2} r F(\tfrac{1}{2}, \tfrac{1}{2}; 2, r^2). \tag{21}$$

For the proof of Lemma 3.4 we need several properties of the functions A and B. They are collected in the following

SUBLEMMA 3.1. (i) *A is a decreasing function on $[0, 1]$ and $A(1) = B(1) = \frac{2}{\pi}$.*

(ii) *B is an increasing function on $[0, 1]$.*

(iii) *$A'(r) = rB'(r)$ for $r \in [0, 1)$.*

(iv) *The following recurrence relations hold:*

$$A\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{1}{1+r} A(r) + \frac{r}{1+r} B(r),$$

$$B\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{\sqrt{r}}{1+r} A(r) + \frac{\sqrt{r}}{1+r} B(r).$$

(v) *Let us put $\lambda(r) = B(r) \cdot A^{-1}(r)$. Then λ is an increasing function in r with the following properties:*

$$\lambda\left(\frac{2\sqrt{r}}{1+r}\right) = \sqrt{r}\, \frac{1+\lambda(r)}{1+r\lambda(r)}, \quad \lambda(1) = 1, \tag{22}$$

$$\lambda(r) \leqslant \frac{3}{5-2r}. \tag{23}$$

Proof. The identity $A(1) = B(1) = 2/\pi$ can be easily checked by direct computation. The remaining assertion in (i), and (ii) follow immediately from

the power series expansion formulae for A and B given in Lemma 3.2. Also from these expansions follows (iii).

To prove (iv) we use several relations between hypergeometric series for which we refer to [1]. We begin with Gauss' quadratic transformation formula (cf. [1], p. 64, formula (24)):

$$F\left(a, b; 2b, \frac{4r}{(1+r)^2}\right) = (1+r)^{2a} F(a, a+\tfrac{1}{2}-b; b+\tfrac{1}{2}, r^2).$$

In particular,

$$F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1, \frac{4r}{(1+r)^2}\right) = (1+r) F(\tfrac{1}{2}, \tfrac{1}{2}, 1, r^2), \tag{24}$$

$$F\left(-\tfrac{1}{2}, \tfrac{1}{2}; 1, \frac{4r}{(1+r)^2}\right) = (1+r)^{-1} F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1, r^2). \tag{25}$$

We also need two other formulae of Gauss (cf. [1], p. 103, formulae (33) and (38)):

$$(c-a-b)F(a, b; c, x) + a(1-x)F(a+1, b; c, x) - (c-b)F(a, b-1, c, x) = 0, \tag{26}$$

$$c(1-x)F(a, b; c, x) - cF(a-1, b; c, x) + (c-b)xF(a, b; c+1, x) = 0. \tag{27}$$

Substituting in (21) $(a, b; c, x) \to (-\tfrac{1}{2}, \tfrac{1}{2}, 1, r^2)$, and in (27) $(a, b; c, x) \to (\tfrac{1}{2}, \tfrac{1}{2}, 1, 4r(1+r)^{-2})$ one gets

$$F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1, r^2) = 2F(-\tfrac{1}{2}, \tfrac{1}{2}; 1, r^2) - (1-r^2) F(\tfrac{1}{2}, \tfrac{1}{2}; 1, r^2), \tag{28}$$

$$\frac{2r}{(1+r)^2} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 2, \frac{4r}{(1+r)^2}\right) = F\left(-\tfrac{1}{2}, \tfrac{1}{2}; 1, \frac{4r}{(1+r)^2}\right) - \left(1 - \frac{4r}{(1+r)^2}\right) F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1, \frac{4r}{(1+r)^2}\right). \tag{29}$$

Combining the first formula of (21) with (25) one gets

$$A\left(\frac{2\sqrt{r}}{1+r}\right) = F\left(-\tfrac{1}{2}, \tfrac{1}{2}; 1, \frac{4r}{(1+r)^2}\right) = (1+r)^{-1} F(-\tfrac{1}{2}, -\tfrac{1}{2}; 1, r^2). \tag{30}$$

The second formula of (21) gives

$$(31)\quad B\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{\sqrt{r}}{1+r}F\left(\tfrac{1}{2},\tfrac{1}{2};2,\frac{4r}{(1+r)^2}\right)$$

$$=\frac{1+r}{2\sqrt{r}}\left[F\left(-\tfrac{1}{2},\tfrac{1}{2};1,\frac{1}{(1+r)^2}\right)-\right.$$

$$\left.-\left(1-\frac{4r}{(1+r)^2}\right)F\left(\tfrac{1}{2},\tfrac{1}{2};1,\frac{4r}{(1+r)^2}\right)\right]\quad \text{(by (29))}$$

$$=\frac{1}{2\sqrt{r}}\left[F(-\tfrac{1}{2},-\tfrac{1}{2};1,r^2)-(1-r)^2F(\tfrac{1}{2},\tfrac{1}{2};1,r^2)\right]$$

(by (24) and (25)).

Using (27) for $(\frac{1}{2},\frac{1}{2};1,r^2)$ and then (21) we get

$$(32)\quad (1-r^2)F(\tfrac{1}{2},\tfrac{1}{2};1,r^2)=F(-\tfrac{1}{2},\tfrac{1}{2};1,r^2)-\frac{r^2}{2}F(\tfrac{1}{2},\tfrac{1}{2};2,r^2)$$

$$=A(r)-rB(r);$$

combining (27) with (23) we obtain

$$(33)\quad F(-\tfrac{1}{2},-\tfrac{1}{2};1,r^2)=A(r)+rB(r).$$

Both formulae in (iv) are immediate consequences of (30), (31), (32) and (33).

To verify (v) observe that (22) is an obvious consequence of the recursive formulae of (iv). Clearly (i) and (ii) implies that λ is increasing. To prove inequality (23) we consider the Taylor expansion of $\lambda(r)$. By Lemma 3.2,

$$\lambda(r)=B(r)A(r)^{-1}=\left(\frac{r}{2}+\frac{r^3}{16}+\ldots\right)\left(1-\frac{r^2}{4}-\frac{3r^4}{64}-\ldots\right)^{-1}$$

$$=\left(\frac{r}{2}+\frac{r^3}{16}+\ldots\right)\left[1+\left(\frac{r^2}{4}+\frac{3r^4}{64}+\ldots\right)+\left(\frac{r^2}{4}+\frac{3r^4}{64}+\ldots\right)^2+\ldots\right]$$

$$=\frac{r}{2}+\sum_{j=1}^{\infty}\lambda_j\cdot r^{2j+1}.$$

Observe that $\lambda_j\geqslant 0$ for $j=1,2,\ldots$, and $\sum_{j=1}^{\infty}\lambda_j=\lambda(1)-\frac{1}{2}=\frac{1}{2}$. Thus

$$\lambda(r)\leqslant\frac{r}{2}+\Big(\sum_{j=1}^{\infty}\lambda_j\Big)r^3=\tfrac{1}{2}(r+r^3).$$

Hence

$$(5-2r)\lambda(r)\leqslant g(r)=(5-2r)\left(\frac{r+r^3}{2}\right)\leqslant g(1)=3$$

(because $g'\geqslant 0$ for $0\leqslant r\leqslant 1$). This completes the proof of the sublemma.

Proof of Lemma 3.4. A straightforward computation gives

$$\Phi(r)=\int_0^1[A(r)+B(r)e(t)]^2[A(r)+B(r)e(-t)]^2\,dt$$

$$=A^4(r)+4A^2(r)B^2(r)+B^4(r)$$

$$=A^4(r)[1+4\lambda^2(r)+\lambda^4(r)].$$

Using Sublemma 3.1 (iii) we get

$$(34)\quad \Phi'(r)=4A^3(r)B'(r)f(r)$$

where

$$f(r)=\lambda^3(r)-2r\lambda^2(r)+2\lambda(r)-r.$$

By Sublemma 3.1 (i) and (ii), $A^3(r)B'(r)\geqslant 0$ for $r\in[0,1)$. Thus to prove that $\Phi'(r)\leqslant 0$ for $r\in[0,1)$ it is enough to show that $f(r)\leqslant 0$ for $r\leqslant[0,1]$. To this end observe first that using Sublemma 3.1 (v) we get

$$f\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{(1-r)\sqrt{r}}{(1+r)(1+r\lambda(r))^3}\{f(r)+(r-1)\lambda^3(r)+r\lambda(r)[(5-2r)\lambda(r)-3]\}$$

$$\leqslant\frac{(1-r)\sqrt{r}}{(1+r)(1+r\lambda(r))^3}f(r).$$

The latter inequality yields that if $f(r)\leqslant 0$ in some interval $[0,\varepsilon]$ then $[0,1]$ is the maximal interval with the property that $f(r)\leqslant 0$ in this interval. Finally observe that by Lemma 3.2

$$\Phi(r)=1-\frac{r^6}{64}+O(r^8).$$

Thus for small $r>0$, $\Phi'(r)\leqslant 0$. Hence, by (34), $f(r)\leqslant 0$ for $r\in[0,\varepsilon]$ and therefore $f'(r)\leqslant 0$ for $0\leqslant r\leqslant 1$. This completes the proof of Lemma 3.4.

Remarks. 1° The function Φ is "practically" almost constant. In fact, we have

$$0{,}985\approx\frac{96}{\pi^4}=\Phi(1)\leqslant\Phi(r)\leqslant\Phi(0)=1.$$

2° Let H^∞ denote the classical Hardy space of boundary values of bounded analytic functions in the open unit disc. Clearly, we can identify in a natural way H^∞ with a subspace of L^∞. We consider the restricted projection $P_{|H^\infty}$ as an operator from H^∞ into L^p. Denote by $\|P_{|H^\infty}\|^{\infty,p}$ the norm of this operator. D. Sarason and A. Shields have proved a result similar to our Theorem 3.3 (private communication):

THEOREM. *One has* $\|P_{|H^\infty}\|^{\infty,p} = 1$ *for* $p \leqslant 4$ *and* $\|P_{|H^\infty}\|^{\infty,p} > 1$ *for* $p > 4$.

Clearly, our Theorem 3.3 implies the first part of their result, while the second part of their result implies that $\|P\|^{\infty,p} > 1$ for $p > 4$. However, one has

PROPOSITION 3.5. *One has* $\|P\|^{\infty,p} > \|P_{|H^\infty}\|^{\infty,p}$ *for* $4 < p \leqslant \infty$.

Proof. It is well-known that H^∞ is the dual space to the space $X = L^1/\text{closed span}\{e^{int}, n > 0\}$. Moreover, $P_{|H^\infty}$ is weak (H^∞, X)-norm continuous. Thus for every $p \in (4, \infty]$ there is a $g_p \in H^\infty$ with $\|P(g_p)\|_p = \|P_{|H^\infty}\|^{\infty,p}$. Clearly $g_p \neq h_r$ for $0 < r \leqslant 1$ because $h_r = \dfrac{1+re}{|1+re|} \notin H^\infty$. Thus, by Proposition 3.4, $\|P(g_p)\|_p < \|P\|^{\infty,p}$ (the case h_0 is also excluded because $\|P(h_0)\|_p = 1 < \|P\|^{\infty,p}$ for $p > 4$).

3° Let $m \neq 0$ and k be integers. Let E be the two dimensional space spanned by $e^{ikt2\pi}$ and $e^{i(m+k)2\pi}$. Then $\mathrm{cr}(P_E) = 4$. This follows from Theorem 3.3 and the fact that the subspace of L^∞ consisting of all $f \in L^\infty$ such that $\int f(t)\, e(-nt)\, dt = 0$ for $n \neq m + lk$, $l = 0, \pm 1, \pm 2, \ldots$ is norm-one complemented in L^∞ via the orthogonal projection.

EXAMPLE 3.4. Let m_n denote the equally distributed probability measure on the set $\{1, 2, \ldots, n\}$. Let

$$ {}^{(n)}P(f) = f - \int f\, dm_n = f - n^{-1} \sum_{j=1}^{n} f(j) \cdot 1. $$

Clearly, $\mathrm{cr}({}^{(2)}P) = \infty$. It follows from Example 3.1 and Corollary 3 that $\mathrm{cr}({}^{(n)}P) \geqslant 3$ for $n = 3, 4, \ldots$ (one embeds $L^2(m_n)$ into $L^2(I)$ so that the projection ${}^{(n)}P$ extends to the projection P^1 where $P^1 f = f - \int_0^1 f\, dx \cdot 1$). Clearly, $\mathrm{cr}({}^{(n)}P_R) \geqslant \mathrm{cr}({}^{(n)}P)$ where ${}^{(n)}P_R$ denotes the restriction of ${}^{(n)}P$ to the real space $L^2_R(m_n)$ regarded as a real operator. We do not know whether $\|{}^{(n)}P\|^{\infty,p} = \|{}^{(n)}P_R\|^{\infty,p}$ for $p > 2$ or $\mathrm{cr}({}^{(n)}P) = \mathrm{cr}({}^{(n)}P_R)$.

Since the unit ball of $L_R(m_n)$ (= the real l_n^∞) has finitely many extreme points, there is a simple method of computing $\mathrm{cr}({}^{(n)}P_R)$. In fact, taking into account that $\|{}^{(n)}P_R(f)\|_p$ depends only on the distribution of f and multiplying if necessary by ± 1 to get $\int f\, dm_n \geqslant 0$, one may assume that a real unimodular f at which the norm $\|{}^{(n)}P_R\|^{\infty,p}$ is attained is one of the functions $f_1^{(n)}, f_2^{(n)}, \ldots, f_{[n+1/2]}^{(n)}$ where $f_k^{(n)}$ is defined by $f_k^{(n)}(j) = 1$ for $j \leqslant [n/2] + k$, $f_k^{(n)}(j) = -1$ for $j > [n/2] + k$. In particular, in the case $n = 3$, 4, 5 the critical exponents $\mathrm{cr}({}^{n}P_R) = p_n^{(R)}$ are determined as the smallest roots $p > 3$ of the equations

$$ \|{}^{(3)}P_R(1, 1, -1)\|_p^p = 1; \quad \|{}^{(4)}P_R(1, 1, 1, -1)\|_p^p = 1; $$
$$ \|{}^{(5)}P_R(1, 1, 1, -1, -1)\|_p^p = 1, $$

respectively. The equivalent equations are

$$ 2 \cdot 2^p + 4^p - 3 \cdot 3^p = 0; \quad 3 + 3^p - 4 \cdot 2^p = 0; \quad 3 \cdot 4^p + 2 \cdot 6^p - 5^{p+1} = 0. $$

This gives $p_3^{(R)} = 3{,}081643\ldots$, $p_4^{(R)} = 3{,}210660\ldots$, $p_5^{(R)} = 3{,}027586\ldots$

Appendix. In this Appendix M is a fixed number with $1 < M < \infty$ and L^p for $1 \leqslant p \leqslant \infty$ denotes the $L^p(\lambda)$ space where λ is the Lebesgue measure on $[0, 1]$. Given $g \in L^\infty$, $g \neq 0$ and $p \in [1, \infty)$ we put

$$ a_p(g) = \|g\|_1 \|g\|_p \cdot \|g\|_2^{-2}, $$

$$ Z_M^0 = \{g \in L^\infty\colon 0 \leqslant g \leqslant 1, \int_0^1 g(x)\, dx = M^{-1}, g \text{ non-increasing}\}, $$

$$ Z_M = \{g \in L^\infty\colon \|g\|_\infty \leqslant 1, \|g\|_1 = M^{-1}\}. $$

Clearly, $a_p(g) = a_p(|g|) = a_p(|g|^*)$ where $|g|^*$ denotes the non-increasing rearrangement of $|g|$. Thus

(A1) $$ \sup\{a_p(g)\colon g \in Z_M\} = \sup\{a_p(g)\colon g \in Z_M^0\}. $$

Next consider the equation

(A2) $$ M^p - 1 = p(M^2 - M). $$

Using standard differential calculus one shows that equation (A2) has exactly one root in the positive half line. We denote this root by $p_0(M)$. In fact, $2 < p_0(M) < 3$.

The purpose of this Appendix is to prove

THEOREM A. (i) *If* $1 \leqslant p \leqslant p_0(M)$ *then*

$$ a_p(g) \leqslant 1 \quad \text{for every} \quad g \in Z_M^0. $$

Moreover, $a_p(g) = 1$ *iff* $g \equiv M^{-1}$ *is a constant function.*

(ii) *If* $p > p_0(M)$ *then*

$$ \sup\{a_p(g)\colon g \in Z_M^0\} > 1. $$

Moreover, the supremum is attained at exactly one function g_p *which is of the form*

$$ g_p(t) = \begin{cases} 1 & \text{for} \quad t \leqslant t(p, M), \\ \dfrac{1 - Mt(p, M)}{M - Mt(p, M)} & \text{for} \quad t > t(p, M) \end{cases} $$

where $t(p, M) \in (0, 1)$ *is uniquely determined.*

Proof. If $p_1 < p_2$ then $a_{p_1}(g) \leqslant a_{p_2}(g)$. Thus we can restrict ourselves to the case $p > 2$. Clearly, Z_M^0 is compact in the pointwise convergence (the Helly Theorem) and the functional $a_p(\cdot)$ is continuous on Z_M^0 (the Lebesgue Theorem). Thus there exists a $g_p \in Z_M^0$ such that

$$a_p(g_p) = \sup\{a_p(g): g \in Z_M^0\}.$$

The description of g_p is given in several steps.

1° *g_p is a step function with at most 4 steps; the intersection of the range of g_p with the open interval (0, 1) is at most a 2-point set.*

Let $h: [0, 1] \to [0, 1]$ be a measurable function (not necessarily non-increasing) with $\int_0^1 h(x)\,dx = M^{-1}$. Let $h_t = th + (1-t)g_p$ for $0 \leqslant t \leqslant 1$. Clearly, $\int_0^1 h_t\,dx = M^{-1}$, $0 \leqslant h_t \leqslant 1$, and $h_0 = g_p$. Thus, by (A1), $a_p(h_t) \leqslant a_p(h_0)$. Let $D(h)$ denote the right derivative of the function $t \to a_p(h_t)$ at the point $t = 0$. Then $D(h) = 0$. The computation gives

$$D(h) = \int_0^1 (\psi \circ g_p)(h - g_p)\,dx$$

where

$$\psi(t) = (\|g_p\|_2^2 \cdot t^{p-1} - 2\|g_p\|_p^p \cdot t)\,pM^{-1}.$$

Now fix ε with $0 < \varepsilon < 2^{-1}$ and let $f \in L_R^\infty$ be an arbitrary function satisfying the conditions

$$\text{(A3)} \qquad \begin{aligned} &f(x) = 0 \quad \text{whenever} \quad g_p(x) \notin [\varepsilon, 1-\varepsilon],\\ &\|f\|_\infty \leqslant 1,\\ &\int_0^1 f(x)\,dx = 0. \end{aligned}$$

Then $h^+ = g_p + \varepsilon f$ satisfies the conditions

$$\|h^+\|_\infty \leqslant 1, \quad \|h^+\|_1 = M^{-1}, \quad h^+: [0, 1] \to [0, 1]$$

and similarly $h^- = g_p - \varepsilon f$. Thus

$$\begin{aligned} \varepsilon^{-1} D(h^+) &= \int (\psi \circ g_p) f\,dx \leqslant 0,\\ \varepsilon^{-1} D(h^-) &= \int (\psi \circ g_p)(-f)\,dx \leqslant 0. \end{aligned}$$

Therefore $\int (\psi \circ g_p) f\,dx = 0$. Thus because of the arbitrariness of f satisfying (A3) we infer that $\psi \circ g_p$ is constant almost everywhere on the set $\{x: g_p(x) \in [\varepsilon, 1-\varepsilon]\}$. Taking into account the arbitrariness of $\varepsilon \in (0, \frac{1}{2})$ we

infer that there exists C such that

$$\psi(g_p(x)) = C \text{ a.e. on } \{x: 0 < g_p(x) < 1\}.$$

Since for $p > 2$ ψ is strictly convex, $\psi^{-1}(C)$ is at most a 2-point set. Furthermore, $0 \leqslant g_p \leqslant 1$. Thus, except may be a set of measure zero, the range of g_p is at most a 4-point set with at most two of these points in the open interval (0, 1). Since g_p is non-increasing, without loss of generality we may assume that g_p is a step function with at most 4 steps.

2° *The number 0 is not a value of g_p; hence the range of g_p is at most a 3-point set.*

Indeed, if 0 is an essential value of a $g \in Z^0$, then there exists a $b > 1$ such that $g(x) = 0$ for $b^{-1} < x \leqslant 1$. Put $g^0(x) = b^{-1} g(b^{-1}x)$. Then

$$\int_0^1 g^0(x)\,dx = \int_0^{b^{-1}} g(x)\,dx = \int_0^1 g(x)\,dx = M^{-1}.$$

Thus $g^0 \in Z_M^0$. A simple computation shows that $a_p(g^0) = b^{1/p} a_p(g) > a_p(g)$. Thus $g \neq g_p$.

3° *The range of g_p is at most a 2-point set.*

It is enough to show that if

$$W = \{\varrho = (\alpha, \beta, \xi, \eta) \in \mathbf{R}^4:\ 0 < \alpha < \beta < 1,\ 0 < \eta < \xi < 1,\ \alpha + \xi(\beta - \alpha) + \eta(1-\beta) = M^{-1}\}$$

then the function $\varphi: W \to \mathbf{R}_+$ defined by

$$\varphi(\alpha, \beta, \xi, \eta) = M^p a_p^p(\chi_{[0,\alpha)} + \xi\chi_{[\alpha,\beta)} + \eta\chi_{[\beta,1]})$$

does not attain its maximum on W. (Observe that

$$\int_0^1 (\chi_{[0,\alpha)} + \xi\chi_{[\alpha,\beta)} + \eta\chi_{[\beta,1]})(x)\,dx = \alpha + \xi(\beta - \alpha) + \eta(1-\beta).)$$

Assume to the contrary that φ attains its maximum on W at the point $\varrho_0 = (a, b, y, z) \in W$. Then by the Lagrange Theorem there exists a $\lambda \in \mathbf{R}$ such that if

$$\Phi(\alpha, \beta, \xi, \eta) = \varphi(\alpha, \beta, \xi, \eta) - \lambda[\alpha + \xi(\beta - \alpha) + \eta(1-\beta) - M^{-1}]$$

then

$$\text{(A4)} \quad \left(\frac{\partial \Phi}{\partial \alpha}\right)_{\varrho=\varrho_0} = 0; \quad \left(\frac{\partial \Phi}{\partial \beta}\right)_{\varrho=\varrho_0} = 0; \quad \left(\frac{\partial \Phi}{\partial \xi}\right)_{\varrho=\varrho_0} = 0; \quad \left(\frac{\partial \Phi}{\partial \eta}\right)_{\varrho=\varrho_0} = 0.$$

Observe that $\varphi = A \cdot B^{-p}$ where

$$A(\varrho) = \alpha + \xi^p(\beta - \alpha) + \eta^p(1-\beta); \quad B(\varrho) = \alpha + \xi^2(\beta - \alpha) + \eta^2(1-\beta).$$

Thus (A4) is equivalent to

$$(1-y^p)B^p-(1-y^2)pB^{p-1}A-\lambda(1-y)B^{2p}=0,$$
$$(y^p-z^p)B^p-(y^2-z^2)pB^{p-1}A-\lambda(y-z)B^{2p}=0,$$
$$py^{p-1}(b-a)B^p-2py(b-a)B^{p-1}A-\lambda(b-a)B^{2p}=0,$$
$$pz^{p-1}(1-b)B^p-2pz(1-b)B^{p-1}A-\lambda(1-b)B^{2p}=0$$

where A and B are taken at $\varrho=\varrho_0$.

Dividing the first equation by $B^p(1-y)$; adding the second to the first and dividing by $B^p(1-z)$; dividing the third by $B^p(b-a)$ and the fourth by $B^p(1-b)$ we get a new system of equations (which is equivalent to the previous one because of our assumptions

$$1\neq b\neq a,\quad 1\neq y,\quad 1\neq z\quad\text{and}\quad B\neq 0):$$

$$\frac{1-y^p}{1-y}-(1+y)p\frac{A}{B}-\lambda=0,$$
$$\frac{1-z^p}{1-z}-(1+z)p\frac{A}{B}-\lambda=0,$$
$$py^{p-1}-2py\frac{A}{B}-\lambda=0,$$
$$pz^{p-1}-2pz\frac{A}{B}-\lambda=0.$$

Put $C=p\frac{A}{B}$. Substracting the new third equation from the new first one, and the new fourth from the new second we get

$$\frac{1-y^p}{1-y}-py^{p-1}-(1-y)C=0,$$
$$\frac{1-z^p}{1-z}-pz^{p-1}-(1-z)C=0.$$

Thus $C=-f'(y)=-f'(z)$ where $f(u)=\frac{1-u^p}{1-u}$. Note that, for $p>2$, $f(u)$ is strictly convex for $0<u<1$. Hence $f'(u)$ is strictly monotone. Thus $y=z$, a contradiction.

4° *Either* $g_p\equiv M^{-1}$ *or the range of* g_p *has two points one of which is* 1.

First we restate the maximum problem. Given $g\in Z_M^0$ there is a unique number c_g with $1\leqslant c_g\leqslant M$ such that $\|c_g g\|_2^2=\|c_g g\|_1$. Let

$$Z_M^*=\{h=c_g g\colon\ g\in Z_M^0\}.$$

Then Z_M^* is compact and there exists an $h_p\in Z_M^*$ such that

$$(*)\qquad \|h_p\|_p=\sup\{\|h\|_p\colon\ h\in Z_M^*\}.$$

Clearly, h_p satisfies $(*)$ iff $h_p=c_{g_p}g_p$ and g_p satisfies:

$$a_p(g_p)=\sup\{a_p(g)\colon\ g\in Z_M^0\}.$$

Thus 4° is equivalent to the following statement:

4_1° *Let*

$$W^*=\{(\alpha,\xi,\eta)\in\boldsymbol{R}^3\colon$$
$$0<\alpha<1,\ 0<\xi<\eta<M(\alpha\xi+(1-\alpha)\eta),\ \alpha\xi+(1-\alpha)\eta=\alpha\xi^2+(1-\alpha)\eta^2\}.$$

Then the function $(\alpha,\xi,\eta)\to\alpha\xi^p+(1-\alpha)\eta^p$ *does not attain its supremum on* W^*.

Assume to the contrary that there is a $\varrho_0=(a,x,y)\in W^*$ such that

$$ax^p+(1-a)y^p=\sup\{\alpha\xi^p+(1-\alpha)\eta^p\colon\ (\alpha,\xi,\eta)\in W^*\}.$$

Then by the Lagrange Theorem there is a $\lambda\in\boldsymbol{R}$ such that if

$$\Phi(\alpha,\xi,\eta)=\alpha\xi^p+(1-\alpha)\eta^p+\lambda[\alpha\xi+(1-\alpha)\eta-\alpha\xi^2-(1-\alpha)\eta^2]$$

then

$$\left(\frac{\partial\Phi}{\partial\alpha}\right)_{\varrho=\varrho_0}=0;\quad \left(\frac{\partial\Phi}{\partial\xi}\right)_{\varrho=\varrho_0}=0;\quad \left(\frac{\partial\Phi}{\partial\eta}\right)_{\varrho=\varrho_0}=0.$$

Equivalently,

$$\text{(A5)}\qquad \begin{aligned} &x^p-y^p+\lambda[(x-y)-x^2+y^2]=0,\\ &px^{p-1}a+\lambda(a-2xa)=0,\\ &py^{p-1}(1-a)+\lambda(1-a)(1-2y)=0.\end{aligned}$$

Multiplying the second equation by $-\frac{x}{pa}$ $(a\neq 0!)$ and the third by $\frac{y}{1-a}$ $(a\neq 1!)$ and adding the results to the first equation we get

$$\lambda\left((x-y)\left(1-\frac{1}{p}\right)-(x^2-y^2)\left(1-\frac{2}{p}\right)\right)=0.$$

Observe that $\lambda\neq 0$ (otherwise from the second equation $xa=0$, a contradiction) and $x\neq y$. Thus

$$\text{(A6)}\qquad x+y=\frac{p-1}{p-2}.$$

From the second and the third equations of (A5) we obtain

$$\text{(A7)} \qquad \frac{x^{p-1}}{2x-1} = \frac{y^{p-1}}{2y-1}.$$

Putting $y = tx$ $(0 < t < 1)$ we get from (A7)

$$x = \frac{1-t^{p-1}}{2t(1-t^{p-2})}, \qquad y = \frac{1-t^{p-1}}{2(1-t^{p-2})}.$$

Thus, in view of (A6),

$$\frac{p-1}{p-2} = \frac{1+t}{2t}\cdot\frac{1-t^{p-1}}{1-t^{p-2}}$$

which yields

$$(p-2)(1-t^{p})+p(t^{p-1}-t) = 0.$$

The desired contradiction follows from the inequality

$$f(t) = (p-2)(1-t^{p})+p(t^{p-1}-t) > 0 \quad \text{for} \quad 0 < t < 1.$$

Indeed,

$$f'(t) = -(p-2)\,pt^{p-1}+p(p-1)\,t^{p-2}-p,$$

$$f''(t) = (p-2)(p-1)\,p(1-t)\,t^{p-3} > 0 \quad \text{for} \quad 0 < t < 1.$$

Thus $f'(t)$ is strictly increasing in $(0, 1)$ and continuous in $[0, 1]$. Since $f'(1) = 0$, we infer that $f'(t) < 0$ in $(0, 1)$. Since $f(1) = 0$, we infer that $f(t) > 0$ for $0 < t < 1$.

We have already shown that any extremal g_p is either a constant function or a step function of the form $\chi_{[0,a)}+x\chi_{[a,1]}$ with $0 < x < 1$. Since

$$\int_0^1 (\chi_{[0,a)}+x\chi_{[a,1]})(t)\,dt = a+x(1-a) = M^{-1},$$

we get $x = (1-Ma)\big(M(1-a)\big)^{-1}$ and $0 < a < M^{-1}$. Observe that for $a = 0$ we get the constant function M^{-1}. Thus all extremals are some members of the one parameter family

$$f_a = \chi_{[0,a)}+\frac{1-Ma}{M(1-a)}\,\chi_{[a,1]} \qquad (0 \leqslant a \leqslant M).$$

Next we show

5° *If* $2 < p \leqslant p_0$ *where* $p_0 = p_0(M)$ *satisfies* $M^{p_0}-1 = p_0(M^2-M)$ *then* $a_p(f_a) < a_p(f_0) = 1$ *for* $0 < a \leqslant M^{-1}$.

Proof. Clearly, it suffices to prove the inequality for $p = p_0$. Since $a_p(f_0) = 1$, it is enough to show that $a_p^p(f_a) < 1$ for $0 < a \leqslant M^{-1}$; equivalently $\|f_a\|_p^p < \|f_a\|_2^{2p}M^p$. We have

$$\|f_a\|_p^p = M^{-p}\left(M_a^p+(1-a)\,\frac{(1-aM)^p}{(1-a)^p}\right)$$

and

$$\|f_a\|_2^{2p} = M^{-2p}\left(M^2a+\frac{(1-aM)^2}{(1-a)^2}\,(1-a)\right)^p = M^{-2p}\left(\frac{1-2aM+aM^2}{1-a}\right)^p.$$

Thus we have to prove that

$$M^p a+\left(\frac{1-aM}{1-a}\right)^p(1-a) < \left(\frac{1-2aM+aM^2}{1-a}\right)^p.$$

Replacing M^p by $1+p(M^2-M)$, using the inequality

$$(1-t)^p < 1-pt+\frac{p(p-1)}{2}\,t^2 \qquad (0 < t < 1),$$

the fact that $p < 3$ (cf. the definition of $p_0(M)$) and the Bernoulli inequality we get

$$\begin{aligned}
M^p a+\left(\frac{1-aM}{1-a}\right)^p(1-a) &= \big(1+p(M^2-M)\big)a+\left(1-a\,\frac{M-1}{1-a}\right)^p(1-a)\\
&< a+p(M^2-M)\,a+1-a-pa(M-1)+\\
&\qquad +p(p-1)\,\frac{(M-1)^2a^2}{2(1-a)}\\
&= 1+pa(M-1)^2\left(1+\frac{(p-1)\,a}{2(1-a)}\right)\\
&< 1+pa(M-1)^2\left(1+\frac{a}{1-a}\right)\\
&= 1+p\,\frac{a(M-1)^2}{1-a} < \left(1+\frac{a(M-1)^2}{1-a}\right)^p\\
&= \left(\frac{1-2aM+aM^2}{1-a}\right)^p.
\end{aligned}$$

6° *If* $p > p_0(M)$ *then* $a_p(f_a) > 1$ *for some* $a \in (0, M^{-1})$.

Proof. It is enough to show that

$$\left(\frac{d}{da}\,a_p^p(f_a)\right)_{a=0} = \lim_{a=+0} a^{-1}\left(\frac{\|f_a\|_p^p}{\|f_a\|_2^{2p}M^p}-1\right) > 0.$$

Since $\lim_{a=+0} \|f_a\|_2^{2p} = M^{-2p}$, we have to show that

$$\lim_{a=+0} a^{-1}(\|f_a\|_p^p - M^p\|f_a\|_2^{2p}) > 0.$$

We have

$$a^{-1}(\|f_a\|_p^p - M^p\|f_a\|_2^{2p})$$
$$= a^{-1}\left[a+\left(\frac{1-Ma}{M(1-a)}\right)^p(1-a)-M^p\left(\frac{1-2Ma+M^2a}{(1-a)M^2}\right)^p\right]$$
$$= [M(1-a)]^{-p}\{M^p-(1-Ma)^p+a^{-1}[(1-Ma)^p-((1-Ma)+(M^2-M)a)^p]\}.$$

Clearly,

$$\lim_{a=+0} a^{-1}[(1-Ma)^p-((1-Ma)+(M^2-M)a)^p] = -p(M^2-M).$$

Thus

$$\lim_{a=+0} a^{-1}(\|f_a\|_p^p - M^p\|f_a\|_2^{2p}) = M^{-p}(M^p-1-p(M^2-M)) > 0$$

because for $p > p_0(M)$, $M^p-1-p(M^2-M) > 0$.

To complete the proof of the theorem, it suffices to show:

7° *The function*

$$\tau(a) = M^p a_p^p(f_a) = \frac{\|f_a\|_p^p}{\|f_a\|_2^{2p}}$$

has exactly one maximum in the interval $[0, M^{-1}]$.

This has already been established in 5° for $p \leqslant p_0(M)$. If $p > p_0$, then, by 6°, $\tau(0)$ is not the maximum; neither is $\tau(M^{-1}) = M^{p-1} < M^p = \tau(0)$. Let us put $a = a(u) = \frac{u-M}{M(u-1)}$. Then $u = \frac{M(1-a)}{1-aM}$; $a = 0$ iff $u = M$; $a \nearrow M^{-1}$ iff $u \nearrow +\infty$. Let $\varphi(u) = \tau(a(u))$. The substitution is monotone and $\varphi(M) = \tau(0) = M^p$, $\lim_{u=+\infty} \varphi(u) = \tau(M^{-1}) = 1$. Thus, by continuity φ attains its maximum somewhere in the open half line $(M, +\infty)$. Hence 7° reduces to

7_1° *The equation* $\varphi'(u) = 0$ *has at most one solution for* $M < u < +\infty$.

We have

$$\tau(a) = aM^p\left(\frac{1-a}{1+a(M^2-2M)}\right)^p+(1-a)\left(\frac{1-aM}{1+a(M^2-2M)}\right)^p,$$

$$\varphi(u) = M^p\left[\frac{u-M}{M(u-1)}\left(\frac{u}{u-M+1}\right)^p+\frac{u(M-1)}{M(u-1)}\frac{1}{(u-M+1)^p}\right]$$
$$= M^{p-1}[u^{p+1}-Mu^p+(M-1)u](u-1)^{-1}(u-M+1)^{-p}.$$

Differentiating, after somewhat long computation we get

$$\varphi'(u) = \frac{M(M-1)}{M^p}\cdot\frac{Q(u)}{(u-1)^2(u-M+1)} \qquad (M < u < +\infty)$$

where

$$Q(u) = (1-p)u^{p+1}+((p-1)(M-1)+2p)u^p-pMu^{p-1}-pu^2+(p-1)u+M-1.$$

Observe that $Q'''(u) = u^{4-p}L(u)$ where

$$L(u) = -p(p-1)[(p^2-1)u^2-(p-2)((p-1)(M-1)+2p)u+(p-2)(p-3)M].$$

Clearly, $L(u) = 0$ has at most one positive root for $p \leqslant 3$. Thus, using the fact

(**) between every two consecutive zeros of the function there is at least one zero of the derivative,

we infer that for $p \leqslant 3$, Q has at most 4 zeros on the half line $(0, \infty)$.

If $p > 3$, then $L(u) = 0$ has two positive roots. Thus Q'' has at most 3 zeros (by (**)) for $u > 0$. On the other hand, if $p > 3$ then $\lim_{u=+\infty} Q''(u) = -\infty$ and $Q''(0) = -2p < 0$. Therefore Q'' has the even number of zeros on the half line $(0, +\infty)$. It cannot have 4 or more because Q''' has only 2 (by (**)). Thus Q'' has at most 2 zeros for $u > 0$. Hence, by (**), Q has at most 4 zeros on the half line $(0, +\infty)$ also for $p > 3$.

Next observe that $u = 1$ is the double zero for Q. Thus on the half line $[M, +\infty)$ Q may have at most two zeros. On the other hand, $Q(M) = M^p - 1 - p(M^2-M) > 0$ because $p > p_0(M)$, and $\lim_{u=+\infty} Q(u) = -\infty$. Thus Q has an odd number of zeros on the half line $[M, +\infty)$. Hence Q has at most one zero on the half line $[M, +\infty)$.

Remarks. 1° It is easy to show that

$$\inf\{a_p(g): g \in Z_M^0\} = M^{-(p-1)/p} \quad \text{for} \quad 1 \leqslant p < \infty.$$

Moreover, $a_p(g) = M^{-(p-1)/p}$ iff $g = \chi_{[1,M^{-1})}$.

Proof. For every $g \in Z_M^0$ the Hölder inequality $\int_0^1 g^2\,dt \leqslant \|g\|_p\,\|g\|_{p'}$ yields

$$a_p(g) = \frac{\|g\|_p}{M\|g\|_2^2} \geqslant \frac{\|g\|_p}{M\|g\|_p\|g\|_{p'}} = M^{-1}\|g\|_{p'}^{-1}.$$

By the inequality $\|g\|_{p'} \leqslant \|g\|_\infty^{1/p}\,\|g\|_1^{1/p'}$ for $g \in Z_M^0$, we then get

$$M^{-1}\|g\|_{p'}^{-1} \geqslant M^{-1}\|g\|_1^{-1/p} = M^{-(p-1)/p}.$$

It is easy to see that the equality in this chain of inequalities holds only for the characteristic function.

2° We have $\sup\{a_\infty(g): g \in Z_M^0\} = M$; the supremum is not attained. One can easily see that the set $\{g \in Z_M^0: \|g\|_\infty = 1\}$ is dense in Z_M^0 in the L^2-norm. Thus

$$\sup\{a_\infty(g): g \in Z_M^0\} = \sup\left\{\frac{1}{M\|g\|_2^2}: g \in Z_M^0\right\} = (\inf\{M\|g\|_2^2: g \in Z_M^0\})^{-1}$$
$$\leqslant (M \cdot \inf\{\|g\|_1^2: g \in Z_M^0\})^{-1} = M.$$

On the other hand, $\lim\limits_{a=+0} a_\infty(f_a) = M$.

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Multipliers along curves

by

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Abstract. The authors analyse boundedness of Fourier multiplier operators which are constant along curves. Boundedness is shown to depend upon a balance between the curvature of the level curves and the lack of smoothness of the multiplier function.

Introduction. In this paper we shall give an analysis of the L^p boundedness of Fourier multiplier operators which are constant along curves in the plane. The three fundamental examples for our study are the following:

(A) $m(x, y) = \varphi(x^2+y^2)$, φ smooth and compactly supported. Then m gives a multiplier operator bounded on all L^p, $1 \leqslant p \leqslant \infty$. If φ has a discontinuity of the first type away from origin, the work of C. Fefferman [1] shows m gives a multiplier bounded on L^p if and only if $p = 2$.

(B) $m(x, y) = \varphi(x^2-y^2)$, φ smooth and compactly supported. Then m gives a multiplier operator bounded on all L^p, $1 < p < \infty$, by the Hörmander–Mihlin multiplier theorem [4].

(C) $m(x, y) = \varphi(y-x^2)$. If φ has any reasonable growth properties, m gives a multiplier of L^p if and only if $p = 2$, from the work of Kenig and Tomas [2].

These examples show that the L^p boundedness of such multipliers depends on a balance between the curvature of the level curves, and the "bumpiness" of the multiplier function. It is these intuitive ideas we shall make precise. Such questions have already been considered by Ruiz [3].

In Section one of the paper we shall use some techniques of Ruiz [3] to give a different geometric characterization of the level curves. In Section two we shall follow Ruiz' [3] proof and show certain restrictions on φ can be removed.

Section one. We shall analyse the $L^p(R^2)$ boundedness of Fourier multiplier operators T, where

$$\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$$

and m is constant along level curves of a function $F: R^2 \to R^1$, that is, $m(\xi) = \varphi \circ F(\xi)$. We shall consider a restricted class of level curves, which we shall call regular.