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The criteria for local uniform rotundity of Orlicz spaces

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Abstract. This paper is concerned with the geometrical properties of Orlicz spaces under Luxemburg norm, particularly with the local uniform rotundity of these spaces. There were found criteria for this property in the case of an atomless measure as well as in the case of a purely atomic measure. It appears that necessary and sufficient conditions for rotundity, midpoint local uniform rotundity and local uniform rotundity coincide when the measure is atomless, but in the case of spaces of sequences it is otherwise, i.e., the conditions for local uniform rotundity lię "between" the conditions for rotundity and uniform rotundity.

0. In the last years the different geometrical properties of Banach spaces are strenuously studied. Therefore it is interesting to investigate these problems in classical spaces, e.g., Orlicz spaces. Uniform rotundity (UR) (uniform convexity) and rotundity (R) (strict convexity) of such spaces, even more generalized called Musielak–Orlicz spaces, were exactly examined in papers [2], [3], [4], [5], [9]. Here we shall consider the relations between some convexity properties of Orlicz spaces, particularly we are interested in the property called the local uniform rotundity.

An arbitrary Banach space $(X, \|\ \|)$ is said to be *locally uniformly rotund* (LUR), if for each $\varepsilon > 0$ and $y \in X$ with $\|y\| = 1$ there is $\delta(\varepsilon, y) > 0$ such that for all $x \in X$ with $\|x\| = 1$ the inequality $\|x-y\| \geqslant \varepsilon$ implies $\|(x+y)/2\| \leqslant 1-\delta(\varepsilon, y)$.

The space $(X, \|\ \|)$ is *midpoint locally uniformly rotund* (MLUR), if for each $\varepsilon > 0$ and $w \in X$ with $\|w\| = 1$ there exists $\delta(\varepsilon, w) > 0$ such that if x and y are in X with $\|x\| = \|y\| = 1$ and $\|x-y\| \geqslant \varepsilon$, then $\|x+y-2w\| \geqslant \delta(\varepsilon, w)$ [8].

Note that in these definitions the equalities $\|x\| = 1$, $\|y\| = 1$ may be replaced by inequalities $\|x\| \leqslant 1$, $\|y\| \leqslant 1$. Moreover, it is known that UR $\to$ LUR $\to$ MLUR $\to$ R. It will be proved that R $\to$ LUR in Orlicz spaces in the case of atomless measure, but not in the case of spaces of sequences.

Let in the following $\boldsymbol{R}$ be the real line, $\boldsymbol{N}$ the set of positive integers, (T, Σ, μ) a measure space, i.e., Σ is a σ-algebra of subsets of an arbitrary set T and μ is a nonnegative measure on Σ. In the sequel let us suppose that μ is only atomless or only purely atomic. By φ let us denote the Orlicz function, i.e., $\varphi: \boldsymbol{R} \to [0, +\infty)$ is an even convex function such that $\varphi(0) = 0$. The functional

$$I_\varphi(x) = \int_T \varphi(x(t))\,d\mu$$

defined on the set of all measurable functions $x: T \to \boldsymbol{R}$ is a *pseudomodular*. This functional defines the modular space called the *Orlicz space* and usually denoted by L_φ ([6], [7]). The norm introduced in L_φ, the so-called *Luxemburg norm*, is defined as follows:

$$\|x\|_\varphi = \inf\{\varepsilon > 0: I_\varphi(x/\varepsilon) \leqslant 1\},$$

where x is a function belonging to L_φ. The conditions of the type of "Δ_2" are very important in the theory of Orlicz spaces. Here they will be used very often. So, we say that φ *satisfies the condition Δ_2 on whole $\boldsymbol{R}$*, or shortly Δ_2, if $\varphi(2u) \leqslant k\varphi(u)$ for every $u \in \boldsymbol{R}$ and some positive k. The function φ *satisfies the condition Δ_2 for large arguments* [*small arguments*] if there exist $k, u_0 > 0$ such that $\varphi(2u) \leqslant k\varphi(u)$ for $|u| \geqslant u_0$ [$|u| \leqslant u_0$], where $\varphi(u_0) > 0$. The condition Δ_2 for small arguments is usually denoted by δ_2 and here it will be used in this sense.

Recall that φ is *strictly convex* on an interval $[a, b]$ if $\varphi((u+v)/2) < (\varphi(u)+\varphi(v))/2$ for each $u, v \in [a, b]$, $u \neq v$. We also say that the pseudomodular I_φ is *locally uniformly rotund* if for every $\varepsilon > 0$ and y with $I_\varphi(y) = 1$ there exists $\delta(\varepsilon, y) > 0$ such that for all x with $I_\varphi(x) = 1$ the inequality $I_\varphi(x-y) \geqslant \varepsilon$ implies $I_\varphi((x+y)/2) \leqslant 1-\delta(\varepsilon, y)$.

0.1. Lemma. *Let $(X, \|\ \|)$ be a Banach space. If $f: X \to [0, +\infty)$ is a convex function satisfying the conditions*

(1) $\|x\| \leqslant 1$ *implies* $f(x) \leqslant 1$ *for* $x \in X$,

(2) *there is $M > 0$ such that $f(2x) \leqslant M$ if $f(x) \leqslant 1$,*

then f is uniformly continuous in the ball $K(0, 1) = \{x \in X: \|x\| \leqslant 1\}$.

The proof of this lemma may be omitted because it is almost the same as the proof of Lemma 4 in [5]. The assumption (2) differs not essentially from that in Lemma 4.

0.2. Lemma. *Let φ satisfy one of the following conditions:*

(i) Δ_2 *if the measure μ is atomless and $\mu T = \infty$,*

(ii) Δ_2 *for large arguments and it vanishes only at zero if μ is atomless and $\mu T < \infty$,*

(iii) δ_2 (*i.e., Δ_2 for small arguments*) *if μ is purely atomic.*

Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $I_\varphi(x-y) < \delta$ implies $|I_\varphi(x) - I_\varphi(y)| < \varepsilon$ for all $x \in L_\varphi$ and every $y \in K_\varphi(0, 1) = \{z \in L_\varphi: \|z\|_\varphi \leqslant 1\} = \{z \in L_\varphi: I_\varphi(z) \leqslant 1\}$.

In the case of (iii), this lemma was formulated and proved as Lemma 5 in [5]. But applying Lemma 0.1 in the place of Lemma 4 from [5], the proof is analogous in the remaining cases.

0.3. Lemma. *The Orlicz space L_φ is locally uniformly convex iff the pseudomodular I_φ is locally uniformly convex and φ satisfies one of conditions* (i), (ii) *or* (iii) *formulated in the above lemma.*

The proof will be omitted because it is identical as the proof of Lemma 1 in [5].

0.4. Lemma. *The function*

$$h(u) = \frac{2\varphi((u+v)/2)}{\varphi(u)+\varphi(v)}$$

is nondecreasing in $[0, v]$ for each $v \in (0, \infty)$.

Proof. Since φ has the derivative almost everywhere,

$$h'(u) = \frac{\varphi'((u+v)/2)[\varphi(u)+\varphi(v)] - 2\varphi((u+v)/2)\varphi'(u)}{[\varphi(u)+\varphi(v)]^2}.$$

If $u \in [0, v]$, then $\varphi'(u) \leqslant \varphi'((u+v)/2)$ and $2\varphi((u+v)/2) \leqslant \varphi(u)+\varphi(v)$. Therefore $h'(u) \geqslant 0$ for $u \in [0, v]$, which shows the lemma.

Remark. The function h defined in the above lemma may behave in the interval $[v, \infty)$ in a very different way; it depends on φ. If we take, for example, $\varphi(u) = u^2$, then h is decreasing on $[v, \infty)$ for every $v > 0$. On the other hand, if we take $\varphi'(u) = \operatorname{arc\,tan} u$, then

$$\lim_{u\to\infty} h(u) = \lim_{u\to\infty} \varphi'((u+v)/2)/\varphi'(u) = \lim_{u\to\infty} (\operatorname{arc\,tan}(u+v)/2)/\operatorname{arc\,tan} u = 1$$

for each $v > 0$. So $h(v) = 1 = \lim_{u\to\infty} h(u)$. But $h(u) < 1$ for all $u \in (v, \infty)$, because φ is strictly convex. This simply implies that h cannot be monotonous on $[v, \infty)$.

0.5. Lemma. *Let φ be strictly convex on an interval $[-a, a]$. Then for each $\varepsilon > 0$, $d_1, d_2 \in (0, a]$, $d_1 < d_2$, there exists $p \equiv p(\varepsilon, d_1, d_2) \in (0, 1)$ such that*

$$\varphi((u+v)/2) \leqslant (1-p)(\varphi(u)+\varphi(v))/2$$

if $|u-v| \geqslant \varepsilon \max(|u|, |v|)$ and $\max(|u|, |v|) \in [d_1, d_2]$.

Proof. Suppose $u > v \geqslant 0$ and $\varepsilon \in (0, 1)$. Then $v \in [0, (1-\varepsilon)u]$ and $u \in [d_1, d_2]$. If we put

$$A = \{(u, v): u \in [d_1, d_2] \wedge v \in [0, (1-\varepsilon)u]\},$$

then $g(u, v) < 1$ for all $(u, v) \in A$, where

$$g(u, v) = \frac{2\varphi((u+v)/2)}{\varphi(u)+\varphi(v)}.$$

But the set A is compact and the function g is continuous on A, so there is $p \in (0, 1)$ such that the inequality from the lemma is fulfilled. To finish the proof it is enough to consider the case when u and v are of different signs. For example, let $u \geqslant 0$, $v < 0$ and $u \leqslant -v$. Then $u \geqslant (1-\varepsilon)v$ and $v \in [-d_2, -d_1]$. But the first inequality is always satisfied by the assumption $0 \leqslant u \leqslant -v$. Let

$$B = \{(u, v): v \in [-d_2, d_1] \wedge u \in [0, -v]\}.$$

The function g is continuous on B and $g(u, v) < 1$ for each $(u, v) \in B$, because φ is strictly convex on $[-a, a]$. Then, similarly as in the previous considerations, there is $p \in (0, 1)$ such that the desired inequality holds.

The following simple lemma is proved as Lemma 6 in [5].

0.6. LEMMA. *Let $u_i, v_i \in \mathbf{R}$, $i = 1, 2$ and $u_2 < u_1 < v_1 < v_2$. If φ is strictly convex on $[u_1, v_1]$, then there exists $p \in (0, 1)$ dependent on u_i, v_i such that*

$$\varphi((u+v)/2) \leqslant (1-p)(\varphi(u)+\varphi(v))/2$$

for all $u \in [u_2, u_1]$ and $v \in [v_1, v_2]$.

0.7. THEOREM ([2], [4], [9]). *The Orlicz space L_φ is rotund iff one of the following conditions is fulfilled:*

φ is strictly convex on whole $\mathbf{R}$ and condition (i) *is satisfied,*

φ is strictly convex on whole $\mathbf{R}$ and (ii) *is satisfied,*

φ is strictly convex on the interval $[-u_0, u_0]$, where $\varphi(u_0) = 1/2$ and (iii) *holds,*

where (i), (ii), (iii) *denote the conditions from Lemma* 0.2.

1. In this part there are given the main results of this paper.

1. THEOREM. *Let the measure μ be atomless. Then the following conditions are equivalent:*

(a) *the Orlicz space L_φ is locally uniformly rotund,*

(b) *L_φ is midpoint locally uniformly rotund,*

(c) *L_φ is rotund,*

(d) *the function φ is strictly convex on $\mathbf{R}$ and it fulfils condition Δ_2 for large arguments if $\mu T < \infty$ and condition Δ_2 if $\mu T = \infty$.*

Proof. The implications (a) $\to$ (b) $\to$ (c) are evident. Also (c) $\to$ (d) by Theorem 0.7 giving criteria for rotundity of Orlicz spaces. So, to complete the proof it is enough to show the implication (d) $\to$ (a).

First suppose $\mu T < \infty$. Let $\varepsilon \in (0, 1)$ be arbitrary and $I_\varphi(x) = I_\varphi(y) = 1$ and $I_\varphi(x-y) \geqslant \varepsilon$, where $x, y \in L_\varphi$. There exist $c_1, k > 0$ such that

$$(1.1) \qquad \varphi(2c_1)\mu T < (1/16)\varepsilon \quad \text{and} \quad \varphi(2u) \leqslant k\varphi(u)+\varphi(2c_1)$$

for all $u \in \mathbf{R}$, by the assumed Δ_2 condition for large arguments. Since $I_\varphi(y) < \infty$, there exist a constant $c > 0$ and a set $T_1 \subset T$ such that

$$(1.2) \quad T_1 = \{t \in T: |y(t)| > c \vee |y(t)| < 1/c\} \text{ and } \int_{T_1} \varphi(y(t))\,d\mu < (1/32k)\varepsilon.$$

Now, we find a constant $c_2 > c$ satisfying

$$(1.3) \qquad \varphi(c)/\varphi(c_2) < (1/32k)\varepsilon.$$

Let

$$T_2 = \{t \in T: |x(t)| > c_2\}.$$

Denote

$$T_0 = T \setminus (T_1 \cup T_2) = \{t \in T: (1/c) \leqslant |y(t)| \leqslant c\} \cap \{t \in T: |x(t)| \leqslant c_2\}.$$

Suppose

$$(1.4) \qquad I_\varphi((x-y)\chi_{T_0}) < (3/4)\varepsilon.$$

However, $I_\varphi(x-y) \geqslant \varepsilon$, and so

$$(1.5) \qquad I_\varphi((x-y)\chi_{T_1 \cup T_2}) > (1/4)\varepsilon.$$

By the definition of T_2, we have

$$\varphi(c_2)\mu T_2 \leqslant \int_{T_2} \varphi(x(t))\,d\mu \leqslant 1,$$

and so $\mu T_2 \leqslant 1/\varphi(c_2)$. However, $|y(t)| \leqslant c$ for $t \in T_2 \setminus T_1$, and so

$$I_\varphi(y\chi_{T_2 \setminus T_1}) \leqslant \varphi(c)\mu T_2 \leqslant \varphi(c)/\varphi(c_2) < (1/32k)\varepsilon,$$

by (1.3). Therefore and by (1.2), we get

$$(1.6) \qquad I_\varphi(y\chi_{T_1 \cup T_2}) < (1/16k)\varepsilon.$$

Thus, from (1.5), (1.1) and (1.6) we obtain

$$\begin{aligned}(1/4)\varepsilon &< I_\varphi((x-y)\chi_{T_1 \cup T_2}) \\ &\leqslant (k/2)I_\varphi(x\chi_{T_1 \cup T_2})+(k/2)I_\varphi(y\chi_{T_1 \cup T_2})+\varphi(2c_1)\mu T \\ &\leqslant (k/2)I_\varphi(x\chi_{T_1 \cup T_2})+(3/32)\varepsilon.\end{aligned}$$

Hence

$$(1.7) \qquad I_\varphi(x\chi_{T_1 \cup T_2}) > (5/16k)\varepsilon.$$

But $I_\varphi(x) = I_\varphi(y) = 1$ and (1.6) and (1.7) are fulfilled, and so $I_\varphi(y\chi_{T_0}) > 1-(1/16k)\varepsilon$ and $I_\varphi(x\chi_{T_0}) < 1-(5/16k)\varepsilon$. Now, apply Lemma 0.2 to the functions $y\chi_{T_0}$, $x\chi_{T_0}$, because $y\chi_{T_0}, x\chi_{T_0} \in K_\varphi(0, 1)$ and $|I_\varphi(y\chi_{T_0}) - I_\varphi(x\chi_{T_0})| > (1/4k)\varepsilon$. Then there exists $\delta > 0$ dependent only on ε, such that

(1.8) $$I_\varphi\big((x-y)\chi_{T_0}\big) \geqslant \delta.$$

If assumption (1.4) is not satisfied, then we have (1.8) immediately, because it must be $\delta \leqslant (3/4)\varepsilon$. Now, let

$$T_3 = \{t \in T_0\colon |x(t) - y(t)| \geqslant (\delta/4)\cdot \max(|x(t)|, |y(t)|)\}.$$

But $1/c \leqslant \max(|x(t)|, |y(t)|) \leqslant c_2$ for $t \in T_3$ and φ is strictly convex on $\boldsymbol{R}$, so applying Lemma 0.5 with $\delta/4$, $1/c$, c_2 in the place of ε, d_1, d_2 we find $p > 0$, dependent only on ε and y, such that

$$\varphi\big((x(t)+y(t))/2\big) \leqslant (1-p)\big(\varphi(x(t)) + \varphi(y(t))\big)/2$$

for $t \in T_3$. Thus

(1.9) $$I_\varphi\big((x+y)/2\big) \leqslant 1 - (p/2)\big(I_\varphi(x\chi_{T_3}) + I_\varphi(y\chi_{T_3})\big).$$

If $t \in T_0 \setminus T_3$, then

$$\varphi\big(x(t) - y(t)\big) \leqslant (\delta/4)\big(\varphi(x(t)) + \varphi(y(t))\big),$$

and so $I_\varphi\big((x-y)\chi_{T_0 \setminus T_3}\big) \leqslant \delta/2$. But $I_\varphi\big((x-y)\chi_{T_0}\big) \geqslant \delta$, by (1.8), and so

(1.10) $$I_\varphi\big((x-y)\chi_{T_3}\big) > \delta/2.$$

By condition Δ_2 for large arguments, we can choose constants c_3, $k_1 > 0$ such that $\varphi(2c_3)\mu T < \delta/4$ and $\varphi(2u) \leqslant k_1 \varphi(u) + \varphi(2c_3)$ for each $u \in \boldsymbol{R}$ (let us note that c_3 and k_1 are dependent only on δ, then they depend only on ε). Therefore and by (1.10) we have

$$\begin{aligned} I_\varphi(x\chi_{T_3}) + I_\varphi(y\chi_{T_3}) &\geqslant (2/k_1)\big(I_\varphi\big((x-y)\chi_{T_3}\big) - \varphi(2c_3)\mu T\big) \\ &\geqslant (2/k_1)(\delta/2 - \delta/4) = \delta/2k_1, \end{aligned}$$

which implies

$$I_\varphi\big((x+y)/2\big) \leqslant 1 - (p\delta/4k_1),$$

by (1.9). Thus we found the constant $p\delta/4k_1$ dependent only on ε and y satisfying the last inequality, which in virtue of Lemma 0.3 finishes the proof in the case of $\mu T < \infty$. If we suppose $\mu T = \infty$, then the proof is analogous; it is even simpler because the assumption of condition Δ_2 allows us to replace condition (1.1) by the inequality $\varphi(2u) \leqslant k\varphi(u)$ which holds for each $u \in \boldsymbol{R}$ and some $k > 0$. In this way the proof of the theorem is finished.

Let us turn to the case of purely atomic measure, i.e., we identify T with $\boldsymbol{N}$. Moreover, generally suppose $\mu\{n\} = 1$ for each $n \in \boldsymbol{N}$. Then the elements of L_φ are real sequences and the integral becomes the sum, i.e.,

$I_\varphi(x) = \sum_{n=1}^{\infty} \varphi(u_n) = \sum_{n \geqslant 1} \varphi(u_n)$ for $x = (u_n)$. Such space L_φ traditionally is called the *Orlicz sequence space* and it is denoted by l_φ.

2. Now we will prove the following theorem:

2. Theorem. *The Orlicz sequence space l_φ is locally uniformly convex if and only if the following conditions are satisfied:*

(1) *the function φ fulfils condition δ_2,*

(2) (a) *the function φ is strictly convex on the interval $[-v_0, v_0]$, where $\varphi(v_0) = 1$,*

or

(b) *φ is strictly convex on the interval $[-u_0, u_0]$, where $\varphi(u_0) = 1/2$ and it fulfils the condition*

$$\overline{\lim_{u\to 0}} \frac{2\varphi(u/2)}{\varphi(u)} < 1.$$

Proof. *Sufficiency:* Let $x = (u_n)$, $y = (v_n) \in l_\varphi$ be such that $I_\varphi(x) = I_\varphi(y) = 1$ and $I_\varphi(x - y) \geqslant \varepsilon$, where $\varepsilon \in (0, 1)$ is arbitrary. Let us consider a few cases.

(A) Suppose φ fulfils condition δ_2 and it is strictly convex on the interval $[-v_0, v_0]$. So we have the inequality

(2.1) $$\varphi(2u) \leqslant k\varphi(u)$$

for all $|u| \leqslant v_0$ and some $k > 0$. There exists a positive constant c such that

(2.2) $$\sum_{E_1} \varphi(v_n) < (1/8k)\varepsilon, \quad \text{where} \quad E_1 = \{n \in \boldsymbol{N}\colon |v_n| < c\}.$$

Let us denote

(2.3) $$E_0 = \boldsymbol{N} \setminus E_1 = \{n \in \boldsymbol{N}\colon |v_n| \geqslant c\}.$$

We shall show that there exists $l \equiv l(\varepsilon) > 0$ satisfying

(2.4) $$\sum_{E_0} \varphi(u_n - v_n) \geqslant l.$$

In order to do this, assume $\sum_{E_0} \varphi(u_n - v_n) < (3/4)\varepsilon$, which implies $\sum_{E_1} \varphi(u_n - v_n) > (1/4)\varepsilon$. Hence and by (2.1) and (2.2), we obtain

$$\begin{aligned} (1/4)\varepsilon &< (k/2)\sum_{E_1} \varphi(u_n) + (k/2)\sum_{E_1} \varphi(v_n) \\ &< (k/2)\sum_{E_1} \varphi(u_n) + (1/16)\varepsilon. \end{aligned}$$

Therefore

$$\sum_{E_1} \varphi(u_n) > (3/8k)\varepsilon.$$

But $I_\varphi(x) = I_\varphi(y) = 1$, and so by the above and (2.2),

$$\sum_{E_0} \varphi(v_n) > 1-(1/8k)\varepsilon \quad \text{and} \quad \sum_{E_0} \varphi(u_n) < 1-(3/8k)\varepsilon.$$

Then

$$\Big|\sum_{E_0} \varphi(v_n) - \sum_{E_0} \varphi(u_n)\Big| > (1/4k)\varepsilon,$$

and so one can apply Lemma 0.2 with $(1/4k)\varepsilon$ in the place of ε. Thus we find a constant $l(\varepsilon) < (3/4)\varepsilon$ such that inequality (2.4) will be satisfied. Let us define the set

$$E_2 = \{n \in E_0 \colon |u_n - v_n| \geqslant (l/4)\max(|u_n|, |v_n|)\}.$$

Since $\max(|u_n|, |v_n|) \in [c, v_0]$ for each $n \in E_2$ and φ is strictly convex on $[-v_0, v_0]$, by Lemma 0.5 there is $p \equiv p(\varepsilon, y)$ such that

$$\varphi((u_n+v_n)/2) \leqslant (1-p)(\varphi(u_n)+\varphi(v_n))/2$$

for each $n \in E_2$. Hence

$$I_\varphi((x+y)/2) \leqslant 1-(p/2)\Big(\sum_{E_2} \varphi(u_n) + \sum_{E_2} \varphi(v_n)\Big). \tag{2.5}$$

If $n \in E_0 \setminus E_2$, then $\varphi(u_n - v_n) \leqslant (l/4)(\varphi(u_n)+\varphi(v_n))$ and hence $\sum_{E_0\setminus E_2} \varphi(u_n - v_n) \leqslant l/2$. Therefore $\sum_{E_2} \varphi(u_n - v_n) \geqslant l/2$, by (2.4). Hence

$$\sum_{E_2} \varphi(u_n) + \sum_{E_2} \varphi(v_n) \geqslant (2/k)\sum_{E_2} \varphi(u_n - v_n) \geqslant l/k.$$

Then in virtue of (2.5),

$$I_\varphi((x+y)/2) \leqslant 1 - pl/2k,$$

where the constant $pl/2k$ depends only on ε and y. Thus, the space l_φ is locally uniformly convex, by Lemma 0.3.

(B) Let us suppose φ is strictly convex on $[-u_0, u_0]$ and it fulfils condition δ_2 and the condition

$$\overline{\lim_{u\to 0}}\, 2\varphi(u/2)/\varphi(u) < 1.$$

The last condition is equivalent to the following

$$\varphi(u/2) \leqslant (1-q)\varphi(u)/2 \tag{2.6}$$

for each $|u| \leqslant v_0$ and some $q \in (0, 1)$. There are at most two indices m, n such that $\varphi(u_m) > 1/2$ and $\varphi(v_n) > 1/2$.

I. First, assume that there are exactly two indices m, n where $m \neq n$. One can put $m = 1$, $n = 2$ without loss of generality. Also we may take the elements u_n, v_n $(n = 1, 2)$ only nonnegative. Indeed, if for example $u_1 < 0$ and $v_1 \geqslant 0$, then $\varphi(u_1) > 1/2$ and $\varphi(v_1) < 1/2$, where $u_1 \in [-v_0, -u_0]$ and $v_1 \in [0, u_0]$. By Lemma 0.6, we find $p \in (0, 1)$ independent of x and y such that

$$\varphi((u_1+v_1)/2) \leqslant (1-p)(\varphi(u_1)+\varphi(v_1))/2.$$

Hence immediately we have the following inequality

$$I_\varphi((x+y)/2) \leqslant 1-(p/2)(\varphi(u_1)+\varphi(v_1)) \leqslant 1-p/4, \tag{2.7}$$

because $\varphi(u_1) > 1/2$.

Let δ be the number from Lemma 0.2, where ε is replaced by $\varepsilon/8k$. Moreover, let $i \in N$ be such that $i \geqslant 5$ and $\varepsilon/2^{i-1} < \delta$.

(a) Let us assume

$$\varphi(v_1) \leqslant 1/2-\varepsilon/2^{i+1} \quad \text{or} \quad \varphi(u_2) \leqslant 1/2-\varepsilon/2^{i+1}.$$

Since $\varphi(u_1) > 1/2$ and $\varphi(v_2) > 1/2$, there is $p \equiv p(\varepsilon) \in (0, 1)$ such that

$$\varphi((u_n+v_n)/2) \leqslant (1-p)(\varphi(u_n)+\varphi(v_n))/2$$

holds for $n = 1$ or $n = 2$. Hence, similarly as (2.7) we get $I_\varphi((x+y)/2) \leqslant 1-p/4$.

(b) Now, let

$$1/2-\varepsilon/2^{i+1} < \varphi(v_1) < 1/2 \quad \text{and} \quad 1/2-\varepsilon/2^{i+1} < \varphi(u_2) < 1/2. \tag{2.8}$$

But $\varphi(u_1)+\varphi(u_2) \leqslant 1$, $\varphi(v_1)+\varphi(v_2) \leqslant 1$, $\varphi(u_1) > 1/2$ and $\varphi(v_2) > 1/2$, then

$$1/2 < \varphi(v_2) < 1/2+\varepsilon/2^{i+1} \quad \text{and} \quad 1/2 < \varphi(u_1) < 1/2+\varepsilon/2^{i+1}. \tag{2.9}$$

Let us denote

$$\tilde{E}_0 = \{n \in N \colon n \neq 1,2 \wedge n \notin E_1\} = \{n \in N \colon n \geqslant 3 \wedge |v_n| \geqslant c\},$$

where E_1 is the set from condition (2.2). Similarly as (2.4) we will show that

$$\sum_{\tilde{E}_0} \varphi(u_n - v_n) \geqslant l_1, \tag{2.10}$$

where l_1 is the constant dependent only on ε. Indeed, if

$$\sum_{\tilde{E}_0} \varphi(u_n - v_n) < (3/4)\varepsilon \quad \text{then} \quad \sum_{E_1\cup\{1,2\}} \varphi(u_n - v_n) > (1/4)\varepsilon.$$

But, by inequalities (2.8), (2.9),

$$\varphi(u_n-v_n) \leqslant |\varphi(u_n)-\varphi(v_n)| \leqslant \varepsilon/2^i \leqslant \varepsilon/2^5 \tag{2.11}$$

for $n = 1, 2$, because $i \geqslant 5$ by the assumption. Then $\sum_{E_1} \varphi(u_n-v_n) > (3/16)\varepsilon$. Hence and by (2.2), we get easily

$$\sum_{E_1} \varphi(u_n) > (1/4k)\varepsilon. \tag{2.12}$$

Then, by Lemma 0.2, in virtue of $I_\varphi(x) = I_\varphi(y) = 1$ and inequalities (2.2), (2.12) there exists $\delta > 0$ chosen for $\varepsilon/8k$ such that $\sum_{N\setminus E_1} \varphi(u_n-v_n) \geqslant \delta$. But $N\setminus E_1 = \tilde{E}_0 \cup \{1, 2\}$ and so

$$\sum_{\tilde{E}_0} \varphi(u_n-v_n) \geqslant \delta-[\varphi(u_1-v_1)+\varphi(u_2-v_2)] \geqslant \delta-\varepsilon/2^{i-1},$$

by (2.11). Putting $l_1 = \delta-\varepsilon/2^{i-1}$ we have $l_1 > 0$ from the assumption $\varepsilon/2^{i-1} < \delta$. Since we can always take $\delta < (3/4)\varepsilon$, inequality (2.10) is thus obtained. In the following the proof will be analogous to the part of the proof of (A). Namely, in the definition of set E_2 we replace E_0 by $\tilde{E}_0$ and l by l_1. Then we have $\max(|u_n|, |v_n|) \in [c, u_0]$ for each $n \in E_2$ and we may apply Lemma 0.5, because φ is strictly convex on $[-u_0, u_0]$. Next, we continue the considerations similarly, replacing inequality (2.4) by (2.10). Thus we will get the desired result.

II. Now, let $m = n = 1$ without loss of generality. Similarly as in I we may suppose that u_1, v_1 are nonnegative. If $\varphi(u_1-v_1) \geqslant (1/2)\varepsilon$, then $|\varphi(u_1)-\varphi(v_1)| \geqslant (1/2)\varepsilon$. But

$$\varphi(u_1) = 1-\sum_{n\geqslant 2} \varphi(u_n) \quad \text{and} \quad \varphi(v_1) = 1-\sum_{n\geqslant 2} \varphi(v_n),$$

and so

$$\Big|\sum_{n\geqslant 2} \varphi(u_n)-\sum_{n\geqslant 2} \varphi(v_n)\Big| \geqslant (1/2)\varepsilon.$$

Then there exists, by Lemma 0.2, a constant $l_2 \leqslant (1/2)\varepsilon$ dependent only on ε such that

$$\sum_{n\geqslant 2} \varphi(u_n-v_n) \geqslant l_2. \tag{2.13}$$

This condition is always true because if $\varphi(u_1-v_1) < (1/2)\varepsilon$, then $\sum_{n\geqslant 2} \varphi(u_n-v_n) > (1/2)\varepsilon$, by the general assumption $I_\varphi(x-y) \geqslant \varepsilon$. Let δ_1 be the constant from Lemma 0.2 chosen for $\varepsilon = ql_2/8k$, where q is the constant from (2.6). It is evident that we always suppose that $\delta_1 < ql_2/8k$. There is $d > 0$ such that

$$\sum_{\bar{E}_1} \varphi(v_n) < \delta_1, \tag{2.14}$$

where

$$\bar{E}_1 = \{n \in N\colon\ n \geqslant 2 \wedge |v_n| < d\}.$$

If we take $x = (u_n)$ such that $\sum_{\bar{E}_1} \varphi(u_n) < l_2/2k$, then

$$\sum_{\bar{E}_1} \varphi(u_n-v_n) \leqslant (k/2)(l_2/2k+\delta_1) \leqslant l_2/2.$$

Therefore and by (2.13), $\sum_{\bar{E}_0} \varphi(u_n-v_n) \geqslant l_2/2$, where

$$\bar{E}_0 = \{n \in N\colon\ n \geqslant 2 \wedge |v_n| \geqslant d\}.$$

This inequality is analogous to (2.4) or (2.10). Thus in the sequel the proof will be similar to the proof of (A) based on inequality (2.4).

Now, let us take $x = (u_n)$ such that $\sum_{\bar{E}_1} \varphi(u_n) \geqslant l_2/2k$. We have

$$\sum_{\bar{E}_1} \varphi((u_n+v_n)/2-u_n/2) \leqslant \sum_{\bar{E}_1} \varphi(v_n) < \delta_1,$$

by (2.14). Therefore

$$\Big|\sum_{\bar{E}_1} \varphi((u_n+v_n)/2)-\sum_{\bar{E}_1} \varphi(u_n/2)\Big| < ql_2/8k. \tag{2.15}$$

Moreover,

$$\sum_{\bar{E}_1} \varphi(u_n/2) \leqslant (1-q)(1/2)\sum_{\bar{E}_1} \varphi(u_n),$$

by (2.6). Then, by the above and (2.15), we obtain

$$\begin{aligned} I_\varphi((x+y)/2) &\leqslant (1/2)\sum_{N\setminus\bar{E}_1} \varphi(u_n)+(1/2)\sum_{N\setminus\bar{E}_1} \varphi(v_n)+\\ &\qquad +(1-q)(1/2)\sum_{\bar{E}_1} \varphi(u_n)+ql_2/8k\\ &\leqslant 1-(q/2)\sum_{\bar{E}_1} \varphi(u_n)+ql_2/8k\\ &\leqslant 1-(q/2)(l_2/2k)+ql_2/8k = 1-ql_2/8k, \end{aligned}$$

where $ql_2/8k$ is only dependent on ε.

Thus, the proof of the case when there are exactly two numbers u_m, v_n such that $\varphi(u_m) > 1/2$ and $\varphi(v_n) > 1/2$, is finished. But the case when only one number, e.g., u_m is such that $\varphi(u_m) > 1/2$, may be considered exactly as

II; while the case when $\varphi(u_n) \leqslant 1/2$ and $\varphi(v_n) \leqslant 1/2$ for all $n \in N$ is similar to the case (A).

Necessity. Supposing that (1) or (2) fail, we get three alternative conditions:

φ does not fulfil condition δ_2,

φ is not strictly convex on $[-u_0, u_0]$,

φ is not strictly convex on $[-v_0, v_0]$ and $\overline{\lim\limits_{u\to 0}} \dfrac{2\varphi(u/2)}{\varphi(u)} = 1$.

But, in virtue of Theorem 0.7 presenting the criterion for rotundity of l_φ, the only essential one is the last condition. However, even here, we also can restrict φ to be not strictly convex only on the interval $[u_0, v_0]$. Then there are numbers $t_1, v_1 \in [u_0, v_0]$, sequences $(u_n), (p_n) \subset (0, 1)$ such that $t_1 < v_1$, $u_n \downarrow 0$, $p_n \downarrow 0$ and

$$\varphi((t_1+v_1)/2) = (1/2)\varphi(t_1) + (1/2)\varphi(v_1), \qquad \varphi(u_n/2) \geqslant (1-p_n)\varphi(u_n)/2 \tag{2.16}$$

for all $n \in N$. Choose $v_2 \geqslant 0$ such that

$$\varphi(v_1) + \varphi(v_2) = 1. \tag{2.17}$$

Since $\varphi(t_1) + \varphi(v_2) < 1$ and $\varphi(u_n) \downarrow 0$, there exist a sequence $(m_n) \subset N$ and a subsequence (w_n) of (u_n) such that

$$1 - 1/n \leqslant \varphi(t_1) + \varphi(v_2) + m_n \varphi(w_n) \leqslant 1 \tag{2.18}$$

for each $n \in N$. Denote by (q_n) the subsequence of (p_n) due to (w_n). Let us put

$$y = (v_1, v_2, 0, \ldots) \quad \text{and} \quad x_n = (t_1, v_2, w_n, \ldots, \overset{m_n+2}{w_n}, 0, \ldots).$$

By (2.17) and (2.18), we have $\|y\|_\varphi = 1$ and $\|x_n\|_\varphi \leqslant 1$ for each $n \in N$. Denoting $c = \varphi(v_1 - t_1)$, we have $c \in (0, 1)$ and

$$I_\varphi((y - x_n)/c) = \varphi((v_1 - t_1)/c) + m_n \varphi(w_n/c) \geqslant 1,$$

which implies $\|y - x_n\|_\varphi \geqslant c$ for all $n \in N$. However,

$$\begin{aligned} I_\varphi((y+x_n)/2(1-q_n)) &\geqslant 1/(1-q_n)\big((1/2)\varphi(v_1) + (1/2)\varphi(t_1) + \varphi(v_2)\big) + m_n\varphi(w_n)/2 \\ &\geqslant (1/2)(\varphi(v_1) + \varphi(v_2)) + (1/2)(\varphi(t_1) + \varphi(v_2) + m_n\varphi(w_n)) \\ &\geqslant 1 - 1/2n \end{aligned}$$

for every $n \in N$, by (2.16), (2.17) and (2.18). Then

$$\|(y+x_n)/2\|_\varphi \geqslant (1-p_n)(1-1/2n) \to 1,$$

as $n \to \infty$ and l_φ is not locally uniformly convex. Thus the proof of the theorem is finished.

Let us recall the criterion for uniform rotundity of l_φ [5]:

The Orlicz sequence space l_φ is uniformly rotund iff φ satisfies condition δ_2 and it is strictly convex on $[-u_0, u_0]$, $\varphi(u_0) = 1/2$, and it is uniformly convex for small arguments, i.e.,

$$\overline{\lim_{u\to 0}} \frac{2\varphi((u+au)/2)}{\varphi(u) + \varphi(au)} < 1$$

for all $a \in (0, 1)$.

Now, let us compare the criteria for rotundity, local uniform rotundity and uniform rotundity of l_φ, presented here respectively as Theorem 0.7, Theorem 2 and the above theorem. First, we note that uniform convexity of φ for small arguments implies the condition

$$\overline{\lim_{u\to 0}} \frac{2\varphi(u/2)}{\varphi(u)} < 1, \tag{$*$}$$

occurring in Theorem 2. This implication is evident, by Lemma 0.4. However, there are examples of Orlicz functions, which are strictly convex near zero without satisfying $(*)$ and strictly convex functions which fulfil $(*)$ without being uniformly convex near zero.

3. Examples. 1. Let us take a function φ for which $\lim\limits_{u\to 0} p(u) = b > 0$, where p is increasing derivative of φ. Then φ is strictly convex, but it does not satisfy condition $(*)$. Indeed,

$$\lim_{u\to 0} 2\varphi(u/2)/\varphi(u) = \lim_{u\to 0} p(u/2)/p(u) = 1.$$

As an example we may take

$$p(u) = \begin{cases} 0, & u = 0, \\ u^2 + 1, & u \neq 0, \end{cases}$$

where $\varphi(u) = \int\limits_0^{|u|} p(t)\,dt$.

2. Now, let

$$p(u) = \begin{cases} u + 1/2, & u \in (1/2, \infty), \\ (1/2)u + 1/2, & u \in (1/2^2, 1/2], \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ (1/2^i)u + 1/2^i + 1/2^{i+1} - 1/2^{2i}, & u \in (1/2^{i+1}, 1/2^i],\ i = 1, 2, \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{cases}$$

This function is increasing and left-continuous. Denoting by p_+ the right-side limit of p, we have

$$\frac{p(1/2^i)}{p_+(1/2^i)} = \frac{1/2^i+1/2^{i+1}}{1/2^{2i-1}-1/2^{2i-2}+1/2^{i-1}+1/2^i} = \frac{3/2}{3-1/2^{i-1}} \leqslant 3/4$$

for each $i = 1, 2, \ldots$ So, if we take $u \in (1/2^{i+1}, 1/2^i]$, then $(1/2)u \in (1/2^{i+2}, 1/2^{i+1}]$ and $p((1/2)u)/p(u) \leqslant p(1/2^{i+1})/p_+(1/2^{i+1}) \leqslant 3/4$. Hence

$$\overline{\lim_{u\to 0}} \frac{p((1/2)u)}{p(u)} \leqslant 3/4. \tag{3.1}$$

Now, let $\varepsilon \in (0, 1/2)$. For such ε the intersection of the intervals $(1/2^{i+1}, 1/2^i]$, $(1/(1-\varepsilon)2^{i+1}, 1/(1-\varepsilon)2^i]$ is nonempty. If

$$u \in (1/2^{i+1}, 1/2^i] \cap (1/(1-\varepsilon)2^{i+1}, 1/(1-\varepsilon)2^i] = (1/(1-\varepsilon)2^{i+1}, 1/2^i],$$

then

$$(1-\varepsilon)u \in (1/2^{i+1}, (1-\varepsilon)/2^i] \subset (1/2^{i+1}, 1/2^i].$$

Therefore

$$\frac{p((1-\varepsilon)u)}{p(u)} = \frac{(1-\varepsilon)u+3/2-1/2^i}{u+3/2-1/2^i} \geqslant 1-1/3\cdot 2^i.$$

Hence

$$\overline{\lim_{u\to 0}} \frac{p((1-\varepsilon)u)}{p(u)} = 1, \tag{3.2}$$

for each $\varepsilon \in (0, 1/2)$. Let us put

$$\varphi(u) = \int_0^{|u|} p(t)\,dt.$$

Condition (3.1) is equivalent to the following one:

$$p((1/2)u) \leqslant (3/4)p(u)$$

for each $u \in [0, a]$ and some $a > 0$. But hence

$$2\varphi(u/2) = 2\int_0^{|u/2|} p(t)\,dt \leqslant (3/4)\int_0^{|u|} p(t)\,dt = (3/4)\varphi(u)$$

for $|u| \leqslant a$, which means condition $(*)$ for φ.

Akimovič showed in [1] that uniform convexity of φ for small arguments is equivalent to the condition

$$\overline{\lim_{u\to 0}} \frac{p((1-\varepsilon)u)}{p(u)} < 1$$

for each $\varepsilon \in (0, 1)$, where p is right or left derivative of φ. Then, in virtue of (3.2), the function φ is not uniformly convex for small arguments, though it is strictly convex and satisfies condition $(*)$.

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