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Stable semi-groups of measures on the Heisenberg group

by

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Abstract. Let $\{\mu_t\}$ be a stable semi-group of measures on the Heisenberg group G. Denote by P the infinitesimal generator of $\{\mu_t\}$. Then the measures μ_t are absolutely continuous with respect to the Haar measure and their densities are square-integrable if and only if for every non-trivial irreducible unitary representation π of G, the closure of the operator $\pi(P)$ is injective. Some other equivalent conditions are given.

Introduction. Recall that a probability measure μ on $\boldsymbol{R}^n$ is said to be *stable* (*in the strict sense*) if for every r, $s > 0$ there exists $t > 0$ such that

$$\delta_r \mu * \delta_s \mu = \delta_t \mu,$$

where $\delta_r \mu$ is defined by

$$\langle f, \delta_r\mu\rangle = \int f(rx)\,\mu(dx), \qquad f \in \mathscr{D}(\boldsymbol{R}^n).$$

If μ is stable, then for some $0 < \theta \leqslant 2$ the probability measures

$$\mu_t = \delta_{t^{1/\theta}}\,\mu$$

form a *continuous semi-group of measures*. Set

$$\psi(\xi) = \frac{d}{dt}\bigg|_{t=0} \hat{\mu}_t(\xi), \qquad \xi \in \boldsymbol{R}^n,$$

where $\hat{\mu}$ denotes the Fourier transform (the characteristic function) of μ. Then ψ is a continuous function. Moreover, it is homogeneous of degree θ and satisfies $\operatorname{Re}\psi(\xi) \leqslant 0$ for $\xi \in \boldsymbol{R}^n$. We also have

$$(*) \qquad \hat{\mu}_t(\xi) = e^{t\psi(\xi)}$$

for $\xi \in \boldsymbol{R}^n$ and $t > 0$. All the above is classical, cf. e.g. [9].

From $(*)$ one can easily deduce that the following conditions are equivalent:

(i) μ_t are absolutely continuous (with respect to the Lebesgue measure) and their densities are smooth functions.

(ii) μ_t are absolutely continuous.

(iii) $\operatorname{Re}\psi(\xi) < 0$ for $\xi \neq 0$.

We would like to generalize this for groups. Namely, let G be a nilpotent connected Lie group with dilations $\{\delta_r\}$. We say that a continuous semi-group of probability measures $\{\mu_t\}$ on G is *stable* if it satisfies $\delta_r \mu_t = \mu_{r^{\theta}t}$ for some $\theta > 0$. Denote by $\hat{G}$ the dual object of G. Our conjecture is:

Let $\{\mu_t\}$ be a stable semi-group of symmetric probability measures on G and $P = P^*$ its infinitesimal generator (cf. the definition in Section 2). Then the following conditions are equivalent:

(i) μ_t are absolutely continuous (with respect to the Haar measure on G) and their densities are smooth functions.

(ii) μ_t are absolutely continuous.

(iii) The closure of $\pi(P)$ is injective for non-trivial $\pi \in \hat{G}$.

In fact we cannot prove it even for the Heisenberg group. However, we have to offer the following result:

THEOREM. *Let $\{\mu_t\}$ be a stable semi-group of symmetric probability measures on the Heisenberg group G and let $P \in \mathscr{D}^*(G)$ be its infinitesimal generator. Then the following are equivalent:*

(i) *μ_t are absolutely continuous and their densities are square-integrable (with respect to the Haar measure on G).*

(ii) *μ_t are absolutely continuous.*

(iii) *The closure of $\pi(P)$ is injective for non-trivial $\pi \in \hat{G}$.*

Before we present an application of the Theorem let us introduce a definition. Following [13] we denote by (S) the smallest class of semi-groups which contains Gaussian semi-groups, i.e. the semi-groups whose infinitesimal generators are of the form

$$P = \sum_{j=1}^{k} X_j^2,$$

where X_j are some elements in the Lie algebra of G, and is closed with respect to taking sums and fractional powers of their generators as well as to multiplying the generators by strictly positive reals. Then our theorem yields:

COROLLARY. *Let $\{\mu_t\}$ be a stable semi-group of measures in (S) on the Heisenberg group G. If the Lie algebra generated by the elements X_j which "appear" in the above inductive definition of the infinitesimal generator P of $\{\mu_t\}$ is the whole Lie algebra of G, then the measures μ_t are absolutely continuous and their densities are square-integrable functions.*

Indications for further applications the reader will find in [13].

Another our result closely related to the theorem is

COROLLARY. *Let $\{\mu_t\}$ be a stable semi-group of symmetric probability measures on the Heisenberg group G. If μ_t are absolutely continuous, then the densities f_t of μ_t are "fractionally differentiable", i.e. there exists an $\varepsilon > 0$ such that $f_t \in H^{\varepsilon}(G)$, where $H^{\varepsilon}(G)$ is the usual Sobolev space.*

Let us now sum up briefly the contents of the paper. In Section 1 we recall elementary properties of a class of pseudo-differential operators whose symbols satisfy the estimates

$$|\partial_x^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)| \leqslant C_{\alpha,\beta}\, m(x, \xi),$$

where m is a weight function in the sense of Hörmander, [18], cf. (1.3) below. We denote the class by $S(m)$. This is, in fact, an "extremely easy" case of the general theory of pseudo-differential operators as presented in [17]. Our symbols, however, are also required to satisfy

$$(**) \qquad |\partial_x^{\alpha} \partial_{\xi}^{\beta} a(x, \xi)| \leqslant C_{\alpha,\beta}\, m'(x, \xi)$$

for $|\alpha|+|\beta| > 0$, where $m' \leqslant m$ is another weight function. The key point of the section is to establish a reasonable symbolic calculus for symbols in $S(m)$ classes, additionally satisfying $(**)$. One would like, for example, the commutator $[a, b]$ to be not only in $S(m_1 m_2)$, but even in $S(m_1' m_2')$, provided $a \in S(m_1)$, $b \in S(m_2)$ satisfy $(**)$ with m_1' and m_2', respectively.

The symbolic calculus together with Proposition 1.21 (proved in [11]) enables us to obtain the goal of the section, Proposition 1.25. Except for Proposition 1.21, Section 1 is self-contained.

Section 2 gives generalities on semi-groups of measures on a Lie group. A part of our main theorem is proved in the context of general Lie group. The standard decomposition of a dissipative distribution to a sum of a compactly supported distribution and a bounded positive measure plays a role.

In Section 3 we introduce basic notions connected with a homogeneous structure on a nilpotent Lie group. A definition of a stable semi-group of measures is given.

Section 4 is devoted to studying some estimates for derivatives of the Fourier transform of a dissipative homogeneous distribution on $\boldsymbol{R}^n$. This section is of completely technical character.

In Section 5 we prove our main theorem. Several conditions on $\{\mu_t\}$ and P are proved to be equivalent to absolute continuity of the measures μ_t. The crucial point is to show that (iii) implies (ii), cf. Theorem above. This is done with a help of the idea (originally applied to $P = X^2 - |Y|^{\gamma}$, cf. [14]) due to J. Cygan and A. Hulanicki: first estimate the growth of the eigenvalues of $\pi(P)$ and then apply the Plancherel theorem. The estimation is obtained by using the pseudo-differential calculus as developped in Section 1. The results of Section 4 show that the calculus is applicable to $\pi(P)$. We also make use of the simple observation that dissipativity of a distribution is connected only with the underlying differentiable manifold and not with the group structure of G.

At last, let us remark that an analogy between the Rockland theorem,

[20], and ours seems to be not only of a formal nature. Anyway, we were influenced by this theorem and its proofs (cf. [1], [20]).

1. Pseudo-differential operators. Let V be an n-dimensional real vector space. Let V^* be the dual space and denote by $x\xi$ the pairing between $x\in V$ and $\xi\in V^*$. We choose and fix an Euclidean norm $\|\cdot\|$ in V and hence the dual norm in V^* and the product norm in $W=V\oplus V^*$ which are denoted in the same way. Let $\{e_j\}$ be an ortho-normal basis in V and $\{e_j^*\}$ the dual basis in V^*. For multi-indices $\alpha=(\alpha_1,\ldots,\alpha_{2n})\in N^{2n}$ we denote

$$\partial^\alpha f=\partial_{e_1}^{\alpha_1}\ldots\partial_{e_n}^{\alpha_n}\partial_{e_1^*}^{\alpha_{n+1}}\ldots\partial_{e_n^*}^{\alpha_{2n}}f,$$

where

$$\partial_v f(w)=\frac{1}{i}\frac{d}{dt}\Big|_{t=0}f(w+tv),$$

$w,\ v\in W$ and f is a smooth function on W. The *length* of a multi-index is defined by $|\alpha|=\sum_{j=1}^{2n}\alpha_j$.

There is a natural simplectic form σ on W:

$$\sigma(w,v)=y\xi-x\eta \tag{1.1}$$

for $w=(x,\xi)$, $v=(y,\eta)\in W=V\oplus V^*$. If $\Delta=\sum_{j=1}^{n}\partial_{e_j}^2+\sum_{j=1}^{n}\partial_{e_j^*}^2$ is the Laplace operator on W, then

$$(\Delta_u+1)^k e^{i\sigma(u,v)}=\langle v\rangle^{2k}e^{i\sigma(u,v)} \tag{1.2}$$

for $u,\ v\in W$ and $k\in N$, where $\langle v\rangle=(1+\|v\|^2)^{1/2}$.

Let $\mathscr{S}(V)$ and $\mathscr{S}(V^*)$ denote the Schwartz spaces. Let dx, $d\xi$ be Lebesgue measures in V and V^*, respectively, normalized so that the relationship between a function $f\in\mathscr{S}(V)$ and its Fourier transform $\hat{f}\in\mathscr{S}^*$ is given by

$$\hat{f}(\xi)=\int e^{-ix\xi}f(x)\,dx,\qquad f(x)=\int e^{ix\xi}\hat{f}(\xi)\,d\xi.$$

By dw we denote the Lebesgue measure $dx\,d\xi$ in $W=V\oplus V^*$.

We say that a strictly positive continuous function m on W is a *temperate weight* or shortly a *weight* if it satisfies

$$m(w+v)\leqslant Cm(w)\langle v\rangle^N \tag{1.3}$$

for $w,\ v\in W$ and some constants C, N (cf. [18], Chap. II, 2.1). In particular, every weight m satisfies

$$(1/C)\langle w\rangle^{-N}\leqslant m(w)\leqslant C\langle w\rangle^N \tag{1.4}$$

for $w\in W$. The weights m_1, m_2 are said to be *eqivalent* if there is a constant C such that

$$(1/C)\,m_1(w)\leqslant m_2(w)\leqslant Cm_1(w)$$

for $w\in W$. Note that weights form a group under multiplication and if m is a weight, then also m^θ is a weight for every real θ. For a given weight m let us denote by $S(m)$ the class of all $a\in C^\infty(W)$ satisfying the estimates:

$$|\partial^\alpha a(w)|\leqslant C_\alpha m(w)$$

for $w\in W$ and all $\alpha\in N^{2n}$. $S(m)$ is a locally convex Fréchet space if endowed with the family of semi-norms:

$$|a|_k=\max_{|\alpha|=k}\sup_W\frac{|\partial^\alpha a(w)|}{m(w)},$$

where $k=0,1,2,\ldots$ Obviously, the definitions of weight and of spaces $S(m)$ do not depend on a norm and a coordinate system in W, respectively. The above defined semi-norms in $S(m)$ obtained in different coordinate systems give the same topology. Moreover, if weights m_1, m_2 are equivalent, then $S(m_1)=S(m_2)$ and the corresponding semi-norms are equivalent.

It follows from the Ascoli theorem that the space $S(m)$ enjoys the following compactness property:

(1.5) if $\{a_n\}$ is a bounded sequence in $S(m)$, then there is a subsequence $\{b_n\}$ of $\{a_n\}$ which is convergent in $C^\infty(W)$-topology to an element b of $S(m)$.

Every $a\in\mathscr{S}(W)$ defines a linear map $A\colon\mathscr{S}(V)\to\mathscr{S}(V)$ by the Weyl prescription:

$$Af(x)=\iint e^{i(x-y)\xi}a\big(\tfrac{1}{2}(x+y),\xi\big)f(y)\,dy\,d\xi. \tag{1.6}$$

If $f,\ g\in\mathscr{S}(V)$, then the function

$$\varphi(x,y)=f(x+\tfrac{1}{2}y)\overline{g(x-\tfrac{1}{2}y)}$$

is in $\mathscr{S}(V\times V)$ and thus

$$\psi(x,\xi)=\int e^{-iy\xi}\varphi(x,y)\,dy$$

belongs to $\mathscr{S}(W)$. Therefore the weak version of the above definition

$$\begin{aligned}\langle Af,g\rangle&=\iiint e^{i(x-y)\xi}a\big(\tfrac{1}{2}(x+y),\xi\big)f(y)\overline{g(x)}\,dy\,d\xi\,dx\\&=\iiint e^{-iy\xi}a(x,\xi)f(x+\tfrac{1}{2}y)\overline{g(x-\tfrac{1}{2}y)}\,dy\,d\xi\,dx\\&=\iiint e^{-iy\xi}a(x,\xi)\varphi(x,y)\,dy\,d\xi\,dx\\&=\iint a(x,\xi)\psi(x,\xi)\,d\xi\,dx=\int a(w)\psi(w)\,dw\end{aligned} \tag{1.7}$$

makes sense for any $a\in\mathcal{S}^*(W)$ and defines a linear operator which maps continuously $\mathcal{S}(V)$ into $\mathcal{S}^*(V)$. The distribution a is called then the *symbol* of A. We shall also write $a^w(x, D)$ or simply a^w for A. The correspondence between the symbols $a\in\mathcal{S}^*(W)$ and the continuous linear operators A from $\mathcal{S}(V)$ into $\mathcal{S}^*(V)$ is, in view of the Schwartz kernel theorem, bijective.

Directly from (1.6) one can obtain (cf. [4], Lemma 1)

PROPOSITION 1.8. *If m is a weight and $a\in S(m)$, then $a^w(x, D)$ is a continuous endomorphism of $\mathcal{S}(V)$.*

From the above proposition we derive as in [4]

COROLLARY 1.9. *Let a_n be a bounded sequence in $S(m)$ converging pointwise to a function a on W. Then $a\in S(m)$ and the sequence of the operators $A_n = a_n^w(x, D)$ converges strongly on $\mathcal{S}(V)$ to $A = a^w(x, D)$.*

To obtain the composition theorem let us start with $a, b\in\mathcal{S}(W)$. Then the symbol of $a^w b^w$ is given by

$$(1.10)\qquad a\circ b(w) = 2^{2n}\int\int a(w+u)\,b(w+v)\,e^{2i\sigma(v,u)}\,du\,dv$$

for $w\in W$ (see [17]). Integration by parts (cf. (1.2)) gives for $k\in N$

$$(1.11)\quad a\circ b(w) = 2^{2n}\int\int\frac{(1+\Delta u)^k a(w+u)}{\langle 2u\rangle^{2k}}(1+\Delta v)^k\left[\frac{b(w+v)}{\langle 2v\rangle^{2k}}\right]e^{2i\sigma(v,u)}\,du\,dv.$$

By using the Taylor expansion for a we get from (1.10)

$$a\circ b(w)-a(w)\,b(w) = 2^{2n-1}\sum_{j=1}^{n}\int_0^1 dt\int\int\partial_{e_j}a(w+tu)\,\partial_{e_j^*}b(w+v)\,e^{2i\sigma(v,u)}\,du\,dv - 2^{2n-1}\sum_{j=1}^{n}\int_0^1 dt\int\int\partial_{e_j^*}a(w+tu)\,\partial_{e_j}b(w+v)\,e^{2i\sigma(v,u)}\,du\,dv.$$

Again integration by parts yields for $k\in N$:

$$(1.12)\quad a\circ b(w)-a(w)\,b(w) = 2^{n-1}\sum_{j=1}^{n}\int_0^1 dt\int\int\frac{(1+t^2\Delta u)^k\,\partial_{e_j}a_k(w+tu)}{\langle 2u\rangle^{2k}}(1+\Delta v)^k\left[\frac{\partial_{e_j^*}b(w+v)}{\langle 2v\rangle^{2k}}\right]e^{2i\sigma(v,u)}\,du\,dv - 2^{n-1}\sum_{j=1}^{n}\int_0^1 dt\int\int\frac{(1+t^2\Delta u)^k\,\partial_{e_j^*}a_k(w+tu)}{\langle 2u\rangle^{2k}}(1+\Delta v)^k\left[\frac{\partial_{e_j}b(w+v)}{\langle 2v\rangle^{2k}}\right]e^{2i\sigma(v,u)}\,du\,dv.$$

THEOREM 1.13. *Let m_1, m_2 be weights on W and let $a\in S(m_1)$, $b\in S(m_2)$. Then $a\circ b\in S(m_1 m_2)$. If, moreover, $\partial^\alpha a\in S(m_1')$, $\partial^\alpha b\in S(m_2')$ for some other weights m_1', m_2' and all $|\alpha|\geqslant 1$, then $a\circ b-ab\in S(m_1' m_2')$.*

Proof. By (1.4) there exists an integer N and a constant $C>0$, such that all the weights under consideration are bounded by $C\langle w\rangle^N$. Thus for $k>n+\frac{1}{2}N$ the right-hand sides of (1.11) and of (1.12) make sense for $a\in S(m_1)$, $b\in S(m_2)$ and all $w\in W$. Denote them by $r_0(a, b)(w)$ and $r_1(a, b)(w)$, respectively.

Now, let f be a positive compactly supported and smooth function on W. Moreover assume $f=1$ in a neighbourhood of 0. Then by a simple application of the Leibniz formula the sequences

$$a_n(w) = f(w/n)\,a(w),\qquad b_n(w) = f(w/n)\,b(w),\qquad n\in N,$$

are bounded in $S(m_1)$ and $S(m_2)$, respectively, and the sequences $a_n\circ b_n$, $a_n b_n$ are bounded in $S(m_1 m_2)$. It is also clear that $a_n\to a$ and $b_n\to b$ in $C^\infty(W)$. Since a_n, b_n are compactly supported, we have by (1.11), (1.12)

$$(1.14)\qquad a_n\circ b_n(w) = r_0(a_n, b_n)(w)$$

and

$$(1.15)\qquad a_n\circ b_n(w)-a_n(w)\,b_n(w) = r_1(a_n, b_n)(w)$$

for $w\in W$ and $n\in N$. By the Lebesgue theorem

$$\lim_{n\to\infty} r_0(a_n, b_n)(w) = r_0(a, b)(w)$$

and

$$\lim_{n\to\infty} r_1(a_n, b_n)(w) = r_1(a, b)(w).$$

Therefore (1.11) and (1.14) together with (1.5) imply $a_n\circ b_n$ is convergent in $C^\infty(W)$ to a member of $S(m_1 m_2)$ which, in view of Cor. 1.9, is equal to $a\circ b$. Thus we have proved $a\circ b\in S(m_1 m_2)$. Since, obviously, $a_n b_n$ converges pointwise (even in C^∞, in fact) to ab, we have also proved (cf. (1.14), (1.15)) that formulae (1.11), (1.12) are valid for $a\in S(m_1)$, $b\in S(m_2)$.

Now assume that not only $a\in S(m_1)$, $b\in S(m_2)$ but also $\partial^\alpha a\in S(m_1')$, $\partial^\alpha b\in S(m_2')$ for $\alpha\in N^{2n}$ of length at least 1. Then using the Leibniz rule and the formula (1.12) for $k\in N$ sufficiently large one checks directly that $a\circ b - ab\in S(m_1' m_2')$.

For a given weight m we denote by $\mathcal{L}(m)$ the space of all the linear operators in $\mathcal{S}(V)$ whose symbols belong to $S(m)$. If A, B are endomorphisms of $\mathcal{S}(V)$ we denote by $[A, B]$ the commutator of A and B.

COROLLARY 1.16. *Let m_1, m_2, m_1', m_2' be weights and let $a\in S(m_1)$, $b\in S(m_2)$ be such that $\partial^\alpha a\in S(m_1')$, $\partial^\alpha b\in S(m_2')$ for $|\alpha|\geqslant 1$. Let A, B and T be the operators in $\mathcal{S}(V)$ corresponding to the symbols a, b and ab, respectively. Then $AB-T\in\mathcal{L}(m_1' m_2')$ and $[A, B]\in\mathcal{L}(m_1' m_2')$.*

For real t let us denote by h_t the weight $h_t(w) = \langle w\rangle^{-t}$.

Corollary 1.17 *Let m be a weight. If the symbol a of $A \in \mathscr{L}(m)$ satisfies*

$$1+|a(w)| \geqslant Cm(w) \tag{1.18}$$

for $w \in W$ and some $C>0$, and if

$$\partial^\alpha a \in S(mh_\varepsilon) \tag{1.19}$$

for $|\alpha| \geqslant 1$ and some $\varepsilon > 0$, then there exists $B \in \mathscr{L}(1/m)$ such that

$$AB-I \in \mathscr{L}(1/m), \qquad BA-I \in \mathscr{L}(1/m),$$

where I stands for the identity operator.

Proof. Let $b_0 \in C^\infty(W)$ be such that $ab_0 = 1$ outside a compact set. Then $b_0 \in S(1/m)$ and $\partial^\alpha b_0 \in S(h_\varepsilon/m)$ for $|\alpha| \geqslant 1$. By Cor.1.16

$$b_0 \circ a = 1-r_0,$$

where $r_0 \in S(h_{2\varepsilon})$. For an integer N set $b_N = (1+r_0)^N \circ b_0$, where the power is understood in the sense of "$\circ$". Then by Th.1.13 $b_N \in S(1/m)$ and

$$b_N \circ a = 1-r_0^{2^N}.$$

By the above $r_0^{2^N} \in S(h_{2^{N+1}\varepsilon})$, so by (1.4) $r_0^{2^N} \in S(1/m)$ for N sufficiently large. Thus we have shown there exist $b \in S(1/m)$ such that $b \circ a = 1-r$, where $r \in S(1/m)$. Similarly we prove there is $b' \in S(1/m)$ such that $a \circ b' = 1-r'$, where $r' \in S(1/m)$. But then $b-b' = b \circ r' - r \circ b' \in S(m^{-2})$. Therefore

$$a \circ b = a \circ b' + a \circ (b-b') = 1-r'+s,$$

where $s = a \circ (b-b') \in S(1/m)$, so we are done.

Let H be a Hilbert space. Denote by $\mathscr{L}(H)$ the algebra of all bounded linear operators on H. For a positive $T \in \mathscr{L}(H)$ let us denote by $\operatorname{Tr} T$ the trace of T which is either a positive number or infinity. For an arbitrary bounded T set

$$|T|_p = \begin{cases} (\operatorname{Tr}|T|^p)^{1/p}, & 1 \leqslant p < \infty, \\ \|T\|, & p = \infty. \end{cases}$$

Then for every $1 \leqslant p \leqslant \infty$

$$C_p(H) = \{T \in \mathscr{L}(H) \colon |T|_p < \infty\}$$

is a Banach space with the norm $|\cdot|_p$. It is, in fact, a two-sided ideal in $\mathscr{L}(H)$. In particular, $C_1(H)$ is the space of trace class operators, $C_2(H)$ is the space of Hilbert-Schmidt operators and $C_\infty(H)$ is the whole of $\mathscr{L}(H)$. If T is in $C_p(H)$ for some $1 \leqslant p < \infty$, then it is compact and

$$\Big(\sum_{n=1}^{\infty} |z_n|^p\Big)^{1/p} \leqslant |T|_p, \tag{1.20}$$

where z_n are all the eigenvalues of T together with their multiplicities (see e.g. [7], Chap.XI.9).

The following proposition was proved in [11] for a different kind of symbols (Th.3.1), but the proof in the case of the Weyl calculus is essentially the same, cf. also [16].

Proposition 1.21. *Let $a \in C^\infty(W)$ and let $1 \leqslant p \leqslant \infty$. If $\partial^\alpha a \in L^p(W)$ for $|\alpha| \leqslant 2(n+1)$, then the operator $a^w(x, D)$ has a unique extension to the operator $A \in C_p(L^2(V))$ and the following estimate holds*:

$$|A|_p \leqslant C_p \max_{|\alpha| \leqslant 2(n+1)} \|\partial^\alpha a\|_{L^p(W)}.$$

Let S be an endomorphism of $\mathscr{S}(V)$. We define a *formal adjoint* to S by

$$\langle S^+ f, g\rangle = \langle f, Sg\rangle,$$

where

$$\langle f, g\rangle = \int f(x)\overline{g(x)}\,dx$$

for $f, g \in \mathscr{S}(V)$. It is straightforward from (1.7) that if $a \in S(m)$ then $(a^w)^+ = (\bar{a})^w$. Therefore Prop.1.8 implies that every $A \in \mathscr{L}(m)$ has a unique extension to a continuous endomorphism $\tilde{A}$ of $\mathscr{S}^*(V)$. It is also clear that $A \in \mathscr{L}(m)$ is closable in $L^2(V)$. Denote by $\bar{A}$ and A^* the closure of A and the adjoint to A in $L^2(V)$, respectively. The domain of $\bar{A}$ in $L^2(V)$ is denoted by $\mathscr{D}(\bar{A})$.

Proposition 1.22. *Let m be a weight such that $1/m \in L^p(W)$ for some $1 \leqslant p \leqslant \infty$ and let the symbol a of $A \in \mathscr{L}(m)$ satisfy* (1.18), (1.19). *Then*

$$\mathscr{D}(\bar{A}) = \{f \in L^2(V) \colon \tilde{A}f \in L^2(V)\}.$$

Proof. First let us show that if $B \in \mathscr{L}(1/m)$ and $f \in L^2(V)$, then $\tilde{B}f \in \mathscr{D}(\bar{A})$ for any $A \in \mathscr{L}(m)$. In fact, let $f_n \in \mathscr{S}(V)$ and $f_n \to f$ in $L^2(V)$. Then Theorem 1.13 and Prop. 1.21 imply $\overline{AB}$ is bounded and so ABf_n is convergent in $L^2(V)$. Thus $\tilde{B}f \in \mathscr{D}(\bar{A})$.

Now, let A satisfy (1.18), (1.19). Let $f \in L^2(V)$ and $g = \tilde{A}f \in L^2(V)$. Let B be as in Cor. 1.17 and $S = BA-I$. By Cor. 1.17 B, $S \in \mathscr{L}(1/m)$ and

$$\tilde{B}g = \tilde{B}(\tilde{A}f) = f + \tilde{S}f.$$

By the above $\tilde{B}g$, $\tilde{S}f \in \mathscr{D}(\bar{A})$. Therefore $f \in \mathscr{D}(\bar{A})$, too. Since the opposite inclusion is trivial, we have completed the proof.

For a given weight m let us define

$$(1.23)\qquad H(m) = \{u\in L^2(V)\colon\ \tilde{A}u\in L^2(V),\ A\in\mathscr{L}(m)\}.$$

COROLLARY 1.24. *Let m be a weight such that $1/m\in L^p(W)$ for some $1\leqslant p\leqslant\infty$. Further, assume there exists an operator $L = l^w(x, D)\in\mathscr{L}(m)$ such that* (1.18), (1.19) *are satisfied. Then for every $A\in\mathscr{L}(m)$,*

$$\mathscr{S}(V)\subseteq H(m)\subseteq\mathscr{D}(\bar{A}).$$

If, moreover, the symbol a of A satisfies (1.18) *and* (1.19), *then we have the equality $H(m) = \mathscr{D}(\bar{A})$.*

Proof. It is sufficient to show that if $L, A\in\mathscr{L}(m)$ and the symbol l of L satisfies (1.18), (1.19), then $\mathscr{D}(\bar{L})\subseteq\mathscr{D}(\bar{A})$. Let $Q\in\mathscr{L}(1/m)$ be such that $QL = I+S$, where $S\in\mathscr{L}(1/m)$. Let $f\in\mathscr{D}(\bar{L})$. Then $Lf = g\in L^2(V)$ and consequently $f = Qg-Sf$. But in the course of the proof of Proposition 1.22 we have seen that vectors of this form belong to $\mathscr{D}(\bar{A})$ for any $A\in\mathscr{L}(m)$.

For an unbounded closable operator A on a Hilbert space we denote by $\operatorname{Sp} A$ the spectrum of A. If $\lambda\notin\operatorname{Sp} A$, then the bounded inverse of $\lambda-A$ is denoted by $R_\lambda(A)$. We conclude this section with the following

PROPOSITION 1.25. *Let m be a weight such that $1/m\in L^p(W)$ for some $1\leqslant p\leqslant\infty$ and let $a\in S(m)$ satisfy* (1.18), (1.19). *Let T be a bounded operator on $L^2(V)$. Set $A_T = A+T$, where $A = a^w(x, D)$. Then*

(i) $(A_T^+)^* = \bar{A}_T$ *in* $L^2(V)$,

(ii) $\mathscr{D}(\bar{A}_T) = H(m)$,

(iii) $R_\lambda(A_T)\in C_p(L^2(V))$ *for* $\lambda\notin\operatorname{Sp} A_T$.

Proof. To prove (i) it is sufficient to show that $\mathscr{D}((A^+)^*)\subseteq\mathscr{D}(\bar{A})$. Let $u\in\mathscr{D}((A^+)^*)$. This means $\tilde{A}u\in L^2(V)$ so, by Prop. 1.22, $u\in\mathscr{D}(\bar{A})$. As for (ii) it follows directly from Cor. 1.24.

Finally, let $\lambda\notin\operatorname{Sp} A_T$. Let B be as in Cor. 1.17. We have

$$R_\lambda(\lambda-A-T)u = u$$

for $u\in\mathscr{D}(\bar{A})$. Let $u = Bv$, where $v\in L^2(V)$. We get then

$$R_\lambda v = (\lambda R_\lambda - R_\lambda T - I)Bv - R_\lambda Sv$$

where $S = AB-I\in\mathscr{L}(L^2(V))$. By Proposition 1.21 $B,\ S\in C_p(L^2(V))$. Since C_p is an ideal in $\mathscr{L}(L^2(V))$, $R_\lambda\in C_p$ and the proof is complete.

2. Dissipative distributions and semi-groups of measures on a Lie group. Let G be a Lie group. Denote by $\mathscr{D}(G)$ the space of smooth functions on G with compact support and by $\mathscr{D}^*(G)$ the dual space, that is, the space of distributions on G. A family $\{\mu_t\}_{t>0}$ of positive measures on G is said to be a *continuous semi-group of measures* if

(i) $\mu_t(G)\leqslant 1,\ t>0$,

(ii) $\mu_t*\mu_s = \mu_{t+s},\ t, s>0$,

(iii) $\lim\limits_{t\downarrow 0}\langle\mu_t, f\rangle = f(e) = \langle\delta, f\rangle$

for $f\in\mathscr{D}(G)$, where e denotes the identity element of G and δ – the Dirac delta.

If $\{\mu_t\}$ is a continuous semi-group of measures on G, then for every $f\in\mathscr{D}(G)$ the limit

$$(2.1)\qquad \lim_{t\downarrow 0}\frac{1}{t}\langle\mu_t-\delta, f\rangle$$

exists and defines a distribution $P\in\mathscr{D}^*(G)$. The distribution P is called the *infinitesimal generator* of the semi-group of measures $\{\mu_t\}$. It follows directly from (2.1) that P is real and satisfies the following maximum principle:

$$(2.2)\qquad \langle P, f\rangle\leqslant 0$$

for real $f\in\mathscr{D}(G)$ such that $f(e) = \sup\limits_{g\in G} f(g)$. A real distribution $P\in\mathscr{D}^*(G)$ which satisfies the maximum principle (2.2) is called *dissipative*. Hence, in other words, the above says that the infinitesimal generator of any continuous semi-group of measures is a dissipative distribution on G.

Conversely, suppose that a dissipative distribution P on G is given. Then there exists a *unique* continuous semi-group of measures $\{\mu_t\}$ such that (2.1) holds ([6], Prop.4 and also [13], Prop. 2.4). Note that the semi-group $\{\mu_t\}$ generated – in the above sense – by P, consists of symmetric measures if and only if $P = P^*$. Another simply observation is that

$$(2.3)\qquad \lim_{t\downarrow 0}\mu_t(G) = 1.$$

For $P\in\mathscr{D}^*(G)$ and $\varphi\in C^\infty(G)$ we denote by φP the distribution $f\mapsto\langle P, \varphi f\rangle$ for $f\in\mathscr{D}(G)$. By a *cut-off function* we shall mean any $[0, 1]$-valued φ in $\mathscr{D}(G)$ such that $\varphi = 1$ in a neighbourhood of e.

LEMMA 2.4. *Let P be a dissipative distribution on G. Then for any cut-off function φ the distribution $(1-\varphi)P$ is a bounded positive measure. Hence P admits a decomposition*

$$P = S+\mu,$$

where S is a compactly supported distribution and μ is a positive bounded measure.

Proof. This is essentially Proposition II.2 of [8].

As a corollary we obtain that every dissipative distribution extends to a linear form on the space of all smooth and bounded functions on G. If f is such a function and if moreover $f(e) = \sup\limits_{g\in G}|f(g)|$, then (2.2) and Lemma 2.4 imply

$$(2.5)\qquad \operatorname{Re}\langle P, f\rangle\leqslant 0.$$

Recall that if π is a strongly continuous representation of G on a Banach space H, then every compactly supported distribution T on G can be represented as a densely defined operator $\pi(T)$ on H. More precisely, let H^∞ denote the space of *smooth vectors* for π, that is, the set of all $\xi \in H$ such that the H-valued function $G \ni g \mapsto \pi_g \xi \in H$ is smooth. We take H^∞ for the domain of $\pi(T)$ and define

$$\langle \pi(T)\xi, \eta^* \rangle = \langle T, \varphi_{\xi,\eta^*} \rangle \tag{2.6}$$

for $\xi \in H^\infty$, $\eta^* \in H^*$, where $\varphi_{\xi,\eta^*}(g) = \langle \pi_g \xi, \eta^* \rangle$ for $g \in G$. $\pi(T)$, as defined by (2.6), is closable. Denote its closure by $\overline{\pi(T)}$. If S is another distribution on G with compact support and $\xi \in H$ is in the intersection of the domains of $\overline{\pi(T)}$, $\overline{\pi(S)}$ and $\overline{\pi(T+S)}$, then

$$\overline{\pi(T+S)}\,\xi = \overline{\pi(T)}\,\xi + \overline{\pi(S)}\,\xi. \tag{2.7}$$

To consider $\pi(P)$ for a dissipative $P \in \mathcal{D}^*(G)$ it is convenient to use the notion of *submultiplicative function*. Following [15] we say that a Borel function m on a Lie group G is *submultiplicative* if

(i) m is locally bounded,
(ii) $m(g) \geqslant 1$,
(iii) $m(gh) \leqslant m(g)\,m(h)$,
(iv) $m(g^{-1}) = m(g)$

for $g, h \in G$. By using the methods of [15] and [6] one can prove

Proposition 2.8. *Let m be a submultiplicative function on G. For a continuous semi-group of measures $\{\mu_t\}$ with the infinitesimal generator P the following are equivalent*:

$$\sup_{0<t\leqslant 1} \int_G m(g)\,\mu_t(dg) < \infty, \tag{2.9}$$

$$\int_{G\setminus V} m(g)\,P(dg) < \infty \tag{2.10}$$

for some compact neighbourhood V of e in G.

Note that by Lemma 2.4 P is a positive measure outside V, so (2.10) makes sense. The same lemma implies that if (2.10) holds for a single compact neighbourhood V of e, then it holds for all such V.

Now let us return to a representation π of G on a Banach space H. Let $P \in \mathcal{D}^*(G)$ be dissipative. Set

$$m_\pi(g) = \max(\|\pi_g\|, \|\pi_{g^{-1}}\|).$$

Then m_π is a submultiplicative function and for any measure μ on G such that

$$\int m_\pi(g)\,|\mu|(dg) < \infty$$

the linear operator $\pi(\mu)$ on H defined by

$$\pi(\mu)\xi = \int \pi_g \xi\,\mu(dg), \qquad \xi \in H,$$

is bounded and $\|\pi(\mu)\| \leqslant \int m_\pi(g)\,|\mu|(dg)$. Assume P and m_π satisfy (2.10). Then we can define $\pi(P)$ on H^∞ by

$$\pi(P) = \pi(S) + \pi(\mu)$$

where the decomposition $P = S + \mu$ is that of Lemma 2.4. By (2.10) $\pi(\mu)$ is a bounded operator on H and S has a compact support, so it can be represented on H^∞. The definition is unambiguous as P and $\pi(P)$ satisfy (2.6). By the above $\pi(P)$ is a closable operator on H and it can be easily deduced from (2.5) and (2.6) that, if $m_\pi(g) \leqslant 1$ for $g \in G$, then the closure of $\pi(P)$ satisfies

$$\operatorname{Re} \langle \overline{\pi(P)}\,\xi, \eta^* \rangle \leqslant 0 \tag{2.11}$$

for ξ in the domain of $\overline{\pi(P)}$ and $\eta^* \in H^*$ such that $\langle \xi, \eta^* \rangle = \|\xi\|$ and $\|\eta^*\| = 1$.

Now we recall the definition of a *strongly continuous semi-group of operators* on a Banach space H (see [21], IX.2). This is a family $\{T_t\}_{t>0}$ of bounded operators on H such that

(i) $\sup_{t>0} \|T_t\| < \infty$,
(ii) $T_t T_s = T_{t+s}$,
(iii) $\|T_t \xi - \xi\| \to 0$ when $t \downarrow 0$ for $\xi \in H$.

If $\{T_t\}$ is a strongly continuous semi-group of operators on H, then the linear subspace $\mathcal{D}$ of H consisting of all vectors $\xi \in H$ for which the limit

$$A\xi = \lim_{t\downarrow 0} \frac{1}{t}(T_t\xi - \xi) \tag{2.12}$$

exists, is dense in H. The linear operator A defined by (2.12) on $\mathcal{D}$ is closed and its spectrum is contained in the half-plane $\operatorname{Re} z \leqslant 0$. It is called the *infinitesimal generator* of the semi-group $\{T_t\}$ ([21], Chap. IX, § 3). The following theorem relates semi-groups of measures to semi-groups of operators. One can obtain it from Prop. 2.8 by using the general theory of semi-groups of operators ([21], [6]).

Theorem 2.13. ([6], Prop. 18, Théorème of § 12.) *Let π be a strongly continuous representation of a Lie group G on a Banach space H. Let P be a dissipative distribution on G such that* (2.10) *holds true for P and m_π. Let $\{\mu_t\}$ be the corresponding continuous semi-group of measures. Then $\{\pi(\mu_t)\}$ is a strongly continuous semi-group of operators on H and the infinitesimal gen-*

erator of it is just the closure of $\pi(P)$. In the case when π is a unitary representation on a Hilbert space H we also have

$$\pi(P^*)^* = \overline{\pi(P)}.$$

In particular, if P is symmetric, then $\pi(P)$ is essentially self-adjoint.

EXAMPLE 2.14. For a submultiplicative m on G and $1 \leqslant p < \infty$ denote by $L^p(m)$ the space of all measurable functions on G which are integrable with pth power with respect to the Radon measure $\mu = m\,dg$. Consider the left quasi-regular representation π on $L^p(m)$:

$$\pi_g f(h) = f(g^{-1}h)$$

for $f \in L^p(m)$ and $g, h \in G$. Then $\|\pi_g\| = \|\pi_{g^{-1}}\| = m(g)^{1/p}$ for $g \in G$, so $m_\pi = m^{1/p}$. If P is a dissipative distribution on G such that (2.10) holds for P and $m^{1/p}$, and $\{\mu_t\}$ is the semi-group of measures generated by P, then Theorem 2.13 yields that the convolution operators

$$L^p(m) \ni f \mapsto \mu_t * f \in L^p(m) \tag{2.15}$$

form a strongly continuous semi-group. It can be shown that the domain of $\overline{\pi(P)}$, which is the infinitesimal generator of the semi-group (2.15), consists of all $f \in L^p(m)$ such that $P * f \in L^p(m)$. (Note that by Lemma 2.4 this convolution always makes sense.) $\overline{\pi(P)}$ can also be thought of as the closure of the convolution operator $f \mapsto P * f$ defined for $f \in \mathcal{D}(G) \subseteq [L^p(m)]^\infty$ (cf. [6], § 7).

The following lemma is due to E. Siebert. It was also independently proved by H. Byczkowska jointly with A. Hulanicki ([5]).

LEMMA 2.16. *Let $\{\mu_t\}$ be a continuous semi-group of symmetric measures on a Lie group G. Then for every $t, s > 0$ $\operatorname{supp}\mu_t = \operatorname{supp}\mu_s = M$. Moreover, M is a closed subgroup of G.*

In case where μ_t are symmetric we shall call the common support M of μ_t the *support of the semi-group of measures*. From [5], Theorem 2 one can easily get

PROPOSITION 2.17. *Let $P, Q \in \mathcal{D}^*(G)$ be dissipative and symmetric. Then the support of the semi-group of measures generated by $P+Q$ is equal to the smallest closed subgroup of G containing the supports of the semi-groups corresponding to P and to Q.*

We define *fractional powers* of a dissipative distribution by the formula

$$\langle P^{(k)}, f\rangle = (1/\Gamma(-k)) \int_0^\infty t^{-(1+k)} \langle \mu_t - \delta, f\rangle\, dt \tag{2.18}$$

for $0 < k < 1$ and $f \in \mathcal{D}(G)$. Here $\{\mu_t\}$ is the semi-group of measures generated by P. (We shall also write $P^{(k)} = -|P|^k$ and $P^{(1)} = -|P| = P$ for symmetric dissipative P.) $P^{(k)}$ is also dissipative and the corresponding semi-group $\{\mu_t^{(k)}\}$ satisfies

$$\langle \mu_t^{(k)}, f\rangle = \int_0^\infty f_t^{(k)}(s) \langle \mu_s, f\rangle\, ds, \tag{2.19}$$

where the functions $f_t^{(k)} \in L^1(\boldsymbol{R}^+)$, $t > 0$, are defined by

$$\int_0^\infty f_t^{(k)}(\lambda) e^{-\lambda a}\, d\lambda = e^{-ta^k}, \quad t, a > 0.$$

It follows that $f_t^{(k)} \geqslant 0$ and $\int_0^\infty f_t^{(k)}(\lambda)\, d\lambda = 1$ (cf. [21], Chap. IX).

EXAMPLE 2.20. Let G be a Lie group and $\mathfrak{G}$ its Lie algebra. Elements of $\mathfrak{G}$ are right-invariant vector fields on G and every $X \in \mathfrak{G}$ defines a distribution supported at e:

$$C^\infty(G) \ni f \mapsto Xf(e).$$

We shall denote this distribution still by X, so we have $Xf = X * f$ for $f \in C^\infty(G)$. Also, $X^2 f = X * X * f$ and so on. If $X \in \mathfrak{G}$, then the distribution X is dissipative, however not symmetric, for $X^* = -X$. On the other hand, X^2 defines a dissipative and symmetric distribution on G. Starting from this point we construct the class of dissipative distributions and hence also the class of semi-groups of measures which is of particular interest for us. Specifically, denote by $(\mathcal{S})$ the smallest class of dissipative distributions on G which contains X^2 *for* $X \in \mathfrak{G}$ and is closed with respect to taking sums and fractional powers as well as to multiplication by strictly positive reals. Let (S) denote the corresponding class of semi-groups of measures. Using the formulae (2.18), (2.19) one can easily see that if $\{\mu_t\} \in (S)$, then μ_t are symmetric. Note also that (S) contains all Gaussian semi-groups i.e., the semi-groups whose generators are of the form $P = \sum X_i^2$, where $X_i \in \mathfrak{G}$, cf. [13], § 6.

For $P \in (\mathcal{S})$ let us define inductively the *Lie subalgebra $\mathfrak{G}_P$ of $\mathfrak{G}$ associated with P*:

$$\begin{aligned} &\text{(i)}\ \mathfrak{G}_P = \boldsymbol{R}X \text{ if } P = X^2,\ X \in \mathfrak{G},\\ &\text{(ii)}\ \mathfrak{G}_{P+Q} = \operatorname{Lie}(\mathfrak{G}_P \cup \mathfrak{G}_Q),\ P, Q \in (\mathcal{S}),\\ &\text{(iii)}\ \mathfrak{G}_{P^{(k)}} = \mathfrak{G}_{\alpha P} = \mathfrak{G}_P,\ P \in (\mathcal{S}),\ \alpha > 0,\ 0 < k < 1. \end{aligned} \tag{2.21}$$

(We denote by $\operatorname{Lie}(\eta)$ the Lie subalgebra of $\mathfrak{G}$ generated by a subset η of $\mathfrak{G}$.)

From Lemma 2.16, Prop. 2.17 and (2.18), (2.19) we get

PROPOSITION 2.22. *Let $\{\mu_t\} \in (S)$ and let $P \in \mathcal{D}^*(G)$ be its infinitesimal generator. Then the support of the semi-group $\{\mu_t\}$ is equal to the subgroup of G generated by $\exp(\mathfrak{G}_P)$.*

In Section 5 we are going to characterize the class of stable semi-groups of measures $\{\mu_t\}$ (cf. Section 3 for the definition) on the Heisenberg group G for which all μ_t are absolutely continuous with respect to the Haar measure dg on G. It turns out that the "easy part" (that is, the necessary condition) of the characterization can be proved in much more general setting, as it is shown by the next lemma.

LEMMA 2.23. *Let G be a connected Lie group and $\{\mu_t\}$ a continuous semi-group of symmetric measures on G. Let P be the infinitesimal generator of $\{\mu_t\}$. Then each of the following conditions implies the next one*:

(i) *μ_t are absolutely continuous with respect to the Haar measure on G, $t > 0$.*

(ii) *The support M of $\{\mu_t\}$ is equal to G.*

(iii) *$\overline{\pi(P)}$ is injective for non-trivial irreducible unitary representations π of G.*

Proof. (i) $\Rightarrow$ (ii): By (i) and Lemma 2.16 M is a closed subgroup of G of positive Haar measure. This implies M is open and hence $M = G$ since G is connected.

(ii) $\Rightarrow$ (iii): Let $\overline{\pi(P)}\,\xi = 0$ for some ξ of norm one in the domain of $\overline{\pi(P)}$, where π is a non-trivial irreducible unitary representation of G. As $\overline{\pi(P)}$ is the infinitesimal generator of the strongly continuous semi-group of contractions $\{\pi(\mu_t)\}$ we have $\pi(\mu_t)\xi = \xi$ for $t > 0$. Set $\varphi(g) = \langle \pi_g \xi, \xi\rangle$ for $g \in G$. Then $\varphi(e) = 1$, $|\varphi(g)| \leqslant 1$ for $g \in G$ and

$$\int \varphi(g)\,\mu_t(dg) = \langle \pi(\mu_t)\xi, \xi\rangle = 1$$

for $t > 0$. Since $\operatorname{supp}\mu_t = M = G$ it follows that (μ_t are probability measures and) $\varphi \equiv 1$. This, in turn, implies $\pi_g\xi = \xi$ for $g \in G$. In view of the fact that π is irreducible this is impossible unless $\xi = 0$ or π is trivial and thus the proof is complete.

Another conclusion can be drawn if the densities of μ_t are square-integrable. But first let us recall the definition of an *analytic vector* for a densely defined operator A on a Banach space H. By this we mean a vector $\xi \in H$ such that ξ is in the domain of A^n for $n \in N$ and the formal series

$$\sum_n \frac{1}{n!}\|A^n\xi\|\,Z^n \tag{2.24}$$

has a strictly positive radius of convergence.

PROPOSITION 2.25. *Let $\{\mu_t\}$ be a continuous semi-group of symmetric measures on a Lie group G such that $\mu_t = f_t\,dg$, where $f_t \in L^2(G)$ for $t > 0$. Let P be the infinitesimal generator of $\{\mu_t\}$ and π the left regular representation of G on $L^2(G)$. Then each f_t is an analytic vector for $\overline{\pi(P)}$.*

Proof. By Thorem 2.13 $\pi(f_t) = \pi(\mu_t)$ form a strongly continuous semi-group of hermitian operators on the Hilbert space $L^2(G)$ with $\overline{\pi(P)}$ as the infinitesimal generator. Still by Theorem 2.13 $\overline{\pi(P)}$ is self-adjoint, so its spectrum has to be contained in the negative half-line. By using the spectral resolution for $\overline{\pi(P)}$ one can see that for any $n \in N$, $t > 0$, $\overline{\pi(P)}^n\pi(f_t)$ is bounded on $L^2(G)$ and

$$\|\overline{\pi(P)}^n\pi(f_t)\| \leqslant t^{-n}n^ne^{-n}.$$

Now, for any $f \in L^2(G)$ we have

$$\overline{\pi(P)}^n\pi(f_t)f = \pi(f_{t/2})\overline{\pi(P)}^n\pi(f_{t/2})f = f_{t/2} * \overline{\pi(P)}^n\pi(f_{t/2})f.$$

Therefore

$$\begin{aligned}|\overline{\pi(P)}^n\pi(f_t)f(e)| &= |f_{t/2} * \overline{\pi(P)}^n\pi(f_{t/2})f(e)|\\ &\leqslant (t/2)^{-n}n^ne^{-n}\|f_{t/2}\|_{L^2(G)}\|f\|_{L^2(G)}\end{aligned}$$

for $n \in N$ and $t > 0$, so $\overline{\pi(P)}^nf_t \in L^2(G)$ and

$$\|\overline{\pi(P)}^nf_t\|_{L^2(G)} \leqslant (t/2)^{-n}n^ne^{-n}\|f_{t/2}\|_{L^2(G)}$$

which shows that for all $t > 0$, $f_t \in L^2(G)$ is an analytic vector for $\overline{\pi(P)}$ and the radius of convergence of the series (2.24) for $A = \overline{\pi(P)}$ and $\xi = f_t$ is equal to, at least, $2/t$.

3. Homogeneous structure on a nilpotent Lie group. Let G be a connected, simply connected n-dimensional nilpotent Lie group and $\mathfrak{G}$ its Lie algebra. A *family of dilations* on $\mathfrak{G}$ is a one-parameter family $\{\delta_r\}_{r>0}$ of automorphisms of $\mathfrak{G}$ of the form

$$\delta_r = r^A = e^{\log rA}$$

where A is a non-degenerate semi-simple linear transformation of $\mathfrak{G}$ with positive eigenvalues. Hence for a given family of dilations there exist a basis $\{X_j\}$ of $\mathfrak{G}$ such that

$$\delta_r X_j = r^{a_j}X_j \tag{3.1}$$

for some $a_j > 0$, $j = 1, \ldots, n$, and $r > 0$. If $\{\delta_r\}$ is a family of dilations, then so is $\{\tilde{\delta}_r\}$, where $\tilde{\delta}_r = \delta_r\alpha$ for any $\alpha > 0$, therefore we shall always assume the smallest eigenvalue of A to be 1. The biggest eigenvalue of A will be denoted by a.

Since the exponential map exp: $\mathfrak{G} \to G$ is a diffeomorphism (cf. e.g. [12]), the dilations $\{\delta_r\}$ lift *via* exp to give a one-parameter group of automorphisms of G, which we still denote by $\{\delta_r\}$. Let us fix a bi-invariant Haar measure dg on G (which is transported by exp from a Lebesgue measure on $\mathfrak{G}$). Then for $f \in L^1(G)$

$$\int_G f(\delta_r g) = r^{-Q}\int_G f(g)\,dg, \tag{3.2}$$

where $Q = \mathrm{Tr}(A) = \sum_{j=1}^{n} a_j \geqslant n$ is called a *homogeneous dimension* of G (with respect to the dilations $\{\delta_r\}$).

A measurable function f on G is said to be *homogeneous* of degree $\theta \in \boldsymbol{R}$ if

(3.3) $$f(\delta_r g) = r^{\theta} f(g)$$

almost everywhere in G. If f is also locally integrable, then

$$\int f(g)\varphi(\delta_r g)\,dg = r^{-Q-\theta}\int f(g)\varphi(g)\,dg$$

for $\varphi \in \mathscr{D}(G)$. This motivates us to call a distribution $T \in \mathscr{D}^*(G)$ *homogeneous* of degree θ if

(3.4) $$\langle T, \varphi\circ\delta_r\rangle = r^{-Q-\theta}\langle T, \varphi\rangle$$

for $\varphi \in \mathscr{D}(G)$ and $r > 0$. Let us also define

(3.5) $$\langle \delta_r T, \varphi\rangle = \langle T, \varphi\circ\delta_r\rangle$$

for $T \in \mathscr{D}^*(G)$ and $\varphi \in \mathscr{D}(G)$. Since every $X \in \mathfrak{G}$ can be regarded as a distribution supported at e, (3.5) can be understood as an extension from $\mathfrak{G}$ to $\mathscr{D}^*(G)$ of the dilations $\{\delta_r\}$. It follows immediately from (3.4) and (3.5) that $T \in \mathscr{D}^*(G)$ is homogeneous of degree $-Q-\theta$ if and only if $\delta_r T = r^{\theta} T$ for $r > 0$. In particular, the eigenvectors X_j of A are, by (3.1), homogeneous of degrees $-Q-a_j$, $j = 1, \ldots, n$.

The exponential map induces an isomorphism

$$\mathscr{D}(G) \ni f \mapsto f\circ\exp \in \mathscr{D}(\mathfrak{G})$$

and hence also an isomorphism between $\mathscr{D}^*(G)$ and $\mathscr{D}^*(\mathfrak{G})$. The distribution corresponding to $T \in \mathscr{D}^*(G)$ is

(3.6) $$\mathscr{D}(\mathfrak{G}) \ni f \mapsto \langle T, f\circ\exp^{-1}\rangle \in C$$

and it will be denoted also by T. A $T \in \mathscr{D}^*(G)$ is said to be *temperate* if it is temperate as a distribution on $\mathfrak{G}$, in the above sense.

The group of dilations $\{\delta_r\}$ on $\mathfrak{G}$ induces a group of dilations $\{\delta_r^*\}$ on the dual vector space $\mathfrak{G}^*$ by

(3.7) $$\langle \delta_r^* \xi, X\rangle = \langle \xi, \delta_r X\rangle$$

for $X \in \mathfrak{G}$, $\xi \in \mathfrak{G}^*$ and $r > 0$. We also have

(3.8) $$(f\circ\delta_r)^\wedge = r^{-Q}\hat{f}\circ\delta^*_{r^{-1}}$$

for $f \in L^1(\mathfrak{G})$ and $r > 0$, where $\hat{f}$ denotes the Fourier transform of f, as defined in Section 1. Using the above definitions it is elementary to prove that if T is temperate and homogeneous of degree θ with respect to $\{\delta_r\}$, then $\hat{T}$ is homogeneous of degree $-Q-\theta$ with respect to $\{\delta_r^*\}$.

A *homogeneous norm* on $\mathfrak{G}$ is defined to be a continuous positive function $|\cdot|$ on $\mathfrak{G}$ which is smooth away from 0, homogeneous of degree 1, and such that $|-X| = |X|$ and $|X| = 0$ only if $X = 0$. Any such a norm satisfies

(3.9) $$(1/C)\|X\| \leqslant |X| \leqslant C\|X\|^{1/a} \quad \text{for} \quad \|X\| \leqslant 1,$$

(3.10) $$(1/C)(1+\|X\|)^{1/a} \leqslant 1+|X| \leqslant C(1+\|X\|),$$

(3.11) $$|X+Y| \leqslant C(|X|+|Y|)$$

for $X, Y \in \mathfrak{G}$ and some positive constant C. By $\|\cdot\|$ we denote an Euclidean norm on $\mathfrak{G}$ (cf. [10]). To show that homogeneous norms exist define

(3.12) $$|X| = t \quad \text{if} \quad \|\delta_{t^{-1}} X\| = 1.$$

The implicit function theorem implies smoothness of $|\cdot|$ on $\mathfrak{G}\setminus\{0\}$ and the other properties are obvious. Note that by a proper choice of an Euclidean norm in (3.12) we can make the homogeneous norm defined by (3.12) to be invariant under *reflections* with respect to the eigenvectors X_j of A. More precisely, if $\|\cdot\|$ is such that $\{X_j\}$ is an orthonormal basis, then the homogeneous norm (3.12) satisfies

(3.13) $$|p_j(X)| = |X|,$$

where p_j are linear transformations of $\mathfrak{G}$ defined by

$$p_j\Big(\sum_{k=1}^{n} \alpha_k X_k\Big) = \sum_{k\neq j} \alpha_k X_k - \alpha_j X_j$$

for $X = \sum \alpha_k X_k \in \mathfrak{G}$ and $1 \leqslant j \leqslant n$. We also define a homogeneous norm on G by $|\exp X| = |X|$ for $X \in \mathfrak{G}$. It is clear that $|\cdot|$ is continuous and positive on G, and smooth away from 0. It is also homogeneous of degree 1 and satisfies $|g^{-1}| = |g|$, and $|g| = 0$ only if $g = e$ for $g \in G$.

The following definition seems to be a natural generalization of the notion of a stable semi-group of measures on the Euclidean space (cf. [9], Chap. IX, § 6).

Definition 3.14. Let G be a connected, simply connected nilpotent Lie group with a family of dilations $\{\delta_r\}$. A continuous semi-group of measures $\{\mu_t\}$ on G is said to be *stable* (*in the strict sense*) with respect to $\{\delta_r\}$ if there exists $\theta > 0$ such that

(3.15) $$\delta_r \mu_t = \mu_{r^{\theta} t}$$

for $r, t > 0$. If it is so, θ is called the *characteristic exponent* of the semi-group $\{\mu_t\}$.

This property can be also formulated in terms of the infinitesimal generator:

Proposition 3.16. *Let $\{\mu_t\}$ be a continuous semi-group of measures on G*

and let P be its infinitesimal generator. Then $\{\mu_t\}$ is stable with the characteristic exponent θ if and only if P is homogeneous of degree $-Q-\theta$.

Proof. For a given $r>0$ let us consider the distributions $\delta_r P$ and $r^\theta P$. It is clear that they both are dissipative and the semi-group generated by $\delta_r P$ is $\{\delta_r \mu_t\}$, while the one corresponding to $r^\theta P$ is $\{v_t\}$, where $v_t = \mu_{r^\theta t}$ for $t>0$. By the uniqueness theorem (cf. Section 2) $\delta_r P = r^\theta P$ if and only if $\delta_r \mu_t = v_t = \mu_{r^\theta t}$ for every $t>0$. Since this holds for every $r>0$, the proof is complete.

Remark 3.17. If a continuous semi-group $\{\mu_t\}$ of measures on G is stable, then $\mu_t (t>0)$ are probability measures. In fact, (3.15) implies $\mu_t = \delta_{t^{1/\theta}} \mu_1$ for $t>0$, so

$$\mu_t(G) = \langle \mu_t, 1\rangle = \langle \mu_1, 1\rangle = \text{const}.$$

Hence, by (2.3), $\mu_t(G) = 1$ for $t>0$.

The definition of a dissipative distribution on a Lie group (cf. Section 2) is given in terms of the underlying manifold, so it does not depend on the group structure. We shall make use of this simple observation in the following

Proposition 3.18. *Let G be a connected simply connected nilpotent Lie group and $\mathfrak{G}$ its Lie algebra. Let P be a dissipative distribution on G. Then P is temperate and the Fourier transform $\psi = \hat{P}$ of it is a continuous function on $\mathfrak{G}^*$ such that*

(3.19) $$\operatorname{Re}\psi(\xi) \leqslant 0 \quad \textit{for} \quad \xi \in \mathfrak{G}^*,$$

(3.20) $$e^{\psi} \textit{ is positive definite in } \mathfrak{G}^*.$$

Proof. According to the above remarks, it follows from (3.6) that P regarded as a distribution on $\mathfrak{G}$ is dissipative, so Lemma 2.4 implies P is temperate. Then (3.19) follows from (2.5) applied to the characters of the abelian vector group $\mathfrak{G}$. Further, let $\{\mu_t\}$ be the continuous semi-group of measures with respect to the abelian convolution on $\mathfrak{G}$ whose infinitesimal generator is P. Then $\hat{\mu}_t(\xi) = e^{t\psi(\xi)}$ and since μ_t are positive, we get (3.20).

4. Estimates for the Fourier transform of a dissipative distribution. In this section we work in n-dimensional real vector space V endowed with a family of dilations $\{\delta_r\}$. We choose a basis $\{e_j\}$ of the eigenvectors of $\{\delta_r\}$, so that

$$\delta_r e_j = r^{a_j} e_j$$

for $1 \leqslant j \leqslant n$. The non-isotropic *length* of a multi-index α is defined by

$$[\alpha] = \sum_{j=1}^{n} a_j \alpha_j$$

We shall also denote

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

for $V \ni x = \sum_{j=1}^{n} x_j e_j$. We start with recalling a well-known fact.

Proposition 4.1 *Let $K \in \mathcal{S}^*(V)$ be homogeneous of degree $-Q-\theta$ and such that restricted to $V \setminus \{0\}$ is a Radon measure. Then there is a measure σ on the unit sphere $\Sigma = \{x \in V\colon |x| = 1\}$ such that for all f locally integrable on $V \setminus \{0\}$*

$$\int_{a \leqslant |x| \leqslant b} f(x) K(dx) = \int_a^b r^{-1-\theta} dr \int_\Sigma f(\delta_r s)\sigma(ds),$$

where $0 < a \leqslant b < \infty$. Moreover,

(4.2) $$\int_\Sigma x^\alpha \sigma(dx) = 0$$

for all $[\alpha] = \theta$.

Recall that by a cut-off function we mean a $[0, 1]$-valued $\varphi \in \mathcal{S}(V)$ which is equal to 1 in a neighbourhood of 0.

Proposition 4.3. *Let K satisfy the assumptions of Proposition 4.1 and let φ be a cut-off function. Then*

$$|\partial^\alpha(\hat{\varphi} * \hat{K})(\xi)| \leqslant C_\alpha (1+|\xi|)^{(\theta-[\alpha])_+}$$

for all α such that $[\alpha] \neq \theta$ and $\xi \in V^*$. Here $t_+ = \max(t, 0)$, $t \in \boldsymbol{R}$.

Proof. First assume $[\alpha] < \theta$. Then the distribution $x^\alpha K$ is homogeneous of degree $-Q-\theta+[\alpha] < -Q$ and so by Prop. 4.1 $x^\alpha(K-\varphi K)$ is a bounded measure. Hence its Fourier transform

$$\xi \mapsto \partial^\alpha(\hat{K} - \hat{\varphi} * \hat{K})(\xi)$$

is a continuous and bounded function. Since $\hat{\varphi} * \hat{K}$ is smooth, $\partial^\alpha \hat{K}$ is a continuous function, too. As it is, of course, homogeneous of degree $\theta - [\alpha]$ we have

$$|\partial^\alpha(\hat{\varphi} * \hat{K})(\xi)| \leqslant C_\alpha (1+|\xi|)^{\theta-[\alpha]}$$

for some $C_\alpha > 0$ and $\xi \in V^*$.

The case $[\alpha] > \theta$ is similar. In fact, then $x^\alpha K$ is by Prop. 4.1 a bounded measure on any compact neighbourhood of 0. This implies $x^\alpha \varphi K$ is a bounded measure, so

$$|\partial^\alpha(\hat{\varphi} * \hat{K})(\xi)| \leqslant C_\alpha$$

for some $C_\alpha > 0$ and $\xi \in V^*$. This ends the proof.

The case when $[\alpha] = \theta$ is a little bit more complicated and the estimate obtained is worse. Nevertheless it is sufficient for our purposes. We shall need the following easy lemma.

Lemma 4.4. *Let u be a compactly supported distribution on V, f a function in $\mathcal{S}(V)$ and h a smooth function of polynomial growth on V. Then*

$$(u * f) * h = u * (f * h).$$

If, moreover, μ is a measure such that

$$\int |h(x)|\, |\mu|(dx) < \infty,$$

then also

$$(\mu * f) * h = \mu * (f * h).$$

Note that if $P = P^*$ is a dissipative distribution on V, then $\hat{P}(\xi) \leqslant 0$ and

$$(|P|^k)^\wedge(\xi) = \left(-\hat{P}(\xi)\right)^k = |\hat{P}(\xi)|^k$$

for $\xi \in V^*$ and $0 < k \leqslant 1$ (cf. (2.18), (3.19)).

PROPOSITION 4.5. *Let $P = P^* \in \mathcal{D}^*(V)$ be dissipative and homogeneous of degree $-Q-\theta$ and let φ be a cut-off function. Then*

$$\lim_{\varepsilon \downarrow 0} \int_{\varepsilon \leqslant |x| \leqslant 1/\varepsilon} |x|^a \, |\varphi P - \delta|^b (dx) < \infty$$

for $a > b\theta$, $0 < b \leqslant 1$.

Proof. For $r > 0$ set $\varphi_r(x) = \varphi(\delta_r x)$. We have then to show that the integral

$$I_n = \int |x|^a (\varphi_{1/n} - \varphi_n)(x) \, |\varphi P - \delta|^b (dx)$$

has a finite limit when n tends to infinity. Denote by K_a the Fourier transform of the locally integrable function $x \mapsto |x|^a$. K_a is homogeneous of degree $-Q-\theta$ and smooth away from the origin, and so by Prop. 4.1 it decomposes

$$K_a = \tilde{K}_a + m_a,$$

where $\tilde{K}_a$ is compactly supported and m_a is a measure such that

$$\int |\xi|^p \, |m_a|(d\xi) < \infty$$

for $p < a$. Therefore, taking the Fourier transforms of the distribution $|\varphi P - \delta|$ and of the function

$$x \mapsto |x|^a (\varphi_{1/n} - \varphi_n)(x)$$

which belongs to $\mathcal{D}(V)$, and applying Lemma 4.4 we get

$$(4.6) \quad I_n = \langle K_a, (\hat{\varphi}_{1/n} - \hat{\varphi}_n) * |1 - \hat{\varphi} * \hat{P}|^b \rangle$$
$$= \langle \tilde{K}_a, \hat{\varphi}_{1/n} * |1 - \hat{\varphi} * \hat{P}|^b \rangle + \langle m_a, \hat{\varphi}_{1/n} * |1 - \hat{\varphi} * \hat{P}|^b \rangle -$$
$$- \langle K_a, \hat{\varphi}_n * |1 - \hat{\varphi} * \hat{P}|^b \rangle.$$

Since $\hat{\varphi}_{1/n}$ is an approximate identity in $\mathcal{S}(V^*)$ and the function

$$\xi \mapsto |1 - \hat{\varphi} * \hat{P}(\xi)|^b$$

is smooth and bounded by $(1 + |\xi|)^{b\theta}$, $b\theta < a$, the first two terms of the right-hand side of (4.6) tend to $\langle K_a, |1 - \hat{\varphi} * \hat{P}|^b \rangle$, so that we have to deal only with

the last one. Because $\hat{\varphi}_r = \delta_r \hat{\varphi}$ and $\delta_r(\hat{\varphi} * \hat{P}) = \delta_r \hat{\varphi} * \delta_r \hat{P}$ (cf. (3.5), (3.8)), we have

$$(4.7) \quad \langle K_a, \hat{\varphi}_n * |1 - \hat{\varphi} * \hat{P}|^b \rangle = \langle K_a, \delta_n(\hat{\varphi} * |1 - n^{Q+\theta} \delta_{1/n} \hat{\varphi} * \hat{P}|^b \rangle$$
$$= n^{b\theta - a} n^{(b-1)Q} \left\langle K_a, \hat{\varphi} * \left| \frac{1}{n^{Q+\theta}} - \hat{\varphi}_{1/n} * \hat{P} \right|^b \right\rangle.$$

Of course, we have made use of homogeneity of both K_a and P. Using again the fact that $\hat{\varphi}_{1/n}$ form an approximate identity and decomposing K_a as above we can see that

$$\lim_{n \to \infty} \left\langle K_a, \hat{\varphi} * \left| \frac{1}{n^{Q+\theta}} - \hat{\varphi}_{1/n} * \hat{P} \right|^b \right\rangle = \langle K_a, \hat{\varphi} * |\hat{P}|^b \rangle$$

and consequently by (4.6) and (4.7)

$$\lim_{n \to \infty} I_n = \langle K_a, |1 - \hat{\varphi} * \hat{P}|^b \rangle$$

which ends the proof.

Before drawing conclusions from the above proposition we shall state a lemma. The proof of the lemma which we omit is based essentially on (4.2).

LEMMA 4.8. *Let P be a dissipative distribution on V. Assume P to be homogeneous of degree $-Q-\theta$. If $\theta = 2a_k$ for some $1 \leqslant k \leqslant n$, then P admits a decomposition*

$$(4.9) \quad P = -\sum_{j \in I_k} C_j \partial_j^2 + S,$$

where $I_k = \{j\colon a_j = a_k\}$, $C_j \geqslant 0$, and S is dissipative and such that the semi-group generated by it is supported in the linear space spanned by e_j, $j \notin I_k$.

PROPOSITION 4.10. *Let $P \in \mathcal{D}^*(V)$ be dissipative and homogeneous of degree $-Q-\theta$. Let φ be a cut-off function. Then for every $\varepsilon > 0$ there is a constant C such that*

$$|\partial^\alpha (\hat{\varphi} * \hat{P})(\xi)| \leqslant C(1 + |\xi|)^\varepsilon$$

for $[\alpha] = \theta$.

Proof. There are only two possibilities: either $\theta = a_k$ for some $1 \leqslant k \leqslant n$, or $\theta = a_i + a_j$ for some $1 \leqslant i, j \leqslant n$. Assume first $\theta = a_k$. For a given $\varepsilon > 0$ set $b = 1 - \frac{\varepsilon}{\theta}$. Then by Prop. 4.5

$$|\partial_k |1 - \hat{\varphi} * \hat{P}|^b(\xi)| \leqslant C$$

for some $C > 0$ and $\xi \in V^*$, whence

$$|\partial_k \hat{\varphi} * \hat{P}(\xi)| \leqslant \left(\frac{C}{b}\right) |1 - \hat{\varphi} * \hat{P}(\xi)|^{1-b} \leqslant C'(1 + |\xi|)^\varepsilon$$

for $\xi \in V^*$ and another constant $C' > 0$.

Now, let $\theta = a_i + a_j$ for some $1 \leqslant i, j \leqslant n$. Since $\theta \leqslant 2a_r$ for every $1 \leqslant r \leqslant n$ (see e.g. [9]), it follows that $a_i = a_j = a_k$ for some $1 \leqslant k \leqslant n$ and we can apply Lemma 4.8 to obtain the decomposition (4.9). According to the same lemma

$$\partial_k^2 (\hat{\varphi} * \hat{S})(\xi) = 0$$

for $\xi \in V^*$ and so we get

$$\partial_k^2 (\hat{\varphi} * \hat{P})(\xi) = -2C_k$$

where C_k is the constant occurring in (4.9). The proof is now complete.

5. The main theorem. The Heisenberg algebra of dimension $2n+1$ is defined to be a $(2n+1)$-dimensional Lie algebra $\mathfrak{G}$ with one-dimensional center $\mathfrak{Z}$ and such that $[\mathfrak{G}, \mathfrak{G}] = \mathfrak{Z}$. On the Heisenberg algebra there exists an alternating (symplectic) form σ such that $[x, y] = \sigma(x, y) z_0$, where z_0 is a fixed non-zero element of $\mathfrak{Z}$. (Henceforth we shall use *small* letters x, y, z to denote the elements of $\mathfrak{G}$.) Let us choose a linear complement W to $\mathfrak{Z}$ in $\mathfrak{G}$. By the above, σ restricted to $W \times W$ is non-degenerate. From the properties of alternating forms it follows that there exist n-dimensional subspaces V, V^* of W such that $W = V \oplus V^*$ and $\sigma | V \times V = 0$, $\sigma | V^* \times V^* = 0$. Then

$$V \times V^* \ni (x, \xi) \mapsto x\xi = \sigma(x, \xi) \in \boldsymbol{R} \tag{5.1}$$

defines a duality between V and V^*, and

$$\sigma(w, v) = y\xi - x\eta, \tag{5.2}$$

where $w = (x, \xi)$, $v = (y, \eta) \in W = V \oplus V^*$.

Having chosen $0 \neq z_0 \in \mathfrak{Z}$ and the complement W to $\mathfrak{Z}$ we can identify $\mathfrak{G}$ with its dual $\mathfrak{G}^*$ by means of the non-degenerate bi-linear form

$$\beta(w + tz_0, v + sz_0) = \sigma(w, v) + ts, \tag{5.3}$$

where $w, v \in W$, $t, s \in \boldsymbol{R}$. Moreover, both $\mathfrak{G}$ and $\mathfrak{G}^*$ can be identified with $W \oplus \boldsymbol{R}$. In terms of the above identification the annihilator

$$\mathfrak{Z}^\perp = \{x \in \mathfrak{G} \cong \mathfrak{G}^*: \beta(x, z) = 0,\ z \in \mathfrak{Z}\}$$

is equal to W and the Fourier transform of a function $f \in \mathcal{S}(\mathfrak{G})$ (cf. Section 1) is given by the following formula

$$\hat{f}(w, t) = \int_W \int_R e^{i\beta(v,s|w,t)} f(v, s)\, dv\, ds \tag{5.4}$$

for $(w, t) \in W \oplus \boldsymbol{R} \cong \mathfrak{G} \cong \mathfrak{G}^*$.

It can easily be seen that $\mathfrak{G}$ admits dilations. In fact, choose a basis $\{x_j\}_{j=1}^n$ in V and a basis $\{x_{n+j}\}_{j=1}^n$ in V^* dual to $\{x_j\}$ with respect to the pairing (5.1). Put also $x_{2n+1} = z_0 \in \mathfrak{Z}$. Then $\{x_j\}_{j=1}^{2n+1}$ is a basis for the vector space $\mathfrak{G}$ and satisfies

$$[x_j, x_{n+j}] = x_{2n+1}, \quad j = 1, \ldots, n, \tag{5.5}$$

the other brackets being zero. Now we define a family of dilations in $\mathfrak{G}$ by

$$\delta_r x_j = r^{a_j} x_j, \quad 1 \leqslant j \leqslant 2n+1, \tag{5.6}$$

for $r > 0$ and positive numbers a_j satisfying $\min\limits_{1 \leqslant j \leqslant 2n+1} a_j = 1$ and

$$a_j + a_{n+j} = a_{2n+1} = a, \quad 1 \leqslant j \leqslant n. \tag{5.7}$$

This is in fact a general form of dilations in $\mathfrak{G}$.

The homogeneous dimension of $\mathfrak{G}$ with respect to the dilations (5.6) is $Q = \sum_{j=1}^{2n+1} a_j = (n+1)a$. Note that V, V^* (and hence W) are invariant under the dilations. According to (3.7) and (5.3) we have

$$\begin{aligned} \delta_r^* x_j &= r^{a_{n+j}} x_j, \quad 1 \leqslant j \leqslant n, \\ \delta_r^* x_{n+j} &= r^{a_j} x_{n+j}, \quad 1 \leqslant j \leqslant n, \\ \delta_r^* x_{2n+1} &= r^{a_{2n+1}} x_{2n+1} \quad \text{for} \quad r > 0. \end{aligned} \tag{5.8}$$

From now on let $\mathfrak{G}$ be the $(2n+1)$-dimensional Heisenberg algebra with a fixed family of dilations $\{\delta_r\}$. Suppose that $0 \neq z_0 \in \mathfrak{Z}$ and a complement $W = V \oplus V^*$ to $\mathfrak{Z}$ in $\mathfrak{G}$ are chosen and fixed. We shall assume V and V^* to be invariant under $\{\delta_r\}$ and such that $\sigma | V \times V = 0 = \sigma | V^* \times V^*$. Note that a choice of W and z_0 determines the bi-linear form β (cf. (5.3)) so the meaning of the identifications $\mathfrak{G} \cong \mathfrak{G}^* \cong W \oplus \boldsymbol{R}$ is clear. Moreover let $|\cdot|$ be a homogeneous norm invariant under reflections with respect to the eigenvectors of the dilations (cf. Section 3).

The simply connected, connected Lie group G corresponding to $\mathfrak{G}$ is called the *Heisenberg group*. According to the Campbell–Hausdorff formula the group law in G is

$$\exp(w + tz_0) \exp(v + sz_0) = \exp\left(w + v + \left(t + s + \tfrac{1}{2}\sigma(w, v)\right) z_0\right),$$

where $w, v \in W$, $s, t \in \boldsymbol{R}$.

The irreducible unitary representations of G are described in the following way. The one-dimensional ones are just the characters:

$$\chi_w(\exp(v + z)) = e^{i\sigma(v,w)}, \tag{5.9}$$

where $w \in \mathfrak{Z}^\perp = W \subseteq \mathfrak{G}^*$, $v \in W$ and $z \in \mathfrak{Z}$. The other irreducibles are infinite-dimensional. They can be realized on $L^2(V)$ and parametrized with $\lambda \in \boldsymbol{R} \setminus \{0\}$. The following realization is convenient for our purposes. For $\lambda = \pm 1$ set

$$\pi^\lambda_{\exp(x,\xi,z)} \varphi(t) = e^{i\lambda z} e^{-\frac{1}{2}\lambda x\xi} e^{i\lambda t\xi} \varphi(t - x), \tag{5.10}$$

where $(x, \xi, z) \in V \oplus V^* \oplus \boldsymbol{R} \cong \mathfrak{G}$, $t \in V$, $\varphi \in L^2(V)$. Now for arbitrary non-zero $\lambda \in \boldsymbol{R}$ we define

$$\pi_g^\lambda = \pi^{\operatorname{sgn}\lambda}_{\delta_{|\lambda|^{(n+1)/Q}} g}, \qquad g \in G \tag{5.11}$$

(cf. e.g. [20]). Recall that for each $\lambda \neq 0$ the space of smooth vectors for π^λ in $L^2(V)$ coincides with the Schwartz class $\mathscr{S}(V)$. Let us denote by $\hat{G}$ the *set* of all unitary irreducible representations of G. All the above notation of this section will be kept and often used without any additional explanation.

The following two propositions give some information about images of distributions on G in representations

PROPOSITION 5.12. *Let P be a dissipative distribution on G. If P is homogeneous of degree $-Q-\theta$, then*

$$\pi^\lambda(P) = |\lambda|^{\theta/a} \pi^{\operatorname{sgn}\lambda}(P).$$

Proof. It follows immediately from (2.6), (5.11) and the homogeneity of P.

PROPOSITION 5.13. *Let T be a compactly supported distribution on G. Then for every real $\lambda \neq 0$*

$$\pi^\lambda(T) = a_\lambda^w(x, D)$$

(see (1.6), (1.7)), *where*

$$a_\lambda(x, \xi) = \hat{T}(\operatorname{sgn}\lambda\, \delta^*_{|\lambda|^{1/a}} x,\ \delta^*_{|\lambda|^{1/a}} \xi,\ \lambda)$$

for $(x, \xi) \in V \oplus V^* = W$.

Proof. Let $f \in \mathscr{S}(G)$. For real λ of modulus 1 and $\varphi \in \mathscr{S}(V)$ we have by (5.10)

$$\begin{aligned}[\pi^\lambda(f)\varphi](t) &= \int_G f(g)(\pi_g^\lambda \varphi)(t)\,dg \\ &= \int_{\mathfrak{G}} f(x, \xi, z)\,\pi^\lambda_{\exp(x,\xi,z)}\varphi(t)\,dx\,d\xi\,dz \\ &= \int_V \int_{V^*} \int_{\boldsymbol{R}} f(x, \xi, z)\, e^{i\lambda(z - \frac{1}{2}x\xi + t\xi)} \varphi(t-x)\,dx\,d\xi\,dz \\ &= \int_V \int_{V^*} \int_{\boldsymbol{R}} f(t-x, \xi, z)\, e^{i\lambda(z + \frac{1}{2}x\xi + \frac{1}{2}t\xi)} \varphi(x)\,dx\,d\xi\,dz \\ &= \int_V \int_{V^*} e^{i\eta(t-x)} F(x, \eta)\,dx\,d\eta,\end{aligned}$$

where

$$\begin{aligned}F(x, \eta) &= \int_V \int_{V^*} \int_{\boldsymbol{R}} f(y, \xi, z)\, e^{\frac{1}{2}i\lambda(x+t)\xi - i\lambda y\eta + i\lambda z}\,dy\,d\xi\,dz \\ &= \int_V \int_{V^*} \int_{\boldsymbol{R}} f(y, \xi, z)\, e^{i\beta(y,\xi,z|\frac{1}{2}\lambda(x+t),\eta,\lambda)}\,dy\,d\xi\,dz \\ &= \hat{f}(\tfrac{1}{2}\lambda(x+t), \eta, \lambda).\end{aligned}$$

Hence by (1.1) the symbol of $\pi^\lambda(f)$, $\lambda = \pm 1$ is

$$a_\lambda(x, \xi) = \hat{f}(\lambda x, \xi, \lambda).$$

Now, by (3.8), (5.11) and the above the symbol of $\pi^\lambda(f)$, $\lambda \in \boldsymbol{R} \setminus \{0\}$, is

$$\begin{aligned}a_\lambda(x, \xi) &= |\lambda|^{-(n+1)}(f \circ \delta_{|\lambda|^{1/a}})^\wedge(\operatorname{sgn}\lambda x, \xi, \operatorname{sgn}\lambda) \\ &= \hat{f}\,(\delta^*_{|\lambda|^{1/a}}(\operatorname{sgn}\lambda x, \xi, \operatorname{sgn}\lambda)) \\ &= \hat{f}(\operatorname{sgn}\lambda\, \delta^*_{|\lambda|^{1/a}} x,\ \delta^*_{|\lambda|^{1/a}} \xi,\ \lambda).\end{aligned}$$

To deal with the case of compactly supported distribution T note first that the Fourier transform of T is a smooth function on $\mathfrak{G}^*$ and satisfies

$$|\partial_w^\alpha \partial_\lambda^k \hat{T}(w, \lambda)| \leqslant C_{\alpha,k}(1 + |\lambda| + \|w\|)^N$$

for $\alpha \in \boldsymbol{N}^{2n}$, $k \in \boldsymbol{N}$, $(w, \lambda) \in W \oplus \boldsymbol{R}$ and some $C_{\alpha,k} > 0$, $N \in \boldsymbol{N}$. Therefore $\hat{T}(\cdot, \lambda) \in S(n_\lambda)$ for every $\lambda \in \boldsymbol{R} \setminus \{0\}$, where

$$n_\lambda(w) = (1 + |\lambda| + \|w\|^N)$$

is a weight. It is clear that there exists a sequence $f_n \in \mathscr{S}(G)$ satisfying

$$\lim_{n \to \infty} \langle f_n, \varphi \rangle = \langle T, \varphi \rangle \tag{5.14}$$

for $\varphi \in C^\infty(G)$ and such that for each $\lambda \in \boldsymbol{R} \setminus \{0\}$ the sequence $\hat{f}_n(\cdot, \lambda)$ is bounded in $S(n_\lambda)$. (5.14) implies $\pi^\lambda(f_n)$ converges weakly on $\mathscr{S}(V)$ to $\pi^\lambda(T)$ for $\lambda \neq 0$. On the other hand it implies $\hat{f}_n(\cdot, \lambda)$ converges pointwise to $\hat{T}(\cdot, \lambda)$ on W for all $\lambda \neq 0$. Therefore, in view of Cor. 1.9 and the above, our assertion follows.

In what follows we shall need weights related to the homogeneous structure of G. Let us restrict the dilations $\{\delta_r\}$ to $W \subseteq \mathfrak{G}$. Then the homogeneous dimension of W is $q = (n/n+1)Q = na$ and $|w| = |(w, 0)|$ is a homogeneous norm on W. Set

$$m_\lambda(w) = |(w, \lambda)| \tag{5.15}$$

for $w \in W$, $\lambda \in \boldsymbol{R} \setminus \{0\}$. It is not difficult to see that due to (3.10), (3.11) m_λ is a weight in W. Moreover, all the weights m_λ, $\lambda \neq 0$ are equivalent. We shall work with $m_1 = m$. From now on m is fixed.

Let us remark that if $p > q$, then $1/m \in L^p(W)$. Note also that m is smooth and belongs to $S(m)$. Moreover, it satisfies (1.18), (1.19), so the hypothesis of Cor. 1.24 is fulfilled.

Now we are going to prove our main theorem.

THEOREM 5.16. *Let $\{\mu_t\}$ be a stable semi-group of symmetric measures on the Heisenberg group G. Let $P \in \mathscr{D}^*(G)$ be its infinitesimal generator. Then the following conditions are equivalent:*

(i) $\overline{\pi(P)}$ *is injective for* $\pi \in \hat{G} \setminus \{1\}$.

(ii) μ_t *are absolutely continuous* (*with respect to the Haar measure on* G) *for* $t > 0$ *and their densities are square-integrable.*

(iii) μ_t *are absolutely continuous for* $t > 0$.

(iv) $\operatorname{supp} \mu_t = G$ *for* $t > 0$.

Proof. Denote by θ the characteristic exponent of $\{\mu_t\}$. We begin with the most important implication (i) $\Rightarrow$ (ii) the proof of which will be carried out in several steps. The first step is to show that there exists $C > 0$ such that

$$|\hat{P}(w, \lambda)| \geqslant C|w|^\theta \tag{A}$$

for $\lambda \in \boldsymbol{R}$ and $|w|$ sufficiently large. We shall prove (A) only for positive λ as the same proof is valid also for $\lambda < 0$. Set $\psi(w, \lambda) = \hat{P}(w, \lambda)$. ψ is a continuous function on $\mathfrak{G}^* = W \times \boldsymbol{R}$. It is homogeneous of degree θ and e^ψ is positive definite (Prop. 3.17). Suppose first $\psi(w, \lambda) \neq 0$ for all (w, λ) such that $\lambda > 0$. Then

$$|\psi(w, \lambda)| \geqslant C|(w, \lambda)|^\theta$$

for $(w, \lambda) \in W \times \boldsymbol{R}$, $\lambda \geqslant 0$, where $C = \inf_{|(w,\lambda)|=1, \lambda \geqslant 0} |\psi(w, \lambda)|$. By our assumption and (i) applied to one dimensional representations of G, C is strictly positive.

Let us now assume there exists a pair $(w_0, \lambda_0) \in W \times \boldsymbol{R}$ such that $\lambda_0 > 0$ and $\psi(w_0, \lambda_0) = 0$. Then the inequality valid for any positive definite function f:

$$|f(x) - f(y)|^2 \leqslant 2f(0)[f(0) - \operatorname{Re} f(x-y)]$$

applied to $f(w, \lambda) = e^{\psi(w,\lambda)}$, $x = (w_0, \lambda_0)$, $y = (w - w_0, 0)$ yields $\psi(w, \lambda_0) = \psi(w - w_0, 0)$ for $w \in W$ whence

$$|\psi(w, \lambda_0)| = |\psi(w - w_0, 0)| \geqslant C|w - w_0|^\theta$$

for $w \in W$. As above, (i) implies $C = \inf_{|w|=1} |\psi(w, 0)| > 0$. Finally, by homogeneity of ψ we get

$$|\psi(w, \lambda_0)| = |\psi(w - w_0, 0)| \geqslant C|w - w_0|^\theta$$

for $w \in W$ and $\lambda \geqslant 0$. Thus for $|w|$ sufficiently large (A) holds. •

(B) For every real $\lambda \neq 0$

$$\pi^\lambda(P) = a_\lambda^w(x, D) + T_\lambda,$$

where $a_\lambda \in S(m^\theta)$ and T_λ is a bounded operator on $L^2(V)$. Moreover, a_λ satisfies (1.18), (1.19).

Let $\varphi \in \mathscr{D}(G)$ be a cut-off function. Set $S = \varphi P$. By Lemma 2.4 $\mu = P - S$ is a bounded measure, and so $T_\lambda = \pi^\lambda(\mu)$ is a bounded operator on $L^2(V)$ for $\lambda \in \boldsymbol{R} \setminus \{0\}$. Now, look at the symbol of $\pi^\lambda(S)$. Since S is compactly supported, the symbol is by Prop. 5.13 smooth and equal to

$$a_\lambda(x, \xi) = \hat{\varphi} * \hat{P}(\operatorname{sgn} \lambda\, \delta^*_{|\lambda|^{1/a}} x, \delta^*_{|\lambda|^{1/a}} \xi, \lambda) \tag{5.17}$$

for $x \in V$, $\xi \in V^*$. Since $\hat{P}$ is homogeneous of degree θ with respect to the dilations $\{\delta_r^*\}$, it satisfies

$$|\hat{P}(w, \lambda)| \leqslant Cm_\lambda^\theta(w)$$

for $w \in W$, $\lambda \in \boldsymbol{R} \setminus \{0\}$. Hence for any $f \in \mathscr{S}(W \times \boldsymbol{R})$

$$|f * \hat{P}(w, \lambda)| \leqslant C' m_\lambda^\theta(w) \leqslant C'' m^\theta(w)$$

which implies $a_\lambda \in S(m^\theta)$. The Fourier transform of the measure $\mu = P - S$ is a continuous and bounded function. Therefore also $\hat{S}$ satisfies (A) and by homogeneity and invariance of $|\cdot|$ under reflections we obtain

$$|a_\lambda(w)| \geqslant C|w|^\theta$$

for $|w|$ sufficiently large and some $C > 0$ which proves (1.18). At last, Prop. 4.3, Prop. 4.10 and (5.17) show that a_λ satisfies (1.19).

(C) For every $\lambda \neq 0$ $\pi^\lambda(P)$ is essentially self-adjoint, its spectrum is discrete and

$$\sum_{z \in \operatorname{Sp} \pi^\lambda(P)} |z|^{-p} < \infty \tag{5.18}$$

for $p > nQ/(n+1)\theta = na/\theta$. The domain of $\overline{\pi^\lambda(P)}$ is equal to $H(m)$.

Fix $\lambda \neq 0$. Denote by R the resolvent operator for $\pi^\lambda(P)$ at $z = 1$. By (B) and Prop. 1.25 (i) $\overline{\pi^\lambda(P)}$ is self-adjoint if only $1/m^\theta \in L^p(W)$ for some $1 \leqslant p \leqslant \infty$. This is the case for $p > nQ/(n+1)\theta$. Moreover, by Prop. 1.25 (iii), $R \in C_p(L^2(V))$ for such p. Hence the spectrum of $\pi^\lambda(P)$ is discrete and by (1.20)

$$\sum_{0 \neq z \in \operatorname{Sp} \pi^\lambda(P)} |z|^{-p} \leqslant C(|R|_p)^p < \infty$$

for $p > nQ/(n+1)\theta$. But since $\overline{\pi^\lambda(P)}$ is injective, $0 \notin \operatorname{Sp} \pi^\lambda(P)$ and so (5.18) holds. The last assertion follows from (B) and Proposition 1.25 (ii).

Denote by z_j^λ the eigenvalues of $\overline{\pi^\lambda(P)}$. Due to Prop. 5.12 we have

$$z_j^\lambda = |\lambda|^{\theta/a} z_j^{\operatorname{sgn}\lambda}; \quad \lambda \neq 0,\ j = 1, 2, \ldots \tag{5.19}$$

By our assumption on P, z_j^λ are real strictly negative numbers.

(D) For every $t > 0$ μ_t is absolutely continuous, its density f_t is square-integrable and

$$\|f_t\|_{L^2(G)}^2 = Ct^{-Q/\theta} \sum_{j=1}^\infty (|z_j^1|^{-Q/\theta} + |z_j^{-1}|^{-Q/\theta}),$$

where

$$C = 2^{-Q/\theta}\frac{Q}{(n+1)\theta}\Gamma(Q/\theta).$$

In fact, $\overline{\pi^\lambda(P)}$ is the infinitesimal generator of the strongly continuous semi-group of contractions $\{\pi^\lambda(\mu_t)\}$ on $L^2(V)$ for every $\lambda \neq 0$. Thus, by (C), $\pi^\lambda(\mu_t)$ is a Hilbert-Schmidt operator ($t > 0$, $\lambda \neq 0$) and

$$\|\pi^\lambda(\mu_t)\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty} e^{2tz_j^\lambda}.$$

Hence, by the Plancherel theorem for the Heisenberg group, it is sufficient to show that the integral

(5.20) $$\int_{\mathbf{R}\setminus\{0\}} \|\pi^\lambda(\mu_t)\|_{\mathrm{HS}}^2 |\lambda|^n \, d\lambda$$

is convergent. Then its value will give a square of L^2-norm of f_t. We have

$$\begin{aligned}\int_{-\infty}^{\infty} \|\pi^\lambda(\mu_t)\|_{\mathrm{HS}}^2 |\lambda|^n \, d\lambda &= \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} e^{2tz_j^\lambda} |\lambda|^n \, d\lambda \\ &= \int_{-\infty}^{\infty} \sum_{j=1}^{\infty} e^{2t|\lambda|^{\theta/a} z_j^{\operatorname{sgn}\lambda}} |\lambda|^n \, d\lambda \\ &= \sum_{j=1}^{\infty} \Big(\int_0^{\infty} e^{2t\lambda^{\theta/a} z_j^1} \lambda^n \, d\lambda + \int_0^{\infty} e^{2t\lambda^{\theta/a} z_j^{-1}} \lambda^n \, d\lambda\Big).\end{aligned}$$

After the change of variables $v = 2t\lambda^{\theta/a} z_j^{\pm 1}$, (5.20) is equal to

$$\begin{aligned}&\frac{a}{\theta}(2t)^{-Q/\theta} \sum_{j=1}^{\infty} \big(|z_j^1|^{-Q/\theta} + |z_j^{-1}|^{-Q/\theta}\big) \int_0^{\infty} v^{Q/\theta - 1} e^{-v} \, dv \\ &\qquad = \frac{1}{n+1}\frac{Q}{\theta}\Gamma(Q/\theta)(2t)^{-Q/\theta} \sum_{j=1}^{\infty} \big(|z_j^1|^{-Q/\theta} + |z_j^{-1}|^{-Q/\theta}\big).\end{aligned}$$

By (C) the series on the right-hand side is convergent and so (D) is proved. This also ends the proof of the implication (i) ⇒ (ii). The remaining implications either are trivial or were proved under much more general assumptions in Section 2 (Lemma 2.23). Therefore the proof of the theorem is complete.

Remark 5.21. The proof of (C) shows that assumption (i) can be replaced by the following, *a priori* weaker, conjunction:

(5.22) $$\hat{P}(w, 0) \neq 0 \quad \text{for} \quad 0 \neq w \in W$$

and

(5.23) $\overline{\pi^\lambda(P)}$ restricted to $H(m^\theta)$ is injective for $\lambda = \pm 1$.

In the following corollaries we assume that a stable semi-group of measures $\mu_t = f_t\,dg$ with a characteristic exponent θ and its infinitesimal generator P satisfy the hypothesis of Theorem 5.16 together with one of the equivalent conditions (i)–(iv).

Let Q be a dissipative distribution on G. In the sequel the operator of left convolution with Q will be denoted also by Q i.e., we shall write $Q * f = Qf$ for functions f on G.

COROLLARY 5.24. *Each f_t, $t > 0$, is an analytic vector for $\bar{P}$ acting on $L^2(V)$.*

Proof. It follows immediately from Theorem 5.16 and Proposition 2.25.

COROLLARY 5.25. *Let $Q \in \mathscr{D}^*(G)$ be dissipative and homogeneous of degree θ. Then there exists a constant C such that*

(5.26) $$\|Qf\|_{L^2(G)} \leqslant C\|Pf\|_{L^2(G)}$$

for $f \in \mathscr{D}(G)$.

Proof. By the Plancherel Theorem and homogeneity of P and Q (cf. Prop. 5.12), it is sufficient to show that

$$\|\pi(Qf)\|_{\mathrm{HS}} \leqslant C\|\pi(Pf)\|$$

for $f \in \mathscr{D}(G)$ and some constant C, where the unitary representation π of G is equal either to π^1 or to π^{-1}. This, in turn, is implied by the inequality

(5.27) $$\|\pi(Q)u\|_{L^2(V)} \leqslant C\|\pi(P)u\|_{L^2(V)}$$

for $u \in \mathscr{D}(V) \subseteq L^2(V)$. We shall show that the operator $\overline{\pi(Q)}\;\overline{\pi(P)}^{-1}$ (cf. (C)) is bounded on $L^2(V)$ which, of course, will give (5.27).

Let φ be a cut-off function on G. Then by (B)

$$\pi(P) = \pi(\varphi P) + T,$$

where $\pi(\varphi P) \in \mathscr{L}(m^\theta)$, $T \in \mathscr{L}(L^2(V))$. Again by (B) and Cor. 1.17 there exists $B \in \mathscr{L}(m^{-\theta})$ such that

$$B\pi(\varphi P) = I + S,$$

where $S \in \mathscr{L}(m^{-\theta})$. Therefore

$$B = B\overline{\pi(P)}\;\overline{\pi(P)}^{-1} = \overline{\pi(P)}^{-1} + S\overline{\pi(P)}^{-1} + BT\overline{\pi(P)}^{-1}$$

whence

$$\overline{\pi(P)}^{-1} = ST_1 + BT_2,$$

where B, $S \in \mathscr{L}(m^{-\theta})$ and T_1, T_2 are bounded on $L^2(V)$. This together with Theorem 1.13 implies $\overline{\pi(Q)}\;\overline{\pi(P)}^{-1}$ is bounded and thus the corollary is proved.

The estimate (5.26) implies that the densities f_t, $t > 0$, are smooth to a certain degree. In fact, let $\{X_j\}_1^{2n+1}$ be a basis of the Lie algebra $\mathfrak{G}$ such that (5.6) holds and, moreover, $X_j * f = D_j f$, $f * X_{n+j} = D_{n+j} f$, $X_{2n+1} * f = D_{2n+1}$, $1 \leqslant j \leqslant n$, where D_k stands for the usual partial derivative. Then (5.26) together with Cor.5.24 imply

$$|D_k|^{\theta/\alpha_k} f_t \in L^2(G), \quad t > 0, \ 1 \leqslant k \leqslant 2n+1,$$

where $|D_k|^\alpha = |D_k^2|^{\alpha/2}$ (see (2.18)). In other words, if $|\cdot|$ denotes a homogeneous norm on $\mathfrak{G}^*$, cf. (3.7), then the functions

$$\mathfrak{G}^* \ni \xi \mapsto |\xi|^\theta \hat{f}_t(\xi), \quad t > 0,$$

are square-integrable on $\mathfrak{G}^*$ and therefore f_t belong to the Sobolev space H^ε with $\varepsilon = \theta/a$.

We do not know whether it is generally true that e.g. $f_t \in C^1(G)$.

COROLLARY 5.28. *Let $Q \in \mathcal{D}^*(G)$ be dissipative, symmetric and homogeneous of degree θ. Then the semi-group of measures generated by $P+Q$ consists of absolutely continuous measures.*

Proof. According to Remark 5.21 it is sufficient to show that $P+Q$ satisfies (5.22) and (5.23). By (3.19) we have

$$(P+Q)^\wedge(w, 0) = \hat{P}(w, 0) + \hat{Q}(w, 0) \leqslant \hat{P}(w, 0) < 0$$

for $w \neq 0$ and so (5.22) is satisfied. Now, let $\pi = \pi^{\pm 1}$ and suppose there is a function $u \in H(m^\theta)$ such that $\overline{\pi(P+Q)}\, u = 0$. Then by Lemma 2.4, (2.7) and Cor.1.24

$$\langle \overline{\pi(P)}\, u, u\rangle + \langle \overline{\pi(Q)}\, u, u\rangle = \langle \overline{\pi(P+Q)}\, u, u\rangle = 0.$$

By (2.11) this implies $\langle \overline{\pi(P)}\, u, u\rangle = 0$ and since $\overline{\pi(P)}$ is self-adjoint and negative definite, we get $\pi(P)u = 0$ and consequently $u = 0$ which proves (5.23) for $P+Q$.

The next corollary shows how to produce a good deal of semi-groups satisfying the conditions of Theorem 5.16. It gives a partial answer to Problem 1 of [13]. It also improves significantly the result of [11], Theorem 4.2.

COROLLARY 5.29. *Let $\{\mu_t\} \in (S)$ be a stable semi-group of measures on G. Let P be its infinitesimal generator. Then μ_t $(t > 0)$ are absolutely continuous with square-integrable densities if and only if $\mathfrak{G}_P = \mathfrak{G}$ (cf. (2.21)).*

Proof. This is an immediate consequence of Proposition 2.22 and Theorem 5.16.

We end with two simple examples. Let $\{X_j\}_1^{2n+1}$ be a basis for the Lie algebra satisfying (5.5). Set

$$P = -\sum_{j=1}^{2n} |X_j|^{\alpha_j}, \tag{5.30}$$

where $0 < \alpha_j \leqslant 2$ and $1/\alpha_j + 1/\alpha_{n+j} = 1/\alpha_{2n+1}$ for $1 \leqslant j \leqslant n$. Let $\theta = \max_{1 \leqslant j \leqslant 2n} (\alpha_j)$. Then P is homogeneous of degree $-Q-\theta$ with respect to the dilations

$$\delta_r X_j = r^{\theta/\alpha_j} X_j, \quad 1 \leqslant j \leqslant 2n+1.$$

It is clear that $P \in (\mathcal{S})$ and $\mathfrak{G}_P = \mathfrak{G}$.

Another example is that considered in [11], Theorem 4.2. Set

$$P = \Big(\sum_{j=1}^{n} X_j^2\Big)^{(\alpha)} + \Big(\sum_{j=1}^{n} X_{n+j}^2\Big)^{(\beta)}, \tag{5.31}$$

where $0 < \alpha, \beta \leqslant 1$. Let $\theta = \max(\alpha, \beta)$ and define the dilations in $\mathfrak{G}$ by

$$\delta_r X_i = r^{\theta/2\alpha} X_i, \quad \delta_r X_{n+i} = r^{\theta/2\beta} X_{n+i}, \quad \delta_r X_{2n+1} = r^{\theta(1/2\alpha + 1/2\beta)} X_{2n+1}$$

for $1 \leqslant i \leqslant n$. Then P is homogeneous of degree $-Q-\theta$, $P \in (\mathcal{S})$ and $\mathfrak{G}_P = \mathfrak{G}$.

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On the order-topological properties of the quotient space L/L_A

by

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Dedicated to Professor Władysław Orlicz on the occasion of his 80th birthday

Abstract. The first part contains some theorems about the order-topological properties of the quotient space of a σ-Dedekind complete and intervally complete locally solid Riesz space (L, τ) by the largest ideal L_A such that $\tau|L_A$ is a Lebesgue topology. These theorems are a generalization of some Lozanovskii's results from [7] and our proofs are slight modifications of Lozanovskii's methods. In the second part it is presented a very simple proof of the fact that L^ψ/L^ψ_A is an abstract M-space (L^ψ denotes a Musielak–Orlicz space and L^ψ_A its subspace of elements with absolutely continuous norm). A broad class of Orlicz spaces L^φ whose quotients L^φ/L^φ_A have no weak units is also indicated.

Let (L, τ) always denote a *Hausdorff locally solid Riesz space*. As concerns the terminology of Riesz spaces (= vector lattices) and locally solid Riesz spaces, we refer to [1]. Moreover, for $x \in L$, let $C(x)$ be the set of components of $|x|$, i.e.,

$$C(x) = \{p \in L:\ p \wedge (|x| - p) = 0\}.$$

The projection onto the band generated by an element $x \in L$ will be denoted by P_x.

1. General case. The theorems presented below were formulated, for Banach lattices, by G. Ja. Lozanovskii in [7]. It appears that Lozanovskii's results remain true also for intervally complete (L, τ) with L being σ-Dedekind complete. Lozanovskii uses in his proofs some facts which are interesting in themselves and which are not proved in [7]. We separate these facts and give their complete proofs under essentially weaker assumptions (Lemmas 1, 3 and 6). The main parts of proofs of our more general theorems are practically the same as Lozanovskii's proofs, but for convenience of the reader we indicate them.

Distinguish the largest ideal L_A in (L, τ) such that $\tau|L_A$ is a Lebesgue topology, i.e.,

$$L_A = \{x \in L:\ |x| \geqslant x_\alpha \downarrow 0 \text{ implies } x_\alpha \xrightarrow{\tau} 0\}.$$