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On the order-topological properties of the quotient space L/L_A

by

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Dedicated to Professor Władysław Orlicz on the occasion of his 80th birthday

Abstract. The first part contains some theorems about the order-topological properties of the quotient space of a σ-Dedekind complete and intervally complete locally solid Riesz space (L, τ) by the largest ideal L_A such that $\tau|L_A$ is a Lebesgue topology. These theorems are a generalization of some Lozanovskii's results from [7] and our proofs are slight modifications of Lozanovskii's methods. In the second part it is presented a very simple proof of the fact that L^ψ/L^ψ_A is an abstract M-space (L^ψ denotes a Musielak–Orlicz space and L^ψ_A its subspace of elements with absolutely continuous norm). A broad class of Orlicz spaces L^φ whose quotients L^φ/L^φ_A have no weak units is also indicated.

Let (L, τ) always denote a *Hausdorff locally solid Riesz space*. As concerns the terminology of Riesz spaces (= vector lattices) and locally solid Riesz spaces, we refer to [1]. Moreover, for $x \in L$, let $C(x)$ be the set of components of $|x|$, i.e.,

$$C(x) = \{p \in L\colon\ p \wedge (|x| - p) = 0\}.$$

The projection onto the band generated by an element $x \in L$ will be denoted by P_x.

1. General case. The theorems presented below were formulated, for Banach lattices, by G. Ja. Lozanovskii in [7]. It appears that Lozanovskii's results remain true also for intervally complete (L, τ) with L being σ-Dedekind complete. Lozanovskii uses in his proofs some facts which are interesting in themselves and which are not proved in [7]. We separate these facts and give their complete proofs under essentially weaker assumptions (Lemmas 1, 3 and 6). The main parts of proofs of our more general theorems are practically the same as Lozanovskii's proofs, but for convenience of the reader we indicate them.

Distinguish the largest ideal L_A in (L, τ) such that $\tau|L_A$ is a Lebesgue topology, i.e.,

$$L_A = \{x \in L\colon\ |x| \geqslant x_\alpha \downarrow 0 \text{ implies } x_\alpha \xrightarrow{\tau} 0\}.$$

The ideal L_A is τ-closed and it may be trivial, in general. We shall always assume that

L_A is a proper ideal in L.

The letter Q will be reserved for the canonical homomorphism from L onto L/L_A, and N will denote the set of natural numbers.

We start with the following lemma:

LEMMA 1. *Let (L, τ) be intervally complete with L having the principal projection property. If $x \in L_+ \setminus L_A$, then there exist a τ-neighbourhood of zero U and a sequence $(q_n) \subset C(x)$ of disjoint elements with $q_n \notin U$ for all n.*

Proof. Denote by $\mathrm{Id}(x)$ the ideal generated by $x \in L_+ \setminus L_A$. We claim $\tau|\mathrm{Id}(x)$ is not a pre-Lebesgue topology. In the contrary case, let $(J, \tilde{\tau})$ be the topological completion of $(\mathrm{Id}(x), \tau|\mathrm{Id}(x))$. The topology $\tau|\mathrm{Id}(x)$ is intervally complete, hence by [1] Theorem 7.3, $\mathrm{Id}(x)$ is an order dense ideal in J; in particular, $\mathrm{Id}(x)$ is a regular sublattice in J. Moreover, $\tilde{\tau}$ is a Lebesgue topology (see [1], Theorem 10.5). Therefore, taking $z_\alpha \in \mathrm{Id}(x)$, $z_\alpha \downarrow 0$, $z_\alpha \leqslant x$, we obtain $z_\alpha \downarrow 0$ in J (regularity) and $z_\alpha \xrightarrow{\tilde{\tau}} 0$. Thus $z_\alpha \xrightarrow{\tau} 0$ and $x \in L_A$, a contradiction.

According to [1] Theorem 10.1, there exists an order bounded sequence (x_n) of positive disjoint elements from $\mathrm{Id}(x)$ with $x_n \not\xrightarrow{\tau} 0$. Let ax be an upper bound of $\{x_n \colon n \in N\}$. The elements $y_n = a^{-1} x_n$ are disjoint, $y_n \leqslant x$ and $y_{n_k} \notin U$ for some solid τ-neighbourhood of zero U and some subsequence (n_k). The elements $q_k = P_{y_{n_k}}(x)$ are the desired components.

Now, using the above lemma we can formulate the first property of L/L_A:

THEOREM 2. *If (L, τ) is intervally complete with L being σ-Dedekind complete, then $(L/L_A)_A = \{0\}$ (in particular, L/L_A is a non-atomic lattice).*

Proof. Suppose $(L/L_A)_A \neq \{0\}$. Take a positive element $x \notin L_A$ such that $Q(x) \in (L/L_A)_A$. By Lemma 1 there exist a sequence (q_n) of disjoint components of x and a solid τ-neighbourhood of zero U with $q_n \notin U$ for all n. Let $\{N_i\}_{i=1}^{\infty}$ be a decomposition of the set N consisting of infinite disjoint sets. Denote $x_i = \sup\{q_n \colon n \in N_i\}$. We have $x_i \notin L_A$ (see [1], Theorem 10.2 and 10.1). The elements $Q(x_i)$ are disjoint and $Q(x)$ is an upper bound of them.

Let V be a solid τ-neighbourhood of zero such that $V + V \subset U$ and let $z \in L_A$. We have $x_i - z \notin V$ for every i, because $|x_i - z| \geqslant |q_n - z \wedge q_n|$ for all $n \in N_i$ and $z \wedge q_n \xrightarrow{\tau} 0$. Thus $Q(x_i) \notin Q(V)$, so $Q(x_i) \nrightarrow 0$. Therefore, see [1], Theorem 10.2, $Q(x) \notin (L/L_A)_A$ and we have got a contradiction.

To the end of the paper we shall work under the additional assumption that

L_A is order dense in L.

In further considerations we shall need the following modified version of Lemma 1:

LEMMA 3. *Let (L, τ) be intervally complete. For every $x \in L_+ \setminus L_A$ there exist a sequence $(q_n) \subset C(x) \cap L_A$ consisting of disjoint elements and a solid τ-neighbourhood of zero U such that $q_n \notin U$ for all n.*

Proof. The interval completeness implies the Dedekind completeness of L_A (see [1], Theorem 10.2 and 10.1), so $x = \sup\{x_\alpha \colon \alpha \in \mathscr{A}\}$, where $0 < x_\alpha \in L_A$ are pairwise disjoint (see [1], Lemma 23.15). The elements x_α are, of course, components of x.

Moreover, $x = \sup\{p_\delta \colon \delta \in \Delta\}$, where Δ is the family of finite subsets of $\mathscr{A}$ and $p_\delta = \sum_{\alpha \in \delta} x_\alpha$. The net (p_δ) increases to x, but it is not τ-convergent (L_A is closed !). Therefore the net (p_δ) does not satisfy the Cauchy condition, i.e., there exists a solid τ-neighbourhood of zero U such that for every finite subset δ_0 there is a finite subset $\delta \supset \delta_0$ with $p_\delta - p_{\delta_0} \notin U$. Take an arbitrary finite δ' and a finite set $\delta_1 \supset \delta'$ such that

$$q_1 = \sum_{\alpha \in \delta_1 \setminus \delta'} x_\alpha = p_{\delta_1} - p_{\delta'} \notin U.$$

In the next step take $\delta_2 \supset \delta_1$ with

$$q_2 = \sum_{\alpha \in \delta_2 \setminus \delta_1} x_\alpha = p_{\delta_2} - p_{\delta_1} \notin U.$$

Proceeding in this way we shall construct, by induction, a desired sequence of components.

The next theorem says about another property of L/L_A:

THEOREM 4. *If (L, τ) is intervally complete with L being σ-Dedekind complete, then every positive element $y \in L/L_A$ majorizes a family having the power of the continuum and consisting of pairwise disjoint elements.*

Proof. Let $y = Q(x) > 0$, where $x > 0$, let $(q_n) \subset C(x) \cap L_A$ be as in Lemma 3. Denote by (N_α) a family of the power of the continuum of infinite subsets of N which are almost disjoint, i.e., $N_\alpha \cap N_\beta$ is finite for all $\alpha \neq \beta$. Put $x_\alpha = \sup\{q_n \colon n \in N_\alpha\}$. Then $x_\alpha \notin L_A$ because x_α majorizes a sequence of disjoint elements which does not converge to zero. Moreover, $y > Q(x_\alpha)$ and

$$Q(x_\alpha) \wedge Q(x_\beta) = Q(\sup\{q_n \colon n \in N_\alpha \cap N_\beta\}) = 0,$$

because $q_n \in L_A$ and $N_\alpha \cap N_\beta$ is finite.

Now we give necessary and sufficient condition for the quotient topology on L/L_A to be σ-Lebesgue. But before doing this we have to prove some sublemma and lemma.

SUBLEMMA 5. *Let L be a Riesz space with the principal projection property. If K is an order dense ideal in L, then every $x \in L_+$ is the supremum of an increasing net (p_δ) of components of x belonging to K.*

Proof. Fix $x \in L_+$ and a net $(x_\beta)_{\beta \in B} \subset K$ with $0 \leqslant x_\beta \uparrow x$. According to Freudenthal's spectral theorem, there exist elements $q_{j,\beta}$ ($j \in N$) such that

$0 \leqslant q_{j,\beta} \uparrow_j x_\beta$ and every $q_{j,\beta}$ has the form $q_{j,\beta} = \sum_{i=1}^{m_j} a_i s_i$, where $s_i \in C(x)$, $s_i \wedge s_k = 0$ for $i \neq k$, and $a_i > 0$. The relation $q_{j,\beta} \leqslant x$ implies $a_i \in (0, 1]$, and so

$$q_{j,\beta} \leqslant t_{j,\beta} = \bigvee_{i=1}^{m_j} s_i \in C(x) \cap K.$$

Moreover, $\sup_{j,\beta} t_{j,\beta} = x$. *Let* Δ be the family of all finite subsets of $N \times B$. Then

$$p_\delta = \sup\{t_{j,\beta}\colon (j, \beta) \in \delta\} \in C(x) \cap K \quad \text{and} \quad p_\delta \uparrow x.$$

LEMMA 6. *Let τ be a Hausdorff Fatou topology on a Riesz space L with the principal projection property. If $x_n \downarrow 0$ and $x_n \stackrel{\tau}{\nrightarrow} 0$, then there exist a sequence (q_n) of disjoint elements with $q_n \in C(x_n) \cap L_A$ and a solid τ-neighbourhood of zero V such that $q_n \notin V$ for all n (we assume, as usual, L_A to be a proper order dense ideal in L).*

Proof. Since $x_n \stackrel{\tau}{\nrightarrow} 0$, then $x_n \notin V+V$ for all n and for some solid order closed τ-neighbourhood of zero V. In particular, $x_n \notin V$, so by Sublemma 5 there exists a projection P_1 onto some principal band with $q_1 = P_1 x_1 \in L_A \setminus V$; the Riesz space L has the principal projection property, thus every component of an element $x \in L_+$ has a form $P_y x$. Moreover, $P_1 x_n \in L_A$ and $P_1 x_n \stackrel{\tau}{\to} 0$. Denote $Q_1 = I - P_1$. Take a solid τ-neighbourhood of zero V_1 such that $V_1 + V_1 \subset V$. It must be $Q_1 x_n \notin V + V_1$. Indeed, the set $V + V_1$ is solid, so in the contrary case $Q_1 x_n \in V + V_1$ for large n. But $x_n = P_1 x_n + Q_1 x_n \in V_1 + V + V_1$ for sufficiently large n. In other words, $x_n \in V + V$ for large n, a contradiction.

Since $Q_1 x_n \notin V$ for all n and $Q_1 x_n \notin L_A$, using Sublemma 5 again we find a projection P_2 onto some principal band such that $q_2 = P_2 Q_1 x_2 \in L_A \setminus V$. We have $q_1 \wedge q_2 = 0$. Moreover, $P_2 Q_1 x_n \in L_A$ for $n \geqslant 2$, and so $P_2 Q_1 x_n \stackrel{\tau}{\to} 0$. Denote $Q_2 = Q_1 - P_2 Q_1$. Let V_2 be a solid τ-neighbourhood of zero with $V_2 + V_2 \subset V_1$. Then $Q_2 x_n \notin V + V_2$ for all n. Of course $Q_2 x_n \notin L_A$ and $Q_2 x_n \notin V$. Proceeding as above we construct, by induction, a desired sequence of components.

Some weaker Hausdorff Fatou topology $\tau^\vee$ can be associated with the topology τ. A base of $\tau^\vee$-neighbourhoods of zero is the following:

$$U^\vee = \{x \in L\colon [0, |x|] \cap L_A \subset U\},$$

where U runs some basis of solid order closed $\tau|L_A$-neighbourhoods of zero in L_A ($\tau^\vee$ is the topology induced by the topology of the maximal topological extension of $(L_A, \tau|L_A)$ — see [11]).

THEOREM 7. *For intervally complete (L, τ) with L being σ-Dedekind complete the following conditions are equivalent:*

(i) *The quotient topology of L/L_A is a σ-Lebesgue topology.*

(ii) *$x_n \downarrow 0$ and $x_n \stackrel{\tau^\vee}{\to} 0$ imply $x_n \stackrel{\tau}{\to} 0$.*

Proof. (i)$\Rightarrow$(ii): Let $x_n \downarrow 0$ and $x_n \stackrel{\tau^\vee}{\to} 0$. It must be $Q(x_n) \downarrow 0$. Indeed, if $Q(x_n) \geqslant Q(x)$, where $x \in L_+ \setminus L_A$, then $x - x \wedge x_n = y_n \in L_A$ and $x - y_n \stackrel{\tau^\vee}{\to} 0$. Thus (y_n) is an order bounded τ-Cauchy sequence because $\tau^\vee | L_A = \tau | L_A$. Therefore $x \in L_A$, a contradiction.

Now we show that $x_n \stackrel{\tau}{\to} 0$. Let U be an arbitrary τ-neighbourhood of zero. Find a solid τ-neighbourhood of zero V with $V + V \subset U$. Since $\tau|L_A$ is a Lebesgue topology, there exists a solid order closed $\tau|L_A$-neighbourhood of zero W included in $V \cap L_A$.

The quotient topology is a σ-Lebesgue topology. Thus $Q(x_n) \in Q(V)$ for large n. In other words $x_n - z_n \in V$ for some $z_n \in L_A$. Moreover,

$$\big|x_n - x_n \wedge |z_n|\big| \leqslant \big|x_n - |z_n|\big| \leqslant |x_n - z_n|$$

and the solidity of V implies $x_n - x_n \wedge |z_n| \in V$. We have also $x_n \in W^\vee$ for large n. Hence $x_n \wedge |z_n| \in W$ for large n. Finally

$$x_n = (x_n - x_n \wedge |z_n|) + x_n \wedge |z_n| \in V + W \subset U$$

for sufficiently large n.

(ii)$\Rightarrow$(i): Suppose $Q(x_n) \downarrow 0$. We can assume $x_n \downarrow 0$ (see [8], Lemma 65.5). It must be $x_n \stackrel{\tau^\vee}{\to} 0$. Indeed, in the contrary case let (q_n) be a sequence as in Lemma 6. We have $q_n \leqslant x_1$, so $p = \sup_n q_n$ exists in L and $p \notin L_A$ because $q_n \stackrel{\tau}{\nrightarrow} 0$. Moreover, for all k,

$$p = \sup_{n \geqslant k} q_n + \sum_{i=1}^{k-1} q_i \leqslant x_k + \sum_{i=1}^{k-1} q_i,$$

and so $Q(p) \leqslant Q(x_k)$ and of course $Q(p) > 0$. Therefore we have got a contradiction.

The conclusion arises now from (ii) and from the continuity of the operator Q.

It is obvious that if τ is a Fatou topology, then $\tau = \tau^\vee$, so in this case the quotient topology is σ-Lebesgue.

The next theorem says that L/L_A is not a σ-Dedekind complete Riesz space. More precisely, we have:

THEOREM 8. *Let τ be a Hausdorff Fatou topology on a σ-Dedekind complete Riesz space L. If (L, τ) is intervally complete, then no non-zero ideal in L/L_A is σ-Dedekind complete.*

Proof. Let Y be an arbitrary non-zero ideal in L/L_A. Fix a positive $x \in L$ with $0 < Q(x) \in Y$. Let (q_n) be a sequence as in Lemma 3. If $\{N_i\}_{i=1}^\infty$ is a decomposition of N consisting of infinite disjoint subsets, then denoting $x_i = \sup\{q_n\colon n \in N_i\}$ we have $Q(x_i) \nrightarrow 0$ (see the proof of Theorem 2). Moreover, $Q(x_i) < Q(x)$ for all i, but the supremum of this sequence does not exist. Indeed, if $w = \sup_i Q(x_i)$ existed, then by virtue of Theorem 7 we should obtain $w - \sum_{i=1}^n Q(x_i) \to 0$, which is impossible because $Q(x_i) \nrightarrow 0$.

Remark. Theorem 8 is proved in [7], for Banach lattices, without the assumption "τ is a Fatou topology".

2. Particular case. The second part will be devoted to special quotients of the form L/L_A, namely to L^ψ/L^ψ_A, where L^ψ is a Musielak–Orlicz space.

Let (S, Σ, μ) be a positive measure space. A function ψ: $[0, \infty)\times S \to [0, \infty]$ is called a *Musielak–Orlicz function* if, for all $s\in S$ and $r\in[0, \infty)$, the following conditions are satisfied:

(ψ1) $\psi(\cdot, s)$: $[0, \infty)\to[0, \infty]$ is left continuous, continuous at zero, non-decreasing and $\psi(r, s) = 0$ iff $r = 0$.

(ψ2) $\psi(r, \cdot)$: $S\to[0, \infty]$ is Σ-measurable.

The class of Musielak–Orlicz functions contains, of course, *Orlicz functions*, i.e., functions φ: $[0, \infty)\to[0, \infty]$ with properties listed in (ψ1).

Every Musielak–Orlicz function ψ generates some space of measurable functions $L^\psi(S, \Sigma, \mu)$ (called a *Musielak-Orlicz space*):

$$L^\psi(S, \Sigma, \mu) = \{x\in L^0(S, \Sigma, \mu)\colon\ m_\psi(tx) = \int_S \psi(t|x(s)|, s)\, d\mu < \infty \text{ for some } t > 0\}.$$

Here $L^0(S, \Sigma, \mu)$ is the space of all μ-equivalence classes of measurable real-valued functions on S. We shall often write L^ψ instead of $L^\psi(S, \Sigma, \mu)$, and l^ψ when μ is the counting measure on N. The symbol 1_B will denote the characteristic function of the set B.

A Musielak–Orlicz space is a super Dedekind complete Riesz space with respect to the standard order: $x\leqslant y$ iff $x(s)\leqslant y(s)$ μ almost everywhere. Moreover, the formula

$$\|x\|_\psi = \inf\{r > 0\colon\ m_\psi(x/r)\leqslant r\}$$

defines a monotone F-norm on L^ψ and L^ψ is $\|\cdot\|_\psi$-complete. Thus $(L, \|\cdot\|_\psi)$ is an F-lattice (= topologically complete, metrizable locally solid Riesz space). For convenience, we shall suppress the letter ψ in the symbol $\|\cdot\|_\psi$. It is easy to see, that $\|\cdot\|$ is a Fatou F-norm, i.e., $0\leqslant x_\alpha\uparrow x$ implies $\|x_\alpha\|\uparrow\|x\|$, and that L^ψ has the σ-Levi property, i.e., whenever $x_n\uparrow$ and (x_n) is topologically bounded, then the sequence (x_n) has the supremum in L^ψ.

The functional $m = m_\psi$ appearing in the definition of a Musielak–Orlicz space is called the *modular* (generated by ψ) and it has one important property:

$$(*)\qquad m(x\vee y)+m(x\wedge y) = m(x)+m(y)\quad \text{for}\quad x, y\geqslant 0.$$

Another property of m is a simply consequence of $(*)$:

$$(**)\quad m(ax+by)\leqslant m(x)+m(y)\quad \text{for all } x, y\in L^\psi,\ a, b\geqslant 0,\ a+b = 1$$

(for details see [9] and [10] or [12]).

The convergence in the F-norm $\|\cdot\|$ can be expressed in terms of the modular m:

$$\|x_n\|\to 0\quad \text{iff}\quad m(rx_n)\to 0 \text{ for every } r > 0.$$

In investigations of Musielak–Orlicz spaces the following subspace is important:

$$L^\psi_f = L^\psi_f(S, \Sigma, \mu) = \{x\in L^\psi\colon\ m(rx) < \infty \text{ for all } r > 0\}.$$

We have $L^\psi_f\subset L^\psi_A$, and this inclusion is proper, in general. Indeed, let (S, Σ, μ) be the direct sum of the measure spaces: $[0, 2]$ with the Lebesgue measure and N with the counting measure. Define $\psi(r, s) = r$ for $(r, s)\in[0, \infty)\times[0, 1]\cup[0, 1]\times(1, 2]\cup[0, \infty)\times\{2, 3, 4, \ldots\}$, $\psi(r, s) = r^2(1-r)^{-1}$ for $(r, s)\in[0, 1)\times\{1\}$ and $\psi(r, s) = \infty$ for $(r, s)\in(1, \infty)\times(1, 2]\cup[1, \infty)\times\{1\}$. Then $L^\psi(S, \Sigma, \mu) = L^1[0, 1]\oplus L^\infty[1, 2]\oplus l^1$, $L^\psi_A = L^1[0, 1]\oplus l^1$, $L^\psi_f = L^1[0, 1]\oplus\{x\in l^1\colon\ x(1) = 0\}$. This example also shows that L^ψ_A need not be order dense in L^ψ.

If ψ takes only finite values, i.e., ψ: $[0, \infty)\times S\to[0, \infty)$, then $L^\psi_f = L^\psi_A$. The assumption that ψ is finite valued ensures also the super order density of L^ψ_f in L^ψ. Indeed, for arbitrary Musielak–Orlicz function ψ the following theorem gives conditions equivalent to the super order density of L^ψ_f in L^ψ:

THEOREM 9. *Let $A_\infty = \{s\colon\ \psi(r, s) = \infty$ for some $r > 0\}$. Then the following conditions are equivalent*:

(a) *L^ψ_f is super order dense in L^ψ.*

(b) *$\mu(A_\infty\cap\{s\colon\ x(s)\neq 0\}) = 0$ for every $x\in L^\psi$.*

(b′) *$\mu(A_\infty) = 0$ or $\mu(\Sigma\cap A_\infty) = \{0, \infty\}$.*

(c) *For every $x\in L^\psi_+$ there exists a sequence (A_n) in Σ such that $x1_{A_n}\uparrow x$ and $1_{A_n}\in L^\psi_f$.*

Proof. See [12], Theorem 1.3.

Therefore, by virtue of conditions (b) and (b′), the assumption that ψ is finite valued is practically equivalent to the super order density of L^ψ_f in L^ψ.

In further parts of the paper we shall always consider only *finite valued* Musielak–Orlicz functions. We shall also assume $L^\psi\neq L^\psi_A$ (so the function ψ cannot satisfy the generalized Δ_2-condition, see [12], or [5], inequality 5.7). The letter Q will denote, as usual, the canonical homomorphism from L^ψ onto L^ψ/L^ψ_A.

We recall that a monotone norm $\|\cdot\|$ defined on a Riesz space L is an *M-norm* if $\|x\vee y\| = \|x\|\vee\|y\|$ for $x, y\in L_+$ (see [1]).

THEOREM 10. *The quotient F-norm $|||\cdot|||$ on L^ψ/L^ψ_A is an M-norm and*

$$|||Q(x)||| = \inf\{r > 0\colon\ m(x/r) < \infty\}\quad \text{for all } x \text{ in } L^\psi.$$

Proof. Let p: $L^\psi\to[0, \infty)$ be the functional defined by the formula

$$p(x) = \inf\{r > 0\colon\ m(x/r) < \infty\}.$$

It is easy to verify that p is a monotone semi-norm on L^ψ and $\operatorname{Ker} p = L^\psi_A$. Moreover, $p(x \vee y) = p(x) \vee p(y)$ for $x, y \geqslant 0$. The inequality $p(x \vee y) \geqslant p(x) \vee p(y)$ holds by the monotonicity of p. To verify the inverse inequality, let $k > p(x) \vee p(y)$. Using $(*)$ we obtain $m(k^{-1}(x \vee y)) \leqslant m(k^{-1}x) + m(k^{-1}y) < \infty$, and so $p(x \vee y) \leqslant k$.

Since $\operatorname{Ker} p = L^\psi_A$, $p(x) = p(x-y) \leqslant \|x-y\|$ for all $x \in L^\psi$ and $y \in L^\psi_A$. Therefore $p(x) \leqslant |||Q(x)|||$. Let $x \in L^\psi_+$ and take any number $r > 0$ such that $m(x/r) < \infty$. The super order density of L^ψ_A in L^ψ implies $m(r^{-1}(x-x_n)) \to 0$ for some increasing sequence (x_n) in L^ψ_A. Thus $m(r^{-1}(x-x_n)) \leqslant r$ for sufficiently large n. In other words $\|x-x_n\| \leqslant r$ for large n. But $x_n \in L^\psi_A$, so $|||Q(x)||| \leqslant r$. Therefore $|||Q(x)||| \leqslant p(x)$, and finally $|||Q(x)||| = p(x)$.

We proved that the quotient F-norm is, in fact, a norm, moreover, it is an M-norm. Indeed, Q is a Riesz homomorphism, so if $Q(x) \geqslant 0$, then we can assume $x \geqslant 0$. Thus we have

$$|||Q(x) \vee Q(y)||| = |||Q(x \vee y||| = p(x \vee y) = p(x) \vee p(y) = |||Q(x)||| \vee |||Q(y)|||$$

for $Q(x), Q(y) \geqslant 0$.

Remark. The equality $|||Q(x)||| = p(x)$ was observed, in the case of Orlicz spaces, by R. Leśniewicz (see also [3]). In this situation the order density of L^ψ_A in L^ψ is trivial because all integrable simple functions belong to L^ψ_A. The above theorem, for convex Orlicz spaces, is due to De Jonge (see [4], § 2).

It is interesting that a Musielak–Orlicz space L^ψ may be highly non-locally convex, but its quotient L^ψ/L^ψ_A is always a Banach space. Hence, whenever $L^\psi \neq L^\psi_A$ then the dual of L^ψ is always non-trivial.

Every Musielak–Orlicz space $L^\psi(S, \Sigma, \mu)$ over a σ-finite measure μ has a weak unit $e \in L^\psi \setminus L^\psi_A$. Indeed, the σ-finiteness implies the existence of a countable complete disjoint system $(x_n) \subset L^\psi_+$. Let (t_n) be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \|t_n x_n\| < \infty$. The series $\sum_{n=1}^{\infty} t_n x_n$ is then convergent and its sum e' is a weak unit in L^ψ. If $e' \in L^\psi_A$, then we can replace it by $e = e' \vee x$, where x is an arbitrary positive element in $L^\psi \setminus L^\psi_A$, to get a weak unit in L^ψ belonging to $L^\psi \setminus L^\psi_A$. But there exist many Orlicz spaces $L^\varphi(S, \Sigma, \mu)$ whose quotient L^φ/L^φ_A does not have any weak unit, as shown by our next theorem.

Before proceeding, let us recall that an Orlicz function $\varphi\colon [0, \infty) \to [0, \infty)$ satisfies:

Δ_2^∞-condition if $\exists\, K > 0\ \exists\, u_0\ \forall\, u \geqslant u_0$: $\varphi(2u) \leqslant K\varphi(u)$,

Δ_2^0-condition if $\exists\, K > 0\ \exists\, u_0 > 0\ \forall\, u \in [0, u_0]$: $\varphi(2u) \leqslant K\varphi(u)$.

THEOREM 11. (i) *Let φ be an Orlicz function not satisfying the Δ_2^∞-condition and assume that $\mu|\Sigma \cap B$ is non-atomic for some set B with $0 < \mu(B) < \infty$. Then the quotient L^φ/L^φ_A has no weak unit.*

(ii) *Let φ be an Orlicz function not satisfying the Δ_2^0-condition and assume that Σ contains a sequence (B_n) of atoms with $0 < \inf_n \mu(B_n) \leqslant \sup_n \mu(B_n) < \infty$. Then the quotient L^φ/L^φ_A has no weak unit.*

Proof. (i): Suppose $0 < Q(x) \in L^\varphi/L^\varphi_A$. We can of course assume $x > 0$. Since φ does not satisfy the Δ_2^∞-condition, we can find a sequence (u_n) with the properties $\varphi(2u_n) > 2^n \varphi(u_n)$ and $\varphi(u_n) > 1$ for all n.

If $\mu(B \setminus \{s\colon x(s) \neq 0\}) > 0$, then we choose a sequence (C_n) in Σ consisting of disjoint sets satisfying the following two conditions:

$$C_n \subset B \setminus \{s\colon x(s) \neq 0\}, \qquad \mu(C_n) = (2^n \varphi(u_n))^{-1} \mu(B \setminus \{s\colon x(s) \neq 0\}).$$

Then the function $z = \sup_n u_n 1_{C_n}$ is in $L^\varphi \setminus L^\varphi_A$ because $m(2z) = \infty$. We have $x \wedge z = 0$ and hence $Q(x) \wedge Q(z) = 0$. However $Q(z) > 0$, so $Q(x)$ cannot be a weak unit.

Let now $\mu(B \setminus \{s\colon x(s) \neq 0\}) = 0$. Find a sequence of sets (C_n) such that $x1_{C_n} \in L^\varphi_A$ and $x1_{C_n} \uparrow x$ (see Sublemma 5). We have $\mu(C_k \cap B) > 0$ for large k. Let (D_n) be a sequence of disjoint subsets of $B \cap C_k$ with the property $\mu(D_n) = (2^n \varphi(u_n))^{-1} \mu(C_k \cap B)$. The function $z = \sup_n u_n 1_{D_n}$ belongs to $L^\varphi \setminus L^\varphi_A$ and $x \wedge z \leqslant x1_{C_k \cap B} \in L^\varphi_A$. Therefore $Q(x) \wedge Q(z) = 0$ although $Q(z) > 0$. Thus $Q(x)$ is not any weak unit.

(ii): Put $B = \bigcup_{n=1}^{\infty} B_n$. The assumptions imply that $L^\varphi(B, \Sigma \cap B, \mu|\Sigma \cap B)$ is Riesz (and thus topologically) isomorphic to the space l^φ, which is essentially larger than l^φ_A. We will identify these isomorphic spaces.

Let $0 < Q(x) \in L^\varphi/L^\varphi_A$. We can assume, as usual, $x > 0$. If $\mu(B \cap \{s\colon x(s) \neq 0\}) < \infty$, then the set $N = \{n\colon \mu(B_n \cap \{s\colon x(s) \neq 0\}) > 0\}$ is finite or empty. Fix a positive element $z \in l^\varphi \setminus l^\varphi_A$. We have $z1_{N \setminus N} \in l^\varphi \setminus l^\varphi_A$, so $Q(z1_{N \setminus N}) > 0$ and $x \wedge z1_{N \setminus N} = 0$. Therefore $Q(x)$ cannot be a weak unit.

If now $\mu(B \cap \{s\colon x(s) \neq 0\}) = \infty$, then the sequence $(x_n) = (x(B_n))$ contains infinitely many non-zero terms. The continuity of φ at zero and the inclusion $l^\varphi \subset c_0$ ensure the existence of a stricly increasing sequence (n_k) such that $\varphi(2^k x_{n_k}) < 2^{-k}$. Denote $C = \{n_k\}_{k=1}^{\infty}$. We have $x1_C \in l^\varphi_A$. Take a positive element $y = (y_k) \in l^\varphi \setminus l^\varphi_A$. Put $z = (z_n)$, where $z_n = y_k$ for $n = n_k$ and $z_n = 0$ for $n \notin C$. The sequence (z_n) does not belong to l^φ_A and $x \wedge z \leqslant x1_C \in l^\varphi_A$, so $Q(x) \wedge Q(z) = 0$.

Therefore no element from L^φ/L^φ_A is a weak unit.

Remark. If $L^\psi(S, \Sigma, \mu)$ is such that L^ψ/L^ψ_A has no weak unit, then L^ψ/L^ψ_A is not isometric to any space $C(K)$ over a compact K (in particular, is not isometric to l^∞/c_0). Indeed, according to Corollary 2 in [6], p. 188, if the spaces L^ψ/L^ψ_A and $C(K)$ were isometric, then they would be Riesz isomorphic. However, these spaces cannot be order isomorphic because $C(K)$ has a strong unit, while L^ψ/L^ψ_A does not even have any weak unit.

Theorem 11 does not cover all quotients of the form L^φ/L^φ_A (φ denotes an Orlicz function). It is not difficult to find an example of an Orlicz space L^φ possessing a strong unit $u \in L^\varphi \setminus L^\varphi_A$; then $Q(u)$ will be of course a strong unit in L^φ/L^φ_A.

Let $\varphi(r) = e^r - 1$ and let μ be the measure on 2^N such that $\mu(\{n\}) = e^{-n}$. The element $u = (u_n)$, where $u_n = n$ for all n, belongs to $L^\varphi \setminus L^\varphi_A$ and it is a strong unit. Indeed, if $x = (x_n) \in L^\varphi$, then

$$\sum_{n=1}^{\infty} (e^{t|x_n|} - 1)e^{-n} = m(tx) < \infty \quad \text{for some } t > 0,$$

so $\exp(t|x_n| - n) \to 0$ as $n \to \infty$. Thus $|x_n| \leqslant t^{-1} n$ for large n. This means that $|x| \leqslant Ku$ for some number K.

Moreover, the Orlicz space constructed above is Riesz and topologically isomorphic to l^∞ (the operator $T: l^\infty \to L^\varphi$ defined by $T((s_n)) = (ns_n)$ is a Riesz isomorphism onto). This fact follows also from the following result:

THEOREM 12. *Let $(L, \|\cdot\|)$ be an infinite dimensional F-lattice with a strong unit u. If L is Dedekind complete and L_A is order dense in L, then L is discrete and Riesz isomorphic to $l^\infty(X)$, where X has the same cardinality as any complete disjoint system of atoms in L.*

Proof. Notice first that $L_A \neq L$. Indeed, assume $L_A = L$. The space is infinite dimensional, so there exists an infinite set C of positive disjoint elements (see, for example, [2], Corollary 1). Take a countable subset $\{x_n\}_{n=1}^{\infty} \subset C$. The elements $y_n = P_{x_n}(u)$ are strictly positive and they converge to zero. Thus $y = \sum_{k=1}^{\infty} k y_{n_k}$ exists for some subsequence (n_k). The element u is a strong unit, so $y \leqslant tu$ for some $t > 0$. Therefore $k y_{n_k} = P_{x_{n_k}}(y) \leqslant t P_{x_{n_k}}(u) = t y_{n_k}$ for all k, a contradiction.

The order density of L_A in L implies that $u = \sup_\alpha u_\alpha$, where $u_\alpha \in L_A$ are pairwise disjoint ([1], Lemma 23.15). Therefore u_α is a strong unit in the band $B(u_\alpha)$ generated by u_α. Hence, by the previous part of the proof, $B(u_\alpha)$ must be finite dimensional. Denoting by X_α a complete disjoint system of atoms in $B(u_\alpha)$ with $u_\alpha = \sup X_\alpha$ we obtain that $X = \bigcup_\alpha X_\alpha$ is a complete disjoint system in L consisting of atoms and $u = \sup X$. The operator $T: l^\infty_+(X) \to L_+$ defined by

$$T((t_x)) = \sup_x t_x x$$

has the unique extension to a Riesz isomorphism onto.

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