contained in a paper of Mocanu [7]. Some further corollaries which belong to this context and which can be proved by an application of the above results can be found in [1] and [5].

Via completion, the theorem and all the subsequent corollaries are also valid for normed complex algebras.

References

[1] B. Aupetit, *Propriétés spectrales des algèbres de Banach*, Springer-Verlag, Berlin–Heidelberg–New York 1979 (Lecture Notes in Mathematics, vol. 735).

[2] J. W. Baker, J. S. Pym, *A remark on continuous bilinear mappings*, Proc. Edinburgh Math. Soc. (2) 17 (1971), 245–248.

[3] F. F. Bonsall, J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London Math. Soc. Lecture Note Series 2, Cambridge 1971.

[4] H. Boyadžiev, *Commutativity in Banach algebras and elements with real spectra*, C. R. Acad. Bulgare Sci. 29 (1976), 1401–1403.

[5] R. A. Hirschfeld, W. Żelazko, *On spectral norm algebras*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 (1968), 195–199.

[6] Le Page, *Sur quelques conditions entraînant la commutativité dans les algèbres de Banach*, C. R. Acad. Sci. Paris, Sér. A-B 265 (1967), 235–237.

[7] G. Mocanu, *A remark on a commutativity criterion in Banach algebra*, Stud. Cerc. Mat. 21 (1969), 947–952.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT
Einsteinstrasse 64, D-4400 Münster, FRG

Received February 28, 1983 (1873)

STUDIA MATHEMATICA, T. LXXIX. (1984)

Multivariate spline functions, *B*-spline bases and polynomial interpolations II*

by

HAKOP HAKOPIAN (Yerevan)

Abstract. In this paper a new notion for the spline functions of several variables is introduced and two constructions of *B*-spline bases for multivariate spline function space are given. We construct a multivariate analogue of Hermite interpolation, which in [2] and [3] was constructed only for $k = 2$. At the end another natural multivariate analogue of the Lagrange–Hermite type interpolation is constructed.

1. Introduction. We begin this paper by giving a natural definition of multivariate spline functions in the case where the knot sequence $\{x^0, \ldots, x^r\} \subset R^k$ is in general position, that is, every subset of $k+1$ points forms a proper simplex (this in one dimension corresponds to the case of distinct knots). This motivates our definition of a multivariate spline function in the general case.

We construct bases for the linear space $S^k_{m,\{x^0,\ldots,x^r\}}$ (of all k-variate splines with a knot sequence $\{x^0, \ldots, x^r\} \subset R^k$ of order m) consisting of *B*-splines with $m+k$ knots from $\{x^0, \ldots, x^r\}$. The first direct construction works only in the case of some restrictions on knot configuration. Instead of this the second one is inductive, works in the general case and seems to be more flexible.

Then we present Hermite's interpolation multivariate analogue in the general case, which in [3] was considered only for $k = 2$.

Finally we give another generalization of Lagrange and Hermite interpolations to the multivariate case, which preserves their pointwise nature.

2. On multivariate spline functions.

Definition. Let $x^0, \ldots, x^r \in R^k$ be in general position, that is, let every subset of $k+1$ points form a proper simplex. A *k-variate spline* with a *knot sequence* $\{x^0, \ldots, x^r\}$ *of order* m ($m \geqslant 2$) or *of degree* $m-1$ is a function of the

---

* This paper is part of my doctoral thesis prepared under the supervision of prof. Z. Ciesielski and submitted at the Gdańsk University during 1981.

class $C^{m-2}(R^k)$ with support in the convex hull of $\{x^0, \ldots, x^r\}$ which reduces to a polynomial of total degree not exceeding $m-1$ in each region bounded but not intersected by the convex hull of k points from $\{x^0, \ldots, x^r\}$.

Let us denote by $S^k_{m,\{x^0,\ldots,x^r\}}$ the linear space of all k-variate splines with knot sequence $\{x^0, \ldots, x^r\}$ of order m. The following theorem gives the first construction of a B-spline basis for $S^k_{m,\{x^0,\ldots,x^r\}}$:

THEOREM 1. *Let $x^0, \ldots, x^r \in R^k$ be in general position, $m \geq 2$ and let $0 \leq i_0, \ldots, i_{m-1} \leq r$ be distinct. Then*

$$\{M(x|x^{i_0}, \ldots, x^{i_{m-1}}, x^{j_0}, \ldots, x^{j_{k-1}})| \quad 0 \leq j_0, \ldots, j_{k-1} \leq r$$
$$\text{are distinct and } \{j_0, \ldots, j_{k-1}\} \cap \{i_0, \ldots, i_{m-1}\} = \emptyset\}$$

is a basis for $S^k_{m,\{x^0,\ldots,x^r\}}$.

COROLLARY. $\dim S^k_{m,\{x^0,\ldots,x^r\}} = \binom{r-m+1}{k}$ *if $x^0, \ldots, x^r$ are in general position.*

First we shall prove the following lemmas. Lemma 1 is due to Micchelli [5]. Here we give another proof.

LEMMA 1. *Let $x^0, \ldots, x^r \in R^k$, $\mathrm{vol}_k[x^{l_0}, \ldots, x^{l_n}] \neq 0$, $\mathrm{vol}_k[\{x^0, \ldots, x^r\} \backslash \{x^{l_i}\}] \neq 0$ for $i = 0, \ldots, n$, where $[A]$ is the convex hull of A. Then*

$$x^l = \sum_{i=0}^{n} \lambda_i x^{l_i}, \qquad \sum_{i=0}^{n} \lambda_i = 1$$

force

$$(1) \qquad M(x|\{x^0, \ldots, x^r\} \backslash \{x^l\}) = \sum_{i=0}^{n} \lambda_i M(x|\{x^0, \ldots, x^r\} \backslash \{x^{l_i}\}).$$

Proof. Let

$$x = \sum_{i=0}^{n} \mu_i x^{l_i} + \mu x^l \quad \text{with} \quad \sum_{i=0}^{n} \mu_i + \mu = 1, \qquad \mu \neq 0.$$

Then by Micchelli's recurrence relation (see Th. 4 of [5])

$$(2) \quad M(x|x^0, \ldots, x^r) = \sum_{i=0}^{n} \mu_i M(x|\{x^0, \ldots, x^r\} \backslash \{x^{l_i}\}) + \mu M(x|\{x^0, \ldots, x^r\} \backslash \{x^l\}).$$

But we have also $x = \sum_{i=0}^{n} (\mu_i + \mu\lambda_i) x^{l_i}$ and of course $\sum_{i=0}^{n} (\mu_i + \mu\lambda_i) = 1$, hence

$$(3) \qquad M(x|x^0, \ldots, x^r) = \sum_{i=0}^{n} (\mu_i + \mu\lambda_i) M(x|\{x^0, \ldots, x^r\} \backslash \{x^{l_i}\}).$$

Now (2) and (3) readily give (1).

LEMMA 2. *A function on R^k which is zero on one side of a $k-1$ dimensional hyperplane L and is a polynomial of total degree not exceeding $m-1 \geq 1$ on the other side, belongs to $C^{(m-2)}(R^k)$ iff it is of the form $c\varrho_+^{m-1}(x, L)$.*

Here $\varrho_+(x, L)$ denotes the distance from $x \in R^k$ to L if x is on the $(+)$ side of L ($(+)$ is the side where f is not zero) and zero otherwise.

Proof. Consider a perpendicular line (l) to L. Of course the function on this line is of the form $c_l \varrho_+^{m-1}(x, L)$. And since on the $(+)$ side it is a polynomial of total degree not exceeding $m-1$, c_l is independent of l.

LEMMA 3. *Let $x^0, \ldots, x^r \in R^k$ be in general position. Then*

$$S^k_{r-k+1,\{x^0,\ldots,x^r\}} = \{cM(x|x^0, \ldots, x^r)| \; c \in R\}.$$

Proof. We shall prove this by induction on k. The case $k = 1$ is familiar. Assume that the lemma is true for $k-1$. Let $x^r \notin [x^0, \ldots, x^{r-1}]$.

Also, let L be a $k-1$ dimensional hyperplane which intersects $[x^r, x^i]$ at $x'^i \neq x^0$ for $i = 0, \ldots, r-1$. Then the inductive assumption shows that every $f \in S^k_{r-k+1,\{x^0,\ldots,x^r\}}$ on the hyperplane L reduces to the form of $c \cdot M(x'|x'^0, \ldots, x'^{r-1})$, $x' \in L$.

On the other hand, on each line l passing through x^0 and $x' \in L$, for all $x \in [x^r, x^i]$ f reduces to the form of $c \cdot \varrho^{m-1}(x, x^r)$, where $\varrho(x, x^r)$ is the distance between x and x^r. Therefore, if $f_1, f_2 \in S^k_{r-k+1,\{x^0,\ldots,x^r\}}$ then $f_1 = \lambda f_2$ in some neighbourhood of x^r and with the help of Lemma 2 we obtain $f_1 - \lambda f_2 \in S^k_{r-k+1,\{x^0,\ldots,x^{r-1}\}}$. Proceeding in this way we obtain $f_1 - \lambda f_2 \in S^k_{r-k+1,\{x^0,\ldots,x^{k+1}\}}$ and again Lemma 2 shows $f_1 = \lambda f_2$. Now it remains to note that

$$M(x|x^0, \ldots, x^r) \in S^k_{r-k+1,\{x^0,\ldots,x^r\}}.$$

Proof of Theorem 1. We divide the proof into three steps.

Step one will show that every B-spline of order m, $M(x|x^{l_0}, \ldots, x^{l_{m+k-1}})$, where $0 \leq l_0, \ldots, l_{m+k-1} \leq r$ are distinct, is in the span of the "basis" (just until the end of the proof).

Let the knot sequence $\{x^{l_0}, \ldots, x^{l_{m+k-1}}\}$ have $\alpha < m$ common points with $\{x^{i_0}, \ldots, x^{i_{m-1}}\}$. Then there are $k+1$ points in $\{x^{l_0}, \ldots, x^{l_{m+k-1}}\} \backslash \{x^{i_0}, \ldots, x^{i_{m-1}}\}$ and Lemma 1 shows that $M(x|x^{l_0}, \ldots, x^{l_{m+k-1}})$ is a linear combination of B-splines of order m with a distinct knot sequence having $\alpha+1$ points from $\{x^{i_0}, \ldots, x^{i_{m-1}}\}$. Hence it is a linear combination of B-splines having $m+k$ distinct knots from $\{x^0, \ldots, x^r\}$ and including $x^{i_0}, \ldots, x^{i_{m-1}}$, that is, $M(x|x^{l_0}, \ldots, x^{l_{m+k-1}})$ is in the span of the "basis".

Step two will show that every spline f from $S^k_{m,\{x^0,\ldots,x^r\}}$ is in the span of all B-splines having $m+k$ distinct points from $\{x^0, \ldots, x^r\}$. Let x^r

$\notin[x^0, \ldots, x^{r-1}]$ and let $[x^r, x^{i_1}, \ldots, x^{i_{k-1}}]$ be an internal side of $[x^0, \ldots, x^r]$. Then Lemmas 2 and 3 show that there is a number λ such that $f(x)-\lambda M(x|x^r, x^{i_1}, \ldots, x^{i_{k-1}}, x^{j_0}, \ldots, x^{j_{m-1}})$ (where M is a B-spline with distinct knots) is zero in every region bounded but not intersected by the convex hull of k points from $\{x^0, \ldots, x^r\}$ and neighbouring on the side $[x^r, x^{i_1}, \ldots, x^{i_{k-1}}]$. Proceeding in this way we find a linear combination

$$f_r = \sum_{\{l\}} \lambda_{\{l\}} M(x|x^r, x^{l_1}, \ldots, x^{l_{m+k-1}})$$

such that $f-f_r \in S^k_{m,\{x^0,\ldots,x^{r-1}\}}$. Hence we have $f-f_r-f_{r-1}- \ldots -f_{k+m} \in S^k_{m,\{x^0,\ldots,x^{k+m-1}\}}$. Now it remains to apply once more Lemma 3.

In step three we shall prove the linear independence of the "basic" functions. If we apply the recurrence relation (see (6) in [4])

$$D_{x^i-x} M(x|x^0, \ldots, x^r)+(r-k) M(x|x^0, \ldots, x^r) = r\, M(x|x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^r), \tag{4}$$

then it is sufficient to show only the linear independence of the functions

$$\{M(x|x^{i_0}, x^{j_0}, \ldots, x^{j_{k-1}})|\ 0 \leqslant j_0, \ldots, j_{k-1} \leqslant r \text{ are distinct and } \{j_0, \ldots, j_{k-1}\}\cap\{i_0, \ldots, i_{m-1}\} = \emptyset\}.$$

Of course the above holds since if

$$f = \sum_{\{j\}} \lambda_{\{j\}} M(x|x^{i_0}, x^{j_0}, \ldots, x^{j_{k-1}}) \quad \text{and} \quad \lambda_{\{j\}_0} = \lambda_{j_0^0, \ldots, j_{k-1}^0} \neq 0,$$

then f is discontinuous at the points of the side $[x^{j_0^0}, \ldots, x^{j_{k-1}^0}]$. Therefore $f \not\equiv 0$. This completes the proof of Theorem 1.

From the point of view of Theorem 1 it is natural to introduce the following

Definition. Let $x^0, \ldots, x^r \in R^k$ and $\mathrm{vol}_k[x^0, \ldots, x^r] \neq 0$. The space $S^k_{m,\{x^0,\ldots,x^r\}}$ of k-variate splines with a knot sequence $\{x^0, \ldots, x^r\}$ of order m is the linear span of the set

$$\{M(x|x^{i_0}, \ldots, x^{i_{m+k-1}})|\ 0 \leqslant i_0, \ldots, i_{m+k-1} \leqslant r \text{ are distinct, } \mathrm{vol}_k[x^{i_0}, \ldots, x^{i_{m+k-1}}] \neq 0\}.$$

The first construction of a B-spline basis can work here if there are $r-m+1$ points from $\{x^0, \ldots, x^r\}$ which are in general position. Otherwise we can use the following construction.

Let us denote a B-spline basis for $S^k_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^r\\ j_0,\ldots,j_r\end{smallmatrix}\right\}}$ by $B^k_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^r\\ j_0,\ldots,j_r\end{smallmatrix}\right\}}$. Here in the knot set $\left\{\begin{matrix}x^0, \ldots, x^r\\ j_0, \ldots, j_r\end{matrix}\right\}$, j_i is the multiplicity of x^i, $i = 0, \ldots, r$ and $\{x^0, \ldots, x^r\}$ are distinct knots.

We shall construct $B^k_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^r\\ j_0,\ldots,j_r\end{smallmatrix}\right\}}$ using induction on k. The basis $B^1_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^r\\ j_0,\ldots,j_r\end{smallmatrix}\right\}}$ is familiar.

(i) First we shall construct a basis for $S^k_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^r\\ j_0,\ldots,j_r\end{smallmatrix}\right\}}$. Let $x^l \notin [x^0, \ldots, x^{l-1}]$, $l = p, \ldots, r$, where p is determined by the conditions $\mathrm{vol}_{k-1}[x^0, \ldots, x^p] \neq 0$ and $\mathrm{vol}_{k-1}[x^0, \ldots, x^{p-1}] = 0$. Also let L_l be a $(k-1)$-dimensional hyperplane which intersects $[x^l, x^i]$ at $x_l^i \neq x^l$ for all $i = 0, \ldots, l-1$. Denote

$$B^{l,j}_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^l\\ j_0,\ldots,j_l\end{smallmatrix}\right\}} = \{M(x|\underbrace{x^l, \ldots, x^l}_{j}, x^{i_1}, \ldots, x^{i_q})|\ M(x|x_l^{i_1}, \ldots, x_l^{i_q}) \in B^{k-1}_{m-j,\left\{\begin{smallmatrix}x_l^0,\ldots,x_l^{l-1}\\ j_0,\ldots,j_{l-1}\end{smallmatrix}\right\}}$$
$$\text{and } x^i = \lambda x^l+(1-\lambda)x^{i_s},\ 0 < \lambda < 1,\ 1 \leqslant s \leqslant q \text{ forces } i \in \{i_1, \ldots, i_q\}\}.$$

Then

$$B^k_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^r\\ j_0,\ldots,j_r\end{smallmatrix}\right\}} = \bigcup_{l=p}^{r} \bigcup_{j=1}^{\min(j_l, m-1)} B^{l,j}_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^l\\ j_0,\ldots,j_l\end{smallmatrix}\right\}}. \tag{5}$$

That is, the left-hand side is a B-spline basis for $S^k_{m,\left\{\begin{smallmatrix}x^0,\ldots,x^r\\ j_0,\ldots,j_r\end{smallmatrix}\right\}}$.

The following lemma, which is crucial in the proof of (5), seems to be a useful recurrence relation.

Lemma 4. *Let* $y^0, \ldots, y^m, x^0, \ldots, x^r \in R^k$, $\mathrm{vol}_k[y^0, \ldots, y^m, x^{j_0^i}, \ldots, x^{j_n^i}] \neq 0$, $i = 0, \ldots, p$, $0 \leqslant j_0^i, \ldots, j_n^i \leqslant r$. *Then if* $\mathrm{vol}_k[x^{j_0^i}, \ldots, x^{j_n^i}] \neq 0$, $i = 0, \ldots, p$, *or if* $x^0, \ldots, x^r$ *belong to some l-dimensional hyperplane,* $l < k$ *and* $\mathrm{vol}_l[x^{j_0^i}, \ldots, x^{j_n^i}] \neq 0$,

$$\sum_{i=0}^{p} \lambda_i M(x|y^0, \ldots, y^m, x^{j_0^i}, \ldots, x^{j_n^i}) = 0$$

is equivalent to

$$\sum_{i=0}^{p} \lambda_i M(x|x^{j_0^i}, \ldots, x^{j_n^i}) = 0.$$

Proof. Without loss of generality we can prove only the case of $l = k-1$ and $m = 0$. Now, if $x^0, \ldots, x^r$ belong to some $(k-1)$-dimensional hyperplane L_{k-1}, we need only apply Theorem 5 from [4] and the equivalence is obvious. Otherwise we can assume that $x^0, \ldots, x^r$ belong to a k-dimensional hyperplane $L_k (=R^k)$ in R^{k+1}. If we now assume $y^0 \in R^{k+1} \backslash L_k$, the problem reduces to the above case.

The case $y_0 \in L_k$ can readily be obtained by a continuity argument. Now let us prove by relation (5) that $B^k_{m,\{x^0,\ldots,x^r \atop j_0,\ldots,j_r\}}$ is a basis. First we show the linear independence of the "basis". Since $x^l \notin [x^0, \ldots, x^{l-1}]$, $l = p, \ldots, r$, the linear independence of $B^k_{m,\{x^0,\ldots,x^r \atop j_0,\ldots,j_r\}}$ reduces to the linear independence of

$$\bigcup_{j=1}^{\min(j_l, m-1)} B^{l,j}_{m,\{x^0,\ldots,x^l \atop j_0,\ldots,j_l\}}, \quad l = p, \ldots, r.$$

This in turn reduces to the linear independence of $B^{l,j}_{m,\{x^0,\ldots,x^l \atop j_0,\ldots,j_l\}}$ because of the recurrence relation (4). Of course the above holds (Lemma 4).

Now take an m-order k-variate B-spline. We can write it in the form of $M(x|\underbrace{x^l, \ldots, x^l}_{j}, x^{i_1}, \ldots, x^{i_q})$, where $x^i \notin \{x^{i_1}, \ldots, x^{i_q}\}$, $i = l, \ldots, r$, $p \leqslant l \leqslant r$, $j < m$. We show now that

$$M(x|\underbrace{x^l, \ldots, x^l}_{j}, x^{i_1}, \ldots, x^{i_q}) \in B^{l,j}_{m,\{x^0,\ldots,x^l \atop j_0,\ldots,j_l\}} \cup S^k_{m,\{x^0,\ldots,x^{l-1} \atop j_0,\ldots,j_{l-1}\}}.$$

Lemma 4 allows us to assume that $j = 1$ and that

$$x^i = \lambda x^l + (1-\lambda) x^{i_s},\ 0 < \lambda < 1,\ 1 \leqslant s \leqslant q \quad \text{forces} \quad i \in \{i_1, \ldots, i_q\}.$$

This completes the proof since the restriction of $M(x|x^l, x^{i_1}, \ldots, x^{i_q})$ to the hyperplane L_l is $M(x|x_l^{i_1}, \ldots, x_l^{i_q})$.

The second construction of a B-spline basis is more flexible and preferable since it allows us to make the supports of basic functions "small".

(ii) If $x^0, \ldots, x^r$ are in general position, we have

$$B^k_{m,\{x^0,\ldots,x^r\}} = \sum_{l=k}^{r} B^{l,1}_{m,\{x^0,\ldots,x^l\}},$$

where

$$B^{l,1}_{m,\{x^0,\ldots,x^l\}} = \{M(x|x^l, x^{i_1}, \ldots, x^{i_q})|\ M(x|x_l^{i_1}, \ldots, x_l^{i_q}) \in B^{k-1}_{m-1,\{x_l^0,\ldots,x_l^{l-1}\}}\}.$$

2. Multivariate divided difference and interpolation.

DEFINITION. Let $x^0, \ldots, x^r \in R^k$, $\mathrm{vol}_k[x^0, \ldots, x^r] \neq 0$, $\alpha = (\alpha_1, \ldots, \alpha_k)$, $|\alpha| = \alpha_1 + \ldots + \alpha_k = r-k+1$ and let f be sufficiently smooth. Then the k-variate α-divided difference of the function f at $x^0, \ldots, x^r$ is

$$[x^0, \ldots, x^r]^\alpha f := \frac{1}{\alpha!} \int_{R^k} M(x|x^0, \ldots, x^r) \cdot D^\alpha f(x)\, dx,$$

where

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}, \quad \alpha! = \alpha_1! \ldots \alpha_k!$$

and

$$[x^0, \ldots, x^{k-1}]^\alpha f := (k-1)! \int_{[x^0,\ldots,x^k]} f := f\{x^0, \ldots, x^{k-1}\}.$$

Above we have denoted, as in [5]

$$\int_{[x^0,\ldots,x^r]} f := \int_{Q^r} f(v_0 x^0 + \ldots + v_r x^r)\, dv_1 \ldots dv_r,$$

where

$$Q^r = \{(v_1, \ldots, v_r)|\ \sum_{i=1}^{r} v_i \leqslant 1,\ v_j \geqslant 0,\ j = 1, \ldots, r\} \quad \text{and} \quad v_0 = 1 - \sum_{i=1}^{r} v_i.$$

If $\mathrm{vol}_k[x^0, \ldots, x^r] \neq 0$ we have (see [4], [5])

$$r! \int_{[x^0,\ldots,x^r]} f = \int_{R^k} f(x) M(x|x^0, \ldots, x^r)\, dx.$$

Taking $\mu = 0$ in (6) of [3], we obtain (see also [6]) Micchelli's recurrence relation

(6) $$\int_{[x^0,\ldots,x^r]} D_y f = -\sum_{i=0}^{r} \mu_i \int_{[x^0,\ldots,x^{i-1},x^{i+1},\ldots,x^r]} f$$

if

$$y = \sum_{i=0}^{r} \mu_i x^i, \quad \sum_{i=0}^{r} \mu_i = 0.$$

Now if we take $x^{l_0}, \ldots, x^{l_k} \in \{x^0, \ldots, x^r\}$ such that $\mathrm{vol}_k[x^{l_0}, \ldots, x^{l_k}] \neq 0$, then

$$y = \sum_{i=0}^{k} \mu_i x^{l_i} \quad \text{and} \quad \sum_{i=0}^{k} \mu_i = 0$$

force of course

(7) $$\mu_i = \frac{\det\begin{bmatrix} y & x^{l_0} & \ldots & x^{l_{i-1}} & x^{l_{i+1}} & \ldots & x^{l_k} \\ 0 & 1 & \ldots & 1 & 1 & \ldots & 1 \end{bmatrix}}{\det\begin{bmatrix} x^{l_i} & x^{l_0} & \ldots & x^{l_{i-1}} & x^{l_{i+1}} & \ldots & x^{l_k} \\ 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \end{bmatrix}},$$

where

$$\det\begin{bmatrix} x & y & \ldots & z \\ \alpha & \beta & \ldots & \gamma \end{bmatrix} = \begin{vmatrix} x_1 & y_1 & \ldots & z_1 \\ \vdots & \vdots & & \vdots \\ x_k & y_k & & z_k \\ \alpha & \beta & \ldots & \gamma \end{vmatrix}.$$

Hence, if μ_i is given by (7), then

$$\int_{[x^0,\ldots,x^r]} D_y f = -\sum_{i=0}^{k} \mu_i \int_{[x^0,\ldots,x^{l_i-1},x^{l_i+1},\ldots,x^r]} f. \tag{8}$$

Now we give a quick proof of Theorems 1 and 2 of [3].

THEOREM 2. *Let $x^0, \ldots, x^r \in R^k$ be in general position, $\alpha = (\alpha_1, \ldots, \alpha_k)$, $|\alpha| = r-k+1$. Then*

(i) $[x^0, \ldots, x^r]^\alpha f$

$$= (-1)^{r-k+1} \frac{r!}{\alpha!\,(k-1)!} \sum_{0 \leqslant i_0,\ldots,i_{k-1} \leqslant r} c^\alpha_{i_0,\ldots,i_{k-1}} f\{x^{i_0}, \ldots, x^{i_{k-1}}\},$$

where

$$c^\alpha_{i_0,\ldots,i_{k-1}} = \frac{\prod_{l=1}^{k} \left[\det \begin{bmatrix} e_l & x^{i_0} & \ldots & x^{i_{k-1}} \\ 0 & 0 & \ldots & 1 \end{bmatrix}\right]^{\alpha_l}}{\prod_{l \neq i_0,\ldots,i_{k-1}} \det \begin{bmatrix} x^l & x^{i_0} & \ldots & x^{i_{k-1}} \\ 1 & 1 & \ldots & 1 \end{bmatrix}}. \tag{9}$$

(ii) *For any real numbers $\gamma_{i_0,\ldots,i_{k-1}}$ ($0 \leqslant i_0, \ldots, i_{k-1} \leqslant r$ being distinct) there exists a unique k-variate polynomial P of total degree not exceeding $r-k+1$ such that*

$$P\{x^{i_0}, \ldots, x^{i_{k-1}}\} = \gamma_{i_0,\ldots,i_{k-1}} \quad \textit{for all distinct } 0 \leqslant i_0, \ldots, i_{k-1} \leqslant r.$$

Proof. First we shall prove (ii). Denote by $\Pi_{r-k+1}(R^k)$ the class of k-variate polynomials of total degree $\leqslant r-k+1$. Since $\dim \Pi_{r-k+1}(R^k) = \binom{r+1}{k}$, it is enough to prove that

$$\gamma_{i_0,\ldots,i_{k-1}=0} = 0 \ \ \forall\, 0 \leqslant i_0, \ldots, i_{k-1} \leqslant r \quad \text{forces} \quad P \equiv 0.$$

The above holds. Indeed, from Micchelli's recurrence relation (6) it follows that $[x^0, \ldots, x^l]^\alpha P$ for all α, $|\alpha| = l-k+1$, and $k-1 \leqslant l \leqslant r$ is a linear combination of

$$\int_{[x^{i_0},\ldots,x^{i_{k-1}}]} P := P\{x^{i_0}, \ldots, x^{i_{k-1}}\} = \gamma_{i_0,\ldots,i_{k-1}}.$$

Now let us prove (i). As we mentioned above $[x^0, \ldots, x^r]^\alpha f$ is a linear combination of

$$f\{x^{i_0}, \ldots, x^{i_{k-1}}\}, \quad 1 \leqslant i_0, \ldots, i_{k-1} \leqslant r.$$

Let us find $c^\alpha_{i_0,\ldots,i_{k-1}}$. From (ii) there exists a polynomial

$P_{i_0,\ldots,i_{k-1}} \in \Pi_{r-k+1}(R^k)$ such that $P_{i_0,\ldots,i_{k-1}}\{x^{i_0}, \ldots, x^{i_{k-1}}\} = 1$ and $P_{i_0,\ldots,i_{k-1}}\{x^{j_0}, \ldots, x^{j_{k-1}}\} = 0$ if $\{j_0, \ldots, j_{k-1}\} \neq \{i_0, \ldots, i_{k-1}\}$. Then we have

$$[x^0, \ldots, x^r]^\alpha P_{i_0,\ldots,i_{k-1}} = \binom{r}{k-1} \cdot c^\alpha_{i_0,\ldots,i_{k-1}}.$$

Now if we choose for the recurrence relation in (8)

$x^{i_0}, \ldots, x^{i_{k-1}}$, x^i, $i \notin \{i_0, \ldots, i_{k-1}\}$ and $y = e_l = (0 \ldots \underset{l}{1} \ldots 0)$, $\alpha_l > 0$, we find

$$[x^0, \ldots, x^r]^\alpha P_{i_0,\ldots,i_{k-1}} = C \cdot [x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^r]^{\alpha-e_l} P_{i_0,\ldots,i_{k-1}}, \tag{10}$$

$$C = -\frac{r}{\alpha_l} \frac{\det \begin{bmatrix} e_l & x^{i_0} & \ldots & x^{i_{k-1}} \\ 0 & 1 & \ldots & 1 \end{bmatrix}}{\det \begin{bmatrix} x_i & x^{i_0} & \ldots & x^{i_{k-1}} \\ 1 & 1 & \ldots & 1 \end{bmatrix}}$$

since all the other summands are equal to zero. If we apply relation (10) $r-k+1$ times, we obtain (9). This completes the proof.

The above polynomial interpolation (when $x^0, \ldots, x^r \in R^k$ are in general position) in one dimension corresponds to interpolation of distinct knots, that is, to the Lagrange interpolation.

Now we are going to present the general case, that is, an analogue of the Hermite interpolation in higher dimension. Let $x^0, \ldots, x^r \in R^k$ and $\mathrm{vol}_k[x^0, \ldots, x^r] \neq 0$. Also let $\mathscr{L}_p = \{L^1_p, \ldots, L^{i_p}_p\}$ be the set of all p-dimensional hyperplanes, $1 \leqslant p \leqslant k-1$, which are spanned by $\geqslant k$ points from $\{x^0, \ldots, x^r\}$. Let us choose $y^{q,p}_{p+1}, \ldots, y^{q,p}_k \in R^k$ such that

$$\operatorname{span}\{L^q_p \cap \{x^0, \ldots, x^r\} \cup \{y^{q,p}_{p+1}, \ldots, y^{q,p}_k\}\} = R^k.$$

Remember also that $B^k_{m,\{x^0,\ldots,x^r\}}$ is a B-spline basis for $S^k_{m,\{x^0,\ldots,x^r\}}$.

Then we define for the set $\{x^0, \ldots, x^r\}$ and a sufficiently smooth function f the following interpolating parameters:

(i) corresponding to each $L^q_p \in \mathscr{L}_p$,

$$\int_{[x^{i_0},\ldots,x^{i_s}]} D^{\alpha_{p+1}}_{y_{p+1}} \ldots D^{\alpha_k}_{y_{p+1}} f,$$

where

$$M(x|x^{i_0}, \ldots, x^{i_s}) \in B^p_{\alpha, L^q_p \cap \{x^0,\ldots,x^r\}},$$

$$\alpha = \alpha_{p+1} + \ldots + \alpha_k \leqslant r-k+1,$$

$$d = \#\{L^q_p \cap \{x^0, \ldots, x^r\}\} - r + k + \alpha - 1 - p,$$

(ii) corresponding to each knot x^i with multiplicity $m_i \geqslant k$,

$$\frac{\partial^{\alpha_1+\ldots+\alpha_k}}{\partial x_1^{\alpha_1}\ldots\partial x_k^{\alpha_k}} f, \quad \alpha_1+\ldots+\alpha_k \leqslant m_i-k.$$

Now the recurrence relation (6) shows that $[x^0, \ldots, x^r]^\alpha f$ for all α, $|\alpha| = r-k+1$ is a linear combination of the interpolating parameters. It is not difficult to prove by induction on k that there are exactly $\binom{r+1}{k}$ parameters. Hence considerations similar to those in the proof of Theorem 2 (ii) provide:

To every set of $\binom{r+1}{k}$ numbers (values of interpolating parameters) there is exactly one polynomial $P \in \Pi_{r-k+1}(R^k)$ whose parameters coincide with those numbers.

3. Another multivariate analogue of Lagrange and Hermite interpolations. The interpolation presented below preserves the pointwise nature of Lagrange and Hermite interpolations.

Let $L_0, \ldots, L_r$ be $(k-1)$-dimensional hyperplanes in R^k such that, for every k, the hyperplanes $L_{i_0}, \ldots, L_{i_{k-1}}$, $0 \leqslant i_0, \ldots, i_{k-1} \leqslant r$, have exactly one common point $x_{i_0,\ldots,i_{k-1}}$. This means that if the equation

$$\lambda_1^{i_m} x_1 + \lambda_2^{i_m} x_2 + \ldots + \lambda_k^{i_m} x_k + \lambda_{k+1}^{i_m} = 0$$

determines L_{i_m}, $m = 0, \ldots, k-1$, then

$$\det \|\lambda_j^{i_m}\| \neq 0, \quad m = 0, \ldots, k-1, \; j = 1, \ldots, k.$$

Let the point $x_{i_0,\ldots,i_{k-1}}$ belong to $m_{i_0,\ldots,i_{k-1}}$ hyperplanes. Also denote by $\varrho(x, L_{i_0})$ the signed distance of x from L_{i_0}, of course

$$\varrho(x, L_{i_0}) = \frac{\lambda_1^{i_0} x_1 + \ldots + \lambda_k^{i_0} x_k + \lambda_{k+1}^{i_0}}{\sqrt{(\lambda_1^{i_0})^2 + \ldots + (\lambda_{k+1}^{i_0})^2}}.$$

Then we have

THEOREM 3. *If $L_0, \ldots, L_r$ are the given hyperplanes and $\{x_{i_0,\ldots,i_{k-1}} | (i_0, \ldots, i_{k-1}) \in J\}$ is the set of all distinct $x_{i_0,\ldots,i_{k-1}}$, $0 \leqslant i_0, \ldots, i_{k-1} \leqslant r$, then for arbitrary real number set*

$$I = \{\gamma_{i_0,\ldots,i_{k-1}}^{\alpha_1,\ldots,\alpha_k} |\ \alpha_1 + \ldots + \alpha_k \leqslant m_{i_0,\ldots,i_{k-1}} - k,\ (i_0, \ldots, i_{k-1}) \in J\}$$

there exists a unique polynomial $P \in \Pi_{r-k+1}(R^k)$, such that

$$\frac{\partial^{\alpha_1+\ldots+\alpha_k}}{\partial x_1^{\alpha_1}\ldots\partial x_k^{\alpha_k}} P(x_{i_0,\ldots,i_{k-1}}) = \gamma_{i_0,\ldots,i_{k-1}}^{\alpha_1,\ldots,\alpha_k} \quad \forall \gamma_{i_0,\ldots,i_{k-1}}^{\alpha_1,\ldots,\alpha_k} \in I.$$

Moreover, if $m_{i_0,\ldots,i_{k-1}} = k$ for all $0 \leqslant i_0, \ldots, i_{k-1} \leqslant r$ then

$$(11) \qquad P(x) = \sum_{0 \leqslant i_0,\ldots,i_{k-1} \leqslant r} \prod_{j \neq i_0,\ldots,i_{k-1}} \frac{\varrho(x, L_j)}{\varrho(x_{i_0,\ldots,i_{k-1}}, L_j)} \cdot P(x_{i_0,\ldots,i_{k-1}}).$$

Proof. Consider the polynomials

$$P_{j_0,\ldots,j_{k-1}}^{\beta_1,\ldots,\beta_k}(x) = \frac{(x - x_{j_0,\ldots,j_{k-1}})_1^{\beta_1} \ldots (x - x_{j_0,\ldots,j_{k-1}})_k^{\beta_k}}{\beta_1! \ldots \beta_k!} \prod_{\substack{n=0 \\ x_{j_0,\ldots,j_{k-1}} \notin L_n}}^{r} \frac{\varrho(x, L_n)}{\varrho(x_{j_0,\ldots,j_{k-1}}, L_n)},$$

where $(j_0, \ldots, j_{k-1}) \in J$, $\beta_1 + \ldots + \beta_k \leqslant m_{j_0,\ldots,j_{k-1}} - k$. They have the following properties:

$$P_{j_0,\ldots,j_{k-1}}^{\beta_1,\ldots,\beta_k} \in \Pi_{r-k+1}(R^k)$$

and

$$\frac{\partial^{\alpha_1+\ldots+\alpha_k}}{\partial x_1^{\alpha_1}\ldots\partial x_k^{\alpha_k}} P_{j_0,\ldots,j_{k-1}}^{\beta_1,\ldots,\beta_k}(x_{i_0,\ldots,i_{k-1}}) = \begin{cases} 1 & \text{if } (i_0, \ldots, i_{k-1}) = (j_0, \ldots, j_{k-1}),\ (\alpha_1, \ldots, \alpha_k) = (\beta_1, \ldots, \beta_k), \\ 0 & \text{if } (i_0, \ldots, i_{k-1}) = (j_0, \ldots, j_{k-1}),\ (\alpha_1, \ldots, \alpha_k) \neq (\beta_1, \ldots, \beta_k), \\ & \qquad \alpha_1 + \ldots + \alpha_k \leqslant \beta_1 + \ldots + \beta_k \\ & \text{or if } (i_0, \ldots, i_{k-1}) \in J,\ (i_0, \ldots, i_{k-1}) \neq (j_0, \ldots, j_{k-1}), \\ & \qquad \alpha_1 + \ldots + \alpha_k \leqslant m_{i_0,\ldots,i_{k-1}} - k. \end{cases}$$

Of course this gives us a way of constructing $P(x)$. On the other hand, $\dim \Pi_{r-k+1}(R^k) = \# I$, and this completes the proof of the first part. Formula (11) can be readily checked directly.

Acknowledgement. I am grateful to my adviser Z. Ciesielski for his very helpful comments and suggestions.

INSTITUTES OF MATHEMATICS
Gdańsk University — Yerevan University
Poland — Armenian SSR, USSR

References

[1] C. de Boor, *Splines as linear combinations of B-splines*, in: *Approximation Theory II*, Edited by G G. Lorentz, C. K. Chui, L. L. Schumaker, Academic Press, 1976, 1–47.
[2] H. Hakopian, *Les différences divisées de plusieurs variables et les interpolations multidimensionnelles de types lagrangien et hermitien*, C. R. Acad. Sci. Paris Sér. A 292 (1981), 453–456.
[3] — *Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type*, J. Approximation Theory 34 (1982), 286–305.

[4] — *Multivariate spline functions, B-spline basis and polynomial interpolations*, SIAM J. Numer. Anal. 19 (1982), 510–517.

[5] C. A. Micchelli, *On a numerically efficient method for computing multivariate B-spline*, Research Report, IBM. T. J. Watson Research Center, Yorktown Heights, New York 1979.

[6] — *A constructive approach to Kergin interpolation in R^k:multivariate B-splines and Lagrange interpolation*, Rocky Mountain J. Math. 10, 3 (1980), 485–497.

Received October 26, 1982
Revised version April 26, 1983 (1827)

STUDIA MATHEMATICA, T. LXXIX. (1984)

Correction à "Représentation fonctionnelle des espaces vectoriels topologiques"

STUDIA MATH. 73 (1982), 1–10

par

PHILIPPE TURPIN (Orsay)

Une faute d'impression dans les définitions préliminaires rend imprécises les conditions de validité du résultat principal.

Il faut donc lire, page 2 ligne 9, $u(ax) \leqslant u(x)$ au lieu de $u(ax) \leqslant au(x)$.

Received February 7, 1983