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A note on criteria of Le Page and Hirschfeld-Żelazko for the commutativity of Banach algebras

by

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Abstract. A theorem of Le Page [6] states that a unital complex Banach algebra A is commutative if and only if there exists a constant $k > 0$ such that $\|xy\| \leqslant k\|yx\|$ for all $x, y \in A$. Hirschfeld and Żelazko [5] proved that a complex Banach algebra A is commutative if there exists a constant $k > 0$ such that $\|x\|^2 \leqslant k\|x^2\|$ for all $x \in A$; in the case that A possesses a unit this result is an easy consequence of the theorem of Le Page. In [3] Bonsall and Duncan showed that a spectral state f of a unital complex Banach algebra A has the following property: $f(xy) = f(yx)$ $(x, y \in A)$. We shall now demonstrate how all these results and even generalizations of these results can be obtained as immediate consequences of a single theorem which is also valid for Banach algebras without unit or without approximate unit. The case that a bounded approximate identity exists has been studied by Baker and Pym [2].

In the following A denotes a complex Banach algebra. Let E be a normed vector space over the field of complex numbers.

THEOREM. *For a continuous linear operator $T\colon A \to E$ the following statements are equivalent*:

(i) *There exists $k > 0$ such that $\|T(xy+y)\| \leqslant k\|yx+y\|$ for all $x, y \in A$.*

(ii) *$T(xy) = T(yx)$ for all $x, y \in A$.*

Proof. The implication (ii)$\Rightarrow$(i) is obvious (with $k = \|T\|$).

(i)$\Rightarrow$(ii): If A is unital, define $\tilde{A} := A$; if A is not unital, let $\tilde{A}$ be the Banach algebra obtained by adjoining an identity to A. Let $x, y \in A$ be fixed. Then we consider the function $G\colon \boldsymbol{C} \to E$ which is defined in the following way:

$$G(\lambda) := T[\exp(-\lambda x)\, y \cdot \exp(\lambda x)] \quad (\lambda \in \boldsymbol{C}).$$

Note that A is a two-sided ideal in $\tilde{A}$ and that $\exp(-\lambda x)\, y \cdot \exp(\lambda x)$ lies in A even if A is not unital. G is an entire (vector-valued) function, and – as we shall see now – G is bounded.

Since $\exp(-\lambda x)-1 = \sum_{n=1}^{\infty}(-\lambda x)^n/n!$ and $y\cdot\exp(\lambda x)$ are elements of A, we obtain by (i):

$$\begin{aligned}\|G(\lambda)\| &= \big\|T\big[(\exp(-\lambda x)-1)\,y\cdot\exp(\lambda x)+y\cdot\exp(\lambda x)\big]\big\|\\ &\leqslant k\big\|y\cdot\exp(\lambda x)\big(\exp(-\lambda x)-1\big)+y\cdot\exp(\lambda x)\big\|\\ &= k\|y\| \qquad (\lambda\in C).\end{aligned}$$

By Liouville's Theorem G is constant, and therefore

$$\begin{aligned}T(y) = G(0) = G(\lambda) &= T[\exp(-\lambda x)\,y\cdot\exp(\lambda x)]\\ &= T(y)+\lambda T(yx-xy)+\ \dots \qquad (\lambda\in C).\end{aligned}$$

Equate coefficients of λ to obtain $T(xy) = T(xy)$.

COROLLARY 1. *If A possesses an approximate identity $\{u_\alpha\}$ (which need not be bounded), and if $T\colon A\to E$ is a continuous linear operator, then the following statements are equivalent:*

(i) $\|T(xy)\| \leqslant k\|yx\|$ *for all* $x, y\in A$ *and some* $k>0$.

(ii) $T(xy) = T(yx)$ *for all* $x, y\in A$.

Proof. We show that condition (i) of this corollary implies condition (i) of the theorem. For all $x, y\in A$ we have:

$$\|T(xy+u_\alpha y)\| = \big\|T\big((x+u_\alpha)y\big)\big\| \leqslant k\|y(x+u_\alpha)\| = k\|yx+yu_\alpha\|$$

for all α, and therefore

$$\|T(xy+y)\| = \lim_\alpha \|T(xy+u_\alpha y)\| \leqslant k\cdot\lim_\alpha \|xy+yu_\alpha\| = k\|yx+y\|.$$

In the case of a bounded approximate identity Corollary 1 can also be derived by an application of a result of Baker and Pym [2].

In the following the spectral radius of an element $a\in A$ is denoted by $\|a\|_\sigma$. The spectral radius has the following property:

$$\|xy+y\|_\sigma = \|yx+y\|_\sigma \text{ for all } x, y\in A.$$

If A is unital, this is immediate since $\|xy+y\|_\sigma = \|(x+1)y\|_\sigma = \|y(x+1)\|_\sigma = \|yx+y\|_\sigma$ $(x, y\in A)$. If A is not unital, adjoin an identity to A.

COROLLARY 2. *If $T\colon A\to E$ is a linear operator such that there exists same $k>0$ with $\|T(x)\|\leqslant k\|x\|_\sigma$ for all $x\in A$, then $T(xy) = T(yx)$ for all $x, y\in A$.*

Proof. Since $\|\cdot\|_\sigma \leqslant \|\cdot\|$, T is continuous. For $x, y\in A$ we have:

$$\|T(xy+y)\| \leqslant k\|xy+y\|_\sigma = k\|yx+y\|_\sigma \leqslant k\|yx+y\|.$$

Now we can apply the theorem. ∎

Corollary 2 is a generalization of a result on spectral states [3], p. 114, and it is equivalent to [1], Lemme 2, p. 46. The following corollaries deal with some topological criteria for the commutativity of Banach algebras.

COROLLARY 3. *A is commutative if and only if there exist a continuous norm $\|\ \|_1$ on A and a constant $k>0$ such that*

$$\|xy+y\|_1 \leqslant k\|yx+y\| \quad \textit{for all } x, y\in A.$$

Proof. Define $E = A$ (endowed with the norm $\|\ \|_1$), and let T be the identity operator. Then apply the theorem. ∎

COROLLARY 4. *If A possesses an approximate identity, then A is commutative if and only if there exist a continuous norm $\|\ \|_1$ on A and a constant $k>0$ such that*

$$\|xy\|_1 \leqslant k\|yx\| \quad \textit{for all } x, y\in A.$$

Proof. Apply Corollary 1 to the identity operator ($E = A$ endowed with the norm $\|\ \|_1$). ∎

If A is unital and if $\|\ \|_1 = \|\ \|$, Corollary 4 reduces to a theorem of Le Page [6]. Since his result is not valid for algebras without approximate identity, Corollary 3 seems to be a suitable formulation of the theorem of Le Page for the general case.

COROLLARY 5. *If there exist another norm $\|\ \|_1$ on A and $k>0$ such that $\|x\|_1\leqslant k\|x\|_\sigma$ for all $x\in A$, then A is commutative.*

Proof. Define $E = A$ endowed with the norm $\|\ \|_1$, and let T be the identity operator. Then apply Corollary 2. ∎

COROLLARY 6. *If there exist a continuous norm $\|\ \|_1$ and a constant $k>0$ such that $\|x\|_1^2\leqslant k\|x^2\|_1$ for all $x\in A$, then A is commutative.*

Proof. By induction it follows that $\|x\|_1^{2^n}\leqslant k^{2^n-1}\|x^{2^n}\|_1$ $(n\in N)$, and since $\|x^{2^n}\|_1\leqslant c\|x^{2^n}\|$ for some $c>0$, we obtain by the spectral radius formula: $\|x\|_1\leqslant k\|x\|_\sigma$ $(x\in A)$. Then apply Corollary 5. ∎

COROLLARY 7. *A is commutative and semi-simple if and only if the spectral radius is a norm on A.*

Proof. If A is commutative, the spectral radius is subadditive, and semi-simplicity means then that $\|a\|_\sigma = 0$ implies $a = 0$ $(a\in A)$, i.e., $\|\cdot\|_\sigma$ is a norm on A. If the spectral radius is a norm, A is commutative by Corollary 5, and A is semi-simple since A contains no quasi-nilpotent elements.

Remarks. From Corollary 6 it follows that a complex Banach algebra A with the property $\|x^2\| = \|x\|^2$ $(x\in A)$ "is" a function algebra (without asuming commutativity); A is automatically commutative, and the Gelfand map is an isometry. The Corollaries 6 and 7 are essentially due to Hirschfeld and Żelazko [5]; they used the estimate $\|x+\lambda\|_\sigma \geqslant (\|x\|_\sigma+|\lambda|)/3$ $(x\in A,\ \lambda\in C)$ for the spectral radius on the unitization $\tilde{A}$ to reduce the general case to the unital case. For unital Banach algebras the last two corollaries are also contained in a paper of Mocanu [7]. Some further corollaries which belong to this context and which can be proved by an application of the above results can be found in [1] and [5].

Via completion, the theorem and all the subsequent corollaries are also valid for normed complex algebras.

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Multivariate spline functions, B-spline bases and polynomial interpolations II*

by

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Abstract. In this paper a new notion for the spline functions of several variables is introduced and two constructions of B-spline bases for multivariate spline function space are given. We construct a multivariate analogue of Hermite interpolation, which in [2] and [3] was constructed only for $k = 2$. At the end another natural multivariate analogue of the Lagrange–Hermite type interpolation is constructed.

1. Introduction. We begin this paper by giving a natural definition of multivariate spline functions in the case where the knot sequence $\{x^0, \ldots, x^r\} \subset R^k$ is in general position, that is, every subset of $k+1$ points forms a proper simplex (this in one dimension corresponds to the case of distinct knots). This motivates our definition of a multivariate spline function in the general case.

We construct bases for the linear space $S^k_{m,\{x^0,\ldots,x^r\}}$ (of all k-variate splines with a knot sequence $\{x^0, \ldots, x^r\} \subset R^k$ of order m) consisting of B-splines with $m+k$ knots from $\{x^0, \ldots, x^r\}$. The first direct construction works only in the case of some restrictions on knot configuration. Instead of this the second one is inductive, works in the general case and seems to be more flexible.

Then we present Hermite's interpolation multivariate analogue in the general case, which in [3] was considered only for $k = 2$.

Finally we give another generalization of Lagrange and Hermite interpolations to the multivariate case, which preserves their pointwise nature.

2. On multivariate spline functions.

Definition. Let $x^0, \ldots, x^r \in R^k$ be in general position, that is, let every subset of $k+1$ points form a proper simplex. A *k-variate spline* with a *knot sequence* $\{x^0, \ldots, x^r\}$ *of order* m ($m \geqslant 2$) or *of degree* $m-1$ is a function of the

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