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On weighted norm inequalities for the Riesz transforms of functions with vanishing moments

by

ERNST ADAMS (Austin, Texas)*

Abstract. Let $R_k f$ be the k-th Riesz transform of f and $1 < p < \infty$.

A necessary and sufficient condition is obtained for a weight function u to satisfy

$$\int |R_k f(x)|^p u(x)\,dx \leqslant C \int |f(x)|^p u(x)\,dx, \qquad k = 1, \ldots, n$$

for all f with $\int x^\beta f(x)\,dx = 0$, $|\beta| \leqslant N$ for some fixed nonnegative integer N.

This characterizes all doubling weights u for which the Riesz transforms are bounded on $S_{0,0}$ in L_u^p norm.

1. Introduction. This paper contains the generalization to higher dimensions of the results of [1]. We refer the reader to that paper for historical remarks. Before we can state the main theorem we need some notational background.

A weight w is a nonnegative measurable function. For $1 < p < \infty$ it is said to belong to the class A_p, if

$$\sup |Q|^{-p} \Big(\int_Q w\Big)\Big(\int_Q w^{1-p'}\Big)^{p-1} < \infty,$$

the sup being taken over all cubes Q in $\boldsymbol{R}^n$, $p' = p/(p-1)$.

For $k \geqslant 0$ we let $\boldsymbol{P}_k$ be the set of polynomials in $\boldsymbol{R}^n$ of degree not exceeding k and $\boldsymbol{P}_{-1} = \{0\}$.

To a given set of points p_j and positive integers m_j, $j = 1, \ldots, J$ we associate the space $\boldsymbol{P}_M$ with $M = \sum_{j=1}^{J} m_j - 1$.

By $\boldsymbol{R}_M$ we denote the set of polynomials R in $\boldsymbol{P}_M$ that satisfy

$$D^\gamma R(p_j) = 0, \qquad |\gamma| < m_j,\ j = 1, \ldots, J.$$

For a nonnegative integer N we let

$$L_N = \{f \in L^1\colon \int |f(x)|(1+|x|)^N\,dx < \infty,\ \int x^\beta f(x)\,dx = 0,\ |\beta| \leqslant N\}$$

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* This paper contains the main results of the author's doctoral dissertation written under the direction of B. Muckenhoupt.

and $L_{-1} = L^1$. The subspace of Schwartz functions whose Fourier transform has compact support not including the origin shall be called $S_{0,0}$.

The k-th Riesz transform of f, $R_k f$, is defined by

$$R_k f(x) = \int f(x-y) \frac{y_k}{|y|^{n+1}} dy.$$

Now we can state our main result.

THEOREM. *Given p, $1 < p < \infty$, and a nonnegative integer N, a weight u satisfies*

$$\int |R_k f(x)|^p u(x)\, dx \leqslant C \int |f(x)|^p u(x)\, dx, \quad k = 1, \ldots, n \tag{1.1}$$

for all f in L_N if and only if u is of the form

$$u(x) = (1+|x|)^{p(N_0 - M)} \prod_{j=1}^{J} |x-p_j|^{pm_j}\, w(x) \tag{1.2}$$

for some positive integers m_j, $-1 \leqslant N_0 \leqslant N$, p_j in $\boldsymbol{R}^n$, $j = 1, \ldots, J$, $M = \sum_{j=1}^{J} m_j - 1$,

$$\boldsymbol{P}_{N_0} \cup \boldsymbol{R}_M \quad \text{spans} \quad \boldsymbol{P}_M, \tag{1.3}$$

where $\boldsymbol{R}_M$ corresponds to $\{(p_j, m_j)\colon j = 1, \ldots, J\}$, and

$$w \quad \text{is in} \quad A_p, \tag{1.4}$$

$$w(x)(1+|x|)^{p(1-n)} \quad \text{is in} \quad A_p, \quad \text{if} \quad N_0 \geqslant 0, \tag{1.5}$$

$$w(x)|x-p_j|^{p(1-n)} \quad \text{is in} \quad A_p, \quad j = 1, \ldots, J. \tag{1.6}$$

Note. If $N_0 = -1$, $\boldsymbol{P}_{N_0} = \{0\}$ and (1.3) forces $M = -1$, hence $J = 0$ and u is in A_p, which is the well-known case.

To see that this theorem is indeed a generalization of the one-dimensional result, we first remark that $\boldsymbol{R}_M = \{0\}$, if $n = 1$ or $J = 1$. This will be shown in Section 2. Hence (1.3) can only hold if $N_0 \geqslant M$, so that $m_0 = N_0 - M$ is nonnegative, $m_0 + \sum_{j=1}^{J} m_j = N_0 + 1 \leqslant N + 1$, and

$$u(x) = (1+|x|)^{pm_0} \prod_{j=1}^{J} |x-p_j|^{pm_j}\, w(x).$$

In one dimension this is equivalent to $|q(x)|^p w(x)$, q a polynomial of degree N_0+1. Also the three conditions (1.4), (1.5) and (1.6) coincide if $n = 1$.

Condition (1.3), which is nontrivial only if $N_0 < M$, $n > 1$ and $J > 1$, is somewhat obscure, and it seems hard to determine, whether $\boldsymbol{P}_{N_0} \cup \boldsymbol{R}_M$ spans $\boldsymbol{P}_M$ for a given N_0 and a set of pairs (p_j, m_j).

In general it does not limit the number of zeros to $N+1$ as in the case of one dimension. If the p_j's are distributed appropriately and if the corresponding orders m_j are not too big, then $\boldsymbol{R}_M$ is a rather large subspace of $\boldsymbol{P}_M$, since the number of conditions on R in $\boldsymbol{R}_M$ decreases as the m_j's decrease. Thus it is conceivable that $\boldsymbol{R}_M$ together with $\boldsymbol{P}_{N_0}$ spans $\boldsymbol{P}_M$.

An example of such a situation is given in $\boldsymbol{R}^3$ by $N_0 = 1$, $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$, $p_3 = (0, 0, 1)$, $m_j = 1$, $j = 1, 2, 3$, hence $M = 2$. First we can easily determine the dimensions of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ as 4 and 10, respectively. Counting conditions we see that $\boldsymbol{R}_2$ has dimension 7. Since, up to a constant factor, there is only one polynomial, $p(x) = 1 - x_1 - x_2 - x_3$, in $\boldsymbol{P}_1$ that vanishes at the p_j's and hence is in $\boldsymbol{R}_2$,

$$\dim(\boldsymbol{P}_1 \cup \boldsymbol{R}_2) = \dim \boldsymbol{P}_1 + \dim \boldsymbol{R}_2 - \dim(\boldsymbol{P}_1 \cap \boldsymbol{R}_2) = 10,$$

so that $\boldsymbol{P}_1 \cup \boldsymbol{R}_2$ spans $\boldsymbol{P}_2$.

Although the number of p_j's is not limited a priori, the orders m_j cannot exceed N_0+1. If otherwise, say $m_1 \geqslant N_0+2$, we use Lemma (2.10) of Section 2, which for given γ, $|\gamma| < m_1$, guarantees the existence of a polynomial P in $\boldsymbol{P}_{N_0}$ with $D^\gamma P(p_1) = 1$. Taking $|\gamma| = N_0+1$ gives a contradiction.

In the special case that all p_j's lie on one line, we can show that as in one dimension, $M \leqslant N_0$. To see this first note that we can assume that the p_j's lie on the x_1-axis. This follows from Remark (3.9) below.

Then let $q(x_1)$ be an arbitrary polynomial of degree not exceeding M, and $q = P + R$, P in $\boldsymbol{P}_{N_0}$, R in $\boldsymbol{R}_M$. The restriction of P to the x_1-axis, $p(x_1)$, is a polynomial in one dimension of degree at most N_0 and it satisfies

$$p^{(k)}(p_{j1}) = q^{(k)}(p_{j1}), \quad k < m_j,\ j = 1, \ldots, J.$$

These are $M+1$ conditions on $p(x_1)$, which can in general not be satisfied unless $M \leqslant N_0$.

Another property of u which was to be expected from the one-dimensional case, is that u grows slower than $|x|^{p(N+1+n)-n}$ at infinity, in the sense that

$$\int \frac{u(x)}{1+|x|^{p(N+1+n)}} dx < \infty. \tag{1.7}$$

This follows easily from (1.2) and the fact that $\int \frac{w(x)}{1+|x|^{pn}} dx < \infty$, since w is in A_p.

Condition (1.7) is also sufficient in a sense made precise by the following corollary.

COROLLARY (1.8). *If a weight u satisfies (1.7) for some integer $N \geqslant 0$, then (1.1) holds for all f in $S_{0,0}$, if and only if u is of the form as stated in the theorem.*

If u satisfies the doubling condition, then (1.7) holds for some N, so that the theorem characterizes all doubling weights for which (1.1) holds for all f in $S_{0,0}$.

As in the one-dimensional case one can show that some growth condition on u has to be imposed to insure the finiteness of $\int |\tilde{f}|^p u$ for any f in $S_{0,0}$.

We only mention the result on the unit circle corresponding to the theorem. The place of the Riesz transforms is taken by the conjugate function

$$\tilde{f}(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\cot\left(\frac{\theta-t}{2}\right)dt.$$

Then we have the following. Given a nonnegative integer N and $p>1$, a weight u on the unit circle satisfies

$$\int_0^{2\pi} |\tilde{f}|^p u \leqslant C \int_0^{2\pi} |f|^p u$$

for all periodic functions f with $\hat{f}(k) = 0$, $k = 0, \ldots, N$, if and only if $u(t) = |\sum_{j=0}^{N} c_j e^{ijt}|^p w(t)$, where the c_j's are complex numbers and w is in A_p of the circle.

For $p = 2$ this was first shown in [4].

2. Preliminaries. Throughout this paper p will be a number greater than 1.

The class A_p was discovered by B. Muckenhoupt in [6]. He showed that

$$w \in A_p \quad \text{if and only if} \quad \|f^*\|_{p,w} \leqslant C\|f\|_{p,w}, \tag{2.1}$$

where f^* is the Hardy–Littlewood maximal function of f.

It was soon found that

$$w \in A_p \quad \text{if and only if} \quad \|R_k f\|_{p,w} \leqslant C\|f\|_{p,w}, \quad k = 1, \ldots, n. \tag{2.2}$$

For a proof of (2.2) see [3].

Some of the consequences of w being in A_p are $w^{1-p'} \in A_{p'}$ and $w \in B_p$, i.e.

$$\int \frac{|Q|^p w(x)}{|x-x_Q|^{np}+|Q|^p}\,dx \leqslant C\int_Q w(x)\,dx,$$

with C independent of the cube Q and x_Q the center of Q.

Combining these two facts it follows that

$$\int_{|x|<r} w(x)\,dx\Big(\int_{|x|>r} w(x)^{1-p'}|x|^{-np'}dx\Big)^{p-1} \leqslant C \quad \text{for all } r>0.$$

A similar condition is encountered in the characterization of weights u

and v satisfying either of the following inequalities, both known as "Hardy's inequalities",

$$\int \Big|\int_{|y|<|x|} f(y)\,dy\Big|^p u(x)\,dx \leqslant C\int |f(x)|^p v(x)\,dx, \tag{2.3}$$

$$\int \Big|\int_{|y|>|x|} f(y)\,dy\Big|^p u(x)\,dx \leqslant C\int |f(x)|^p v(x)\,dx. \tag{2.4}$$

In one dimension it is well known that u and v satisfy (2.3) or (2.4), if and only if for some B

$$\int_{|x|>r} u(x)\,dx\Big(\int_{|x|<r} v(x)^{1-p'}dx\Big)^{p-1} \leqslant B, \tag{2.5}$$

$$\int_{|x|<r} u(x)\,dx\Big(\int_{|x|>r} v(x)^{1-p'}dx\Big)^{p-1} \leqslant B, \tag{2.6}$$

respectively, for all positive r. For a proof see [5].

The same result holds in n dimensions. We do not know of any reference, so we give a proof of the sufficiency of (2.5) for (2.3) to hold, which is due to B. Muckenhoupt. Similarly, (2.4) can be shown to follow from (2.6).

We will show that for a given positive function f and $p>1$ (2.3) holds if

$$\int_{|x|>r} u(x)\,dx\Big(\int_{|x|<r} v(x)^{1-p'}\chi_{\operatorname{supp} f}(x)\,dx\Big)^{p-1} \leqslant C4^{-p}. \tag{2.7}$$

Let c_k be such that $\int_{|y|<c_k} f(y)\,dy = 2^k$ and $c_k = c_{k+1} = \ldots = \infty$ if $\int f < 2^k$. Then the expression on the left of (2.3) equals

$$\sum_{k=-\infty}^{\infty} \int_{c_k<|x|<c_{k+1}} \Big(\int_{|y|<|x|} f(y)\,dy\Big)^p u(x)\,dx. \tag{2.8}$$

Using that $\int_{|y|<c_{k+1}} f(y)\,dy \leqslant 4\int_{c_{k-1}<|y|<c_k} f(y)\,dy$, (2.8) can be estimated by

$$4^p \sum_k \int_{c_k<|x|<c_{k+1}} u(x)\,dx\Big(\int_{c_{k-1}<|y|<c_k} f(y)\,dy\Big)^p$$
$$\leqslant 4^p \sum_k \int_{c_k<|x|} u(x)\,dx\Big(\int_{|y|<c_k} v(y)^{1-p'}\chi_{\operatorname{supp} f}(y)\,dy\Big)^{p-1} \int_{c_{k-1}<|y|<c_k} (f(y))^p v(y)\,dy$$

by Hölder's inequality. After using (2.7) and summing over k, (2.3) follows.

The following lemma will be used frequently.

LEMMA (2.9). *Let E be a measurable subset of $\boldsymbol{R}^n$ with positive measure and φ_k, $k = 1, \ldots, m$ be functions in $L^\infty(E)$, linearly independent over E. Then for every m-tuple $(\lambda_k)_1^m$ of complex numbers there exists α in $L^\infty(E)$ such that*

$$\int_E \varphi_k(x)\alpha(x)\,dx = \lambda_k, \quad k = 1, \ldots, m.$$

If in addition E is compact and the φ_k's are C^∞ in the interior of E, α can be chosen to be C^∞.

The proof is a simple adaptation of the proof of Lemma (2.6) in [2] and is omitted here.

We will use the following notations.

If not otherwise specified, $j, k, l, m, n, J, K, L, M$ and N will denote nonnegative integers; β, γ and η are multiindices of order n with nonnegative integral coefficients β_k etc., and for example $|\beta|$ will mean $\sum_{k=1}^{n} \beta_k$.

We will write the Taylor polynomial of order k of a function g at $x = 0$ in direction y as

$$T_k(g, y) = \sum_{j=0}^{k} (y \cdot \nabla)^j g(0)/j!, \quad \text{where} \quad \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right).$$

In our terminology a cone around a vector y in $\boldsymbol{R}^n$, $|y| = 1$, with opening a, $0 < a < \pi/2$, is the subset of $\boldsymbol{R}^n$ containing all x that satisfy $\sum_{k=1}^{n} x_k y_k \geqslant |x| \cos a$.

We say that a function f in the Schwartz space S is in S_k if $\int x^\beta f(x)\,dx = 0$, $|\beta| \leqslant k$. Note that $S_{0,0}$ is a subset of S_k for all k.

For $t > 0$ and a function g we set $g_t(x) = t^{-n} g(x/t)$.

The subscripts c and loc as in L_c^1 and L_{loc}^1 indicate compact support or local integrability of the functions considered.

The letter C will always denote a constant not necessarily the same at each occurrence. We will further need some facts about the existence of certain interpolating polynomials. Let p_j, m_j, $\boldsymbol{P}_k$, $\boldsymbol{R}_M$ be as in the first section and $\{R_k\colon k = 1, \ldots, K\}$ be a basis of $\boldsymbol{R}_M$. We have the following lemmas.

LEMMA (2.10). *For $j = 1, \ldots, J$ and $|\gamma| < m_j$ there exists a polynomial $P_{j,\gamma}$ in $\boldsymbol{P}_M$ with*

$$\text{(2.11)} \qquad D^\eta P_{j,\gamma}(p_i) = \delta_{ij}\delta_{\eta\gamma}, \quad |\eta| < m_i,\ i = 1, \ldots, J.$$

If $\boldsymbol{P}_N \cup \boldsymbol{R}_M$ generates $\boldsymbol{P}_M$, $P_{j,\gamma}$ can be chosen to be in $\boldsymbol{P}_N \cap \boldsymbol{P}_M$.

COROLLARY (2.12). *For $j = 1, \ldots, J$ and $|\gamma| \leqslant M$ there exists a polynomial $Q_{j,\gamma}$ in $\boldsymbol{P}_M$ with*

$$\text{(2.13)} \qquad D^\eta Q_{j,\gamma}(p_j) = 0, \quad |\eta| < m_j,$$

$$\text{(2.14)} \qquad D^\eta \big(Q_{j,\gamma} - (\cdot - p_j)_\gamma\big)(p_i) = 0, \quad |\eta| < m_i,\ i \neq j.$$

In other words

$$|Q_{j,\gamma}(x)| \leqslant C|x - p_j|^{m_j}, \quad |x - p_j| \leqslant 1$$

and

$$|Q_{j,\gamma}(x) - (x - p_j)^\gamma| \leqslant C\,|x - p_i|^{m_i}, \quad |x - p_i| \leqslant 1,\ i \neq j.$$

If $\boldsymbol{P}_N \cup \boldsymbol{R}_M$ spans $\boldsymbol{P}_M$, $Q_{j,\gamma}$ can be chosen to be in $\boldsymbol{P}_N \cap \boldsymbol{P}_M$.

LEMMA (2.15). *A basis of $\boldsymbol{P}_M$ is given by $\{P_{j,\gamma}\}_{j,\gamma} \cup \{R_k\}_k$.*

LEMMA (2.16). *$\boldsymbol{R}_M = \{0\}$ if and only if $n = 1$ or $J = 1$.*

Proof of Lemma (2.10) and Corollary (2.12). We will prove the assertion assuming that the first coefficients of the p_j's are all different. This can always be achieved by a rotation. A moment's reflection shows that this is no loss of generality.

If $J = 1$ the lemma and the corollary are trivial. If $J > 1$ we fix j and γ. Then note that there exist polynomials in x_1, $q_1(x_1)$ and $r_1(x_1)$, such that

$$\text{(2.17)} \qquad (x_1 - p_{j1})^{m_j} q_1(x_1) = 1 + \prod_{i \neq j} (x_1 - p_{i1})^{m_i} r_1(x_1)$$

and $\deg r_1 < m_j$. This is an elementary algebraic fact, which follows from the above assumption.

Multiplying (2.17) by $(x - p_j)^\gamma/\gamma!$ gives a polynomial $P(x)$ which can be written as

$$\sum_{|\eta| \geqslant m_j} (x - p_j)^\eta q_\eta(x) = (x - p_j)^\gamma/\gamma! + \prod_{i \neq j} (x_1 - p_{i1})^{m_i} r(x).$$

We can assume that $\deg r < m_j$, Otherwise subtract all terms of order $\geqslant m_j$ from the last expression on the right and include them in the sum on the left.

Then the degree of P does not exceed $\max(|\gamma|, M)$. If $|\gamma| < m_j - 1 \leqslant M$, $P(x) - (x - p_j)^\gamma/\gamma!$ is the desired $P_{j,\gamma}$, and if we let $Q_{j,\gamma} = \gamma! P$, then (2.13) and (2.14) hold.

The additional assertions are easily verified.

Proof of Lemma (2.15). Let P be in $\boldsymbol{P}_M$. Then using (2.11) shows that $P - \sum_{j=1}^{J} \sum_{|\gamma| < m_j} D^\gamma P(p_j) P_{j,\gamma}$ is in $\boldsymbol{R}_M$. Thus we only have to show that a representation $P = \sum_j \sum_\gamma C_{j,\gamma} P_{j,\gamma} + \sum_k \lambda_k R_k$ is unique.

But taking $D^\gamma P$ at p_j gives $C_{j,\gamma} = D^\gamma P(p_j)$, and the λ_k's are unique, because $\{R_k\}_k$ is a basis of $\boldsymbol{R}_M$.

Proof of Lemma (2.16). Let C_i denote the number of multiindices in dimension n of order i. Then $\dim \boldsymbol{P}_M = \sum_{i=0}^{M} C_i$.

If $\boldsymbol{R}_M = \{0\}$, then by Lemma (2.15) $\sum_{i=0}^{M} C_i = \sum_{j=1}^{J} \sum_{i < m_j} C_i$, since this is the number of $P_{j,\gamma}$'s.

Since $M = \sum_{j=1}^{J} m_j - 1$, this is only possible if either $J = 1$ and $m_1 = M + 1$, or if $n = 1$ so that $C_i = 1$ for all i.

On the other hand, if $n=1$, then a polynomial R in $\boldsymbol{R}_M$ has to satisfy the $M+1$ conditions

$$R^{(k)}(p_j)=0, \quad k<m_j,\ j=1,\ldots,J,$$

which implies that $R\equiv 0$.

Also, if $J=1$

$$D^\gamma R(p_1)=0, \quad |\gamma|\leqslant m_1-1=M,$$

and again $R\equiv 0$.

3. Sufficiency. In this section we show that conditions (1.2) through (1.6) are sufficient for (1.1) to hold for all f in L_N.

We fix k and let $R(x)$ denote the k-th Riesz kernel, $x_k/|x|^{n+1}$, and $Rf=R*f$ the k-th Riesz transform of f.

It is easy to verify that for $x\neq 0$, $|\beta|\leqslant N+1$

$$(3.1)\qquad |D^\beta R(x)|\leqslant C|x|^{-n-|\beta|}.$$

This implies that for all $k=1,\ldots,N$

$$(3.2)\qquad \big|R(x-y)-T_k\big(R(x-\cdot),y\big)\big|\leqslant C|y|^{k+1}|x|^{-n-k-1}, \quad |y|<\tfrac{1}{2}|x|.$$

We will need the following lemma.

LEMMA (3.3). *If $1<p<\infty$, $w\in A_p$, then there exists a constant C such that for all b in $\boldsymbol{R}$*

$$(3.4)\qquad \int\Big|\int_{\frac{|x|}{2}<|y|<2|x|} f(y)R(x-y)\,dy\Big|^p|x|^b w(x)\,dx\leqslant C\int|f(x)|^p|x|^b w(x)\,dx.$$

Proof. Let $I_k=\{x\colon 2^k\leqslant|x|<2^{k+1}\}$, $I_k^*=\{x\colon 2^{k-1}\leqslant|x|<2^{k+2}\}$, $f_k=f\cdot\chi_{I_k^*}$.

Then the left side of (3.4) can be written as

$$\sum_{k=-\infty}^{\infty}\int_{I_k}\Big|\int_{\frac{|x|}{2}<|y|<2|x|} f_k(y)R(x-y)\,dy\Big|^p|x|^b w(x)\,dx.$$

Using (3.1) with $|\beta|=0$ this is seen to be bounded by the sum of

$$(3.5)\qquad C\sum_k\int|Rf_k(x)|^p 2^{kb}w(x)\,dx,$$

$$(3.6)\qquad C\sum_k\int\Big(\int_{|y|<|x|}|f_k(y)|\,dy\Big)^p 2^{kb}w(x)|x|^{-np}\,dx,$$

and

$$(3.7)\qquad C\sum_k\int\Big(\int_{|y|>|x|}|f_k(y)|\,|y|^{-n}\,dy\Big)^p 2^{kb}w(x)\,dx.$$

Since w is in A_p, (3.5) is less than

$$(3.8)\qquad C\sum_k\int|f_k(x)|^p 2^{kb}w(x)\,dx.$$

The following inequalities hold because $w\in A_p$ and $w^{1-p'}\in A_{p'}$:

$$\int_{|x|<r}w(x)\,dx\Big(\int_{|x|>r}w(x)^{1-p'}|x|^{-np'}\,dx\Big)^{p-1}\leqslant C,$$

$$\int_{|x|>r}w(x)|x|^{-np}\,dx\Big(\int_{|x|<r}w(x)^{1-p'}\,dx\Big)^{p-1}\leqslant C,$$

with C independent of $r>0$. Thus we can apply Hardy's inequalities to (3.6) and (3.7), respectively, to get a multiple of (3.8) as an upper bound for both.

This completes the proof, since (3.8) is less than a constant times the right side of (3.4).

Now let $H(N,R)$ be the set of weights u such that (1.1) holds for all f in L_N.

Remark (3.9). If u is in $H(N,R)$ with constant C_u in (1.1), then every translation, dilation and rotation of u is also in $H(N,R)$ with a constant not exceeding nC_u.

Only the case of a rotation is nontrivial. Let $\varrho=(\varrho_{ij})_{i,j=1,\ldots,n}$ be the matrix of a rotation of $\boldsymbol{R}^n$. The following formula holds for all functions f and rotations ϱ:

$$R_jf(\varrho^{-1}x)=\sum_{k=1}^{n}\varrho_{jk}R_k[f(\varrho^{-1}\cdot)](x), \qquad \text{see [8], p. 58.}$$

A change of variable and this formula show that

$$(3.10)\qquad \int|R_jf(x)|^p u(\varrho x)\,dx=\int\Big|\sum_{k=1}^{n}\varrho_{jk}R_k[f(\varrho^{-1}\cdot)](x)\Big|^p u(x)\,dx.$$

It is easy to see that $f(\varrho^{-1}\cdot)$ is in L_N, if and only if f is in L_N. Use this, $|\varrho_{jk}|\leqslant 1$, to see that if f is in L_N and u is in $H(N,R)$ with constant C_u, then the left side of (3.10) does not exceed

$$nC_u\int|f(\varrho^{-1}x)|^p u(x)\,dx=nC_u\int|f(x)|^p u(\varrho x)\,dx.$$

We now come to the proof of (1.1). As noted after the theorem, $N_0=-1$ is the case $u\in A_p$, and we only consider the case $N_0\geqslant 0$.

We first treat the case $J=0$. Then $M=-1$, $u(x)=(1+|x|)^{p(N_0+1)}w(x)$ and (1.4) and (1.5) hold.

Since $N_0\leqslant N$, $H(N_0,R)\subset H(N,R)$ and we can assume that $N_0=N$. Under these assumptions we show that (1.1) holds for f in L_N.

For any f in L^1

$$\int_{|x|\leqslant 1}|Rf(x)|^p(1+|x|)^{p(N+1)}w(x)\,dx\leqslant C\int|Rf(x)|^p w(x)\,dx.$$

Since w is in A_p, this is less than $C\int|f|^p w$, which is less than $C\int|f(x)|^p(1+|x|)^{p(N+1)}w(x)\,dx$.

Before we estimate the part where $|x|>1$, we introduce the notation $\Phi_k(x, y) = R(x-y)-T_k(R(x-\cdot), k = 0, 1, \ldots$

Then we have for f in L_N

$$\int_{|x|>1} |Rf(x)|^p u(x)\,dx = \int_{|x|>1} |\int f(y)\,\Phi_N(x, y)\,dy|^p u(x)\,dx. \tag{3.11}$$

This is less than a constant times the sum of

$$\int\Big|\int_{|y|<\frac{|x|}{2}} f(y)\,\Phi_N(x, y)\,dy\Big|^p |x|^{p(N+1)} w(x)\,dx, \tag{3.12}$$

$$\int\Big|\int_{\frac{|x|}{2}<|y|<2|x|} f(y)\,\Phi_N(x, y)\,dy\Big|^p |x|^{p(N+1)} w(x)\,dx, \tag{3.13}$$

and

$$\int_{|x|>1}\Big|\int_{|y|>2|x|} f(y)\,\Phi_N(x, y)\,dy\Big|^p |x|^{p(N+1)} w(x)\,dx. \tag{3.14}$$

Using (3.2) with $k = N$ we have the following estimates for (3.12):

$$\begin{aligned} &C\int\Big(\int_{|y|<|x|} |f(y)|\,|y|^{N+1}|x|^{-n-N-1}\,dy\Big)^p |x|^{p(N+1)} w(x)\,dx \\ &\quad\leqslant C\int\big((|y|^{N+1} f(y))^*(x)\big)^p w(x)\,dx \leqslant C\int|f(x)|^p |x|^{p(N+1)} w(x)\,dx, \end{aligned} \tag{3.15}$$

since w is in A_p.

We use (3.1) to get the following upper bound for (3.13):

$$\begin{aligned} &C\int\Big|\int_{\frac{|x|}{2}<|y|<2|x|} f(y)\,R(x-y)\,dy\Big|^p |x|^{p(N+1)} w(x)\,dx+ \\ &\quad+C\sum_{k=0}^{N}\int\Big(\int_{\frac{|x|}{2}<|y|<2|x|} |f(y)|\,|y|^k|x|^{-n-k}\,dy\Big)^p|x|^{p(N+1)} w(x)\,dx. \end{aligned}$$

By Lemma (3.3) the first summand is bounded by (3.16). The second one is less than $C\int\big(\int_{|y|<2|x|} |f(y)|\,|y|^{N+1}|x|^{-n}\,dy\big)^p w(x)\,dx$, which is treated like (3.15).

Again by (3.1) (3.14) is seen to be bounded by

$$\begin{aligned} &C\int_{|x|>1}\Big(\int_{|y|>|x|} |f(y)|\,|y|^{-n}+\sum_{k=0}^{N}|y|^k|x|^{-n-k}\big)\,dy\Big)^p|x|^{p(N+1)} w(x)\,dx \\ &\quad\leqslant C\int_{|x|>1}\Big(\int_{|y|>|x|} |f(y)|\,|y|^N\,dy\Big)^p|x|^{p(1-n)} w(x)\,dx \\ &\quad\leqslant C\int\Big(\int_{|y|>|x|} |f(y)|\,(1+|y|)^N\,dy\Big)^p(1+|x|)^{p(1-n)} w(x)\,dx. \end{aligned}$$

Hardy's inequality shows that this is less than $C\int|f(x)|^p(1+|x|)^{p(N+1)}w(x)\,dx$, if

$$\int_{|x|<r} w(x)(1+|x|)^{p(1-n)}\,dx\Big(\int_{|x|>r} w(x)^{1-p'}(1+|x|)^{-p'}\,dx\Big)^{p-1}\leqslant C.$$

But this follows from $w(x)^{1-p'}(1+|x|)^{p'(n-1)}\in A_{p'}\subset B_{p'}$, which is a consequence of (1.5).

This completes the proof of (1.1) for $J = 0$.

Let now $J > 0$. As before we can assume that $N_0 = N$. By Remark (3.9) we may assume that $|p_j-p_k|>5$, $j\neq k$.

We claim that it suffices to consider the case that $M\geqslant N$. If otherwise $M<N$, observe that since

$$(1+|x|)^{p(N-M)}\sim(1+|x-p_1|)^{p(N-M)}\sim 1+|x-p_1|^{p(N-M)},$$

$u\sim u_1+u_2$ and neither of the u_i's has the factor $1+|x|$. For these u_i's $M = N$, and if (1.1) holds with u replaced by u_i, then it holds for u also.

We will decompose an arbitrary function f in L_N into a sum of functions f_j, $j = 0, \ldots, J$, each of which is in L_N and supported either in a neighborhood of one p_j or away from all p_j's.

If $M = N$, let $f = f_0+\sum_{j=1}^{J} f_j$, where

$$f_j(x) = f(x)\,\chi_{\{|x-p_j|<2\}}-\sum_{|\gamma|\leqslant N}\alpha_\gamma(x-p_j)\int_{|y-p_j|<2}(y-p_j)^\gamma f(y)\,dy \tag{3.17}$$

for $j>0$, $\alpha_\gamma\in C^\infty\{1<|x|<2\}$, $\int x^\beta\alpha_\gamma(x)\,dx = \delta_{\beta\gamma}$, $|\beta|$, $|\gamma|\leqslant N$. Then $\int(x-p_j)^\gamma f_j(x)\,dx = 0$, $|\gamma|\leqslant N$, hence $f_j\in L_N$, and since $f\in L_N$, also $f_0\in L_N$. If $M>N$, we write an f in $L_N\cap L_c^1$ as $f_0+\sum_{j=1}^{J} f_j$, where for $j>0$

$$\begin{aligned} f_j(x) &= f(x)\,\chi_{\{|x-p_j|<2\}}- \\ &-\sum_{i=1}^{J}\sum_{|\gamma|<m_i}\alpha_{i,\gamma}^j(x)\int_{|y-p_j|<2} f(y)\,P_{i,\gamma}(y)\,dy-\sum_{k=1}^{K}\alpha_k^j(x)\int_{|y-p_j|<2} f(y)\,R_k(y)\,dy, \end{aligned} \tag{3.18}$$

$$\alpha_{i,\gamma}^j,\ \alpha_k^j\in C^\infty\{1<|x-p_j|<2\},\quad \int\alpha_{i,\gamma}^j P_{l,\eta} = \delta_{il}\,\delta_{\gamma\eta},\quad \int\alpha_{i,\gamma}^j R_k = 0,$$
$$\int\alpha_k^j P_{l,\gamma} = 0,\quad \int\alpha_k^j R_m = \delta_{km},\quad i, j, l = 1, \ldots, J;\ k, m = 1, \ldots, K.$$

These functions exist, since $\{P_{j,\gamma}\}_{j,\gamma}$ and $\{R_k\}_k$ are linearly independent. Then $\int f_j P_{i,\gamma} = \int f_j R_k = 0$. Since $\{P_{j,\gamma}\}_{j,\gamma}\cup\{R_k\}_k$ spans $\mathbf{P}_M$, f_j is in L_M, hence in L_N. This is also true for f_0, since f is in L_N.

$$\operatorname{supp} f_0\subset\{x\colon |x-p_j|\geqslant 1,\ j = 1, \ldots, J\},$$
$$\operatorname{supp} f_j\subset\{x\colon |x-p_j|\leqslant 2\},\ j = 1, \ldots, J\},$$

and we will show that

$$\int |Rf_j|^p u \leqslant C \int |f_j|^p u, \quad j = 0, \ldots, J \tag{3.19}$$

and.

$$\int |f_j|^p u \leqslant C \int |f|^p u, \quad j = 1, \ldots, J. \tag{3.20}$$

It is in the proof of (3.20), where the arguments are essentially different according to whether $M > N$ or not. Note that (3.20) implies $\int |f_0|^p u \leqslant C \int |f|^p u$, so that (3.19) and (3.20) will complete the proof of (1.1).

To show (3.19) for $j \neq 0$, let $j = 1$ and we can assume that $p_1 = 0$ by Remark (3.9). First we have

$$\int |Rf_1|^p u = \int_{|x|<4} |Rf_1|^p u + \int_{|x|>4} |Rf_1|^p u = A + B.$$

The following estimate holds for A:

$$A \leqslant C \int_{|x|<4} \left|\int f_1(y) \Phi_{m_1-1}(x, y)\, dy\right|^p |x|^{pm_1} w(x)\, dx.$$

We used here that f_1 is in L_{m_1-1}, since $m_1 - 1 \leqslant N$. This can be treated like (3.11) to get the upper bound $C \int |f_1(x)|^p |x|^{pm_1} w(x)\, dx$, if we use $w(x)^{1-p'} |x|^{p'(n-1)} \in A_{p'} \subset B_{p'}$ for the part corresponding to (3.14).

Since $f_1(y) = 0$ for $|y| > 2$, we have

$$B = \int_{|x|>4} \left| \int_{|y|<\frac{|x|}{2}} f_1(y) \Phi_N(x, y)\, dy\right|^p |x|^{p(N+1)} w(x)\, dx.$$

The fact that w is in A_p gives as for (3.12) the bound $C \int |f_1(x)|^p |x|^{p(N+1)} w(x)\, dx$. Thus $A + B$ is bounded by $C \int |f_1(x)|^p (|x|^{pm_1} + |x|^{p(N+1)}) w(x)\, dx$, which is less than $C \int |f_1|^p u$, since f_1 is supported around p_1.

This completes the proof of (3.19) for $j \neq 0$.

To show (3.19) for $j = 0$ we can again assume that $p_1 = 0$. The following inequalities imply (3.19):

$$\sum_{j=1}^{J} \int_{|x-p_j|<1} |Rf_0(x)|^p u(x)\, dx \leqslant C \int |f_0(x)|^p u(x)\, dx, \tag{3.21}$$

$$\int_G |Rf_0(x)|^p u(x)\, dx \leqslant C \int |f_0(x)|^p u(x)\, dx, \quad G = \{x \colon |x - p_j| \geqslant 1,\ j = 1, \ldots, J\}. \tag{3.22}$$

The left side of (3.21) is less than $C \int |Rf_0|^p w$, which is bounded by $C \int |f_0|^p w$, since w is in A_p. Since f_0 is supported away from the p_j's, this is bounded by

$$C \int |f_0|^p u. \tag{3.23}$$

For the left side of (3.22) we get the upper bound

$$C \int_{|x|\geqslant 1} |Rf_0(x)|^p (1+|x|)^{p(N+1)} w(x)\, dx \leqslant C \int \left|\int f_0(y) \Phi_N(x, y)\, dy\right|^p |x|^{p(N+1)} w(x)\, dx.$$

This can be treated like (3.11) to get the upper bound (3.23), using that $w(x)|x|^{p(1-n)} \in A_p$ for the part corresponding to (3.14).

This completes the proof of (3.19).

To show (3.20) in the case $M = N$ let again $j = 1$, $p_1 = 0$. The definition of f_1 shows that we only need to show that

$$\left| \int_{|y|<2} y^\gamma f(y)\, dy\right|^p \leqslant C \int |f|^p u, \quad |\gamma| \leqslant M.$$

Fix γ and use Corollary (2.12) to get a polynomial $P = Q_{1,\gamma}$ in $\boldsymbol{P}_N$ with

$$|P(x)| \leqslant C|x|^{m_1}, \quad |x| < 2, \tag{3.24}$$

$$|P(x) - x^\gamma| \leqslant C|x - p_j|^{m_j}, \quad |x - p_j| < 2,\ j \neq 1. \tag{3.25}$$

Using that $\int y^\gamma f(y)\, dy = \int f(y) P(y)\, dy = 0$ we have

$$\left|\int_{|y|<2} y^\gamma f(y)\, dy\right|^p \leqslant C\left|\int_{|y|<2} f(y) P(y)\, dy\right|^p + C\left|\int_{|y|>2} (P(y) - y^\gamma) f(y)\, dy\right|^p.$$

Use (3.24) in the first summand and then Hölder's inequality on both the last two expressions, to see that this sum is bounded by the sum of

$$C \int |f(y)|^p u(y)\, dy \Big(\int_{|y|<2} w(y)^{1-p'} \prod_{j=2}^{J} |y - p_j|^{-p' m_j}\, dy\Big)^{p-1}$$

and

$$C \int |f(y)|^p u(y)\, dy \Big(\int_{|y|>2} w(y)^{1-p'} |P(y) - y^\gamma|^{p'} \prod_{j=1}^{J} |y - p_j|^{-p' m_j}\, dy\Big)^{p-1}.$$

The first of these expressions is less than $C \int |f|^p u$, since $w^{1-p'}$ is in $A_{p'}$, hence locally integrable and $|p_j| > 5$, $j \neq 1$.

To see that the second one has the same bound, note that by (3.25) for $|y| > 2$

$$|P(y) - y^\gamma|^{p'} \prod_{j=1}^{J} |y - p_j|^{-p' m_j} \leqslant C(1 + |y|)^{-p'},$$

since $\deg(P(y) - y^\gamma) \leqslant N = \sum_{j=1}^{J} m_j - 1$. The fact that $w(y)^{1-p'} |y|^{p'(n-1)}$ is in $B_{p'}$ and hence $\int_{|y|>2} w(y)^{1-p'} (1 + |y|)^{-p'}\, dy < \infty$, gives the desired bound.

To show that (3.20) holds in the case $M > N$, recall the definition of f_j to see that it suffices to prove

(3.26) $$\Big|\int_{|y-p_j|<2} f(y)P_{i,\gamma}(y)\,dy\Big|^p \leqslant C\int|f|^p u, \quad |\gamma|<m_i,\ i,j\leqslant J,$$

and

(3.27) $$\Big|\int_{|y-p_j|<2} f(y)R_k(y)\,dy\Big|^p \leqslant C\int|f|^p u, \quad k\leqslant K,\ j\leqslant J.$$

If $i\neq j$ in (3.26), we use $|P_{i,\gamma}(y)|\leqslant C|y-p_j|^{m_j}$ and Hölder's inequality to get the upper bound

$$C\int|f(y)|^p u(y)\,dy\Big(\int_{|y-p_j|<2} u(y)^{1-p'}|y-p_j|^{p'm_j}\,dy\Big)^{p-1}$$

for the expression on the left of (3.26). Since the second integral is finite, we get the desired bound.

If $i=j$, observe that $\int fP_{i,\gamma}=0$, since $P_{j,\gamma}\in \boldsymbol{P}_N\cap\boldsymbol{P}_M$. Then the left side of (3.26) is seen to be less than

(3.28) $$C\Big|\sum_{i\neq k}\int_{|y-p_k|<2} f(y)P_{i,\gamma}(y)\,dy\Big|^p + C\Big|\int_G f(y)P_{i,\gamma}(y)\,dy\Big|^p,$$
$$G=\{x\colon |x-p_j|\geqslant 2,\ j=1,\ldots,J\}.$$

The first term is treated as above. The second one is less than $C\int|f|^p u\big(\int_G u(x)^{1-p'}(1+|x|)^{p'N}\,dx\big)^{p-1}$ by Hölder's inequality. This is bounded by $C\int|f|^p u$, since the second integral is finite. This completes the proof of (3.26).

Use $D^\gamma R_k(p_j)=0$, $|\gamma|<m_j$ to see that the expression on the left of (3.27) is less than

$$C\Big|\int_{|y-p_j|<2}|f(y)|\,|y-p_j|^{m_j}\,dy\Big|^p.$$

As before this has upper bound $C\int|f|^p u$. This completes the proof of (3.27) and hence of (1.1).

We remark here that (3.20) also holds if $p=1$. Then w and $w(x)(1+|x|)^{1-n}$ are in A_1, which is defined as the class of functions v satisfying $v^*(x)\leqslant cv(x)$ a.e. This implies that

$$\operatorname*{ess\,sup}_{x\in Q}\frac{1}{v(x)} \quad\text{and}\quad \operatorname*{ess\,sup}_{x\in \boldsymbol{R}^n}\frac{1}{(1+|x|)^n v(x)}$$

are both finite for any cube Q.

With these two facts in mind only simple modifications of the above proof show that (3.20) is true for $p=1$, too.

4. Preliminaries to the necessity part. We will show the necessity of conditions (1.2) through (1.6) for (1.1) to hold for more convolutions than the Riesz transforms. We will assume that for some $C>0$ and all f in L_N

(4.1) $$\int\sup_{t>0}|f*\varphi_t(x)|^p u(x)\,dx\leqslant C\int|f(x)|^p u(x)\,dx$$

either for all $\varphi(x)=x_k/|x|^{n+1}$, $k=1,\ldots,n$ or for some positive φ in S, radial and decreasing in $|x|$.

Note that if $\varphi(x)=x_k/|x|^{n+1}$, then $f*\varphi_t=R_kf$ for every $t>0$, if we interpret the convolution in the principal value sense.

Note also that if φ in S is of the above form, then it is easily seen that, for positive f, $\sup_{t>0}|f*\varphi_t|\geqslant Cf^*$, with C independent of f.

For the remaining part of this paper we let $H(N)=H(N,R)\cup H(N,\varphi)$, where $H(N,R)$ and $H(N,\varphi)$ are the sets of weights u that satisfy (1.1) and (4.1), respectively, for all f in L_N.

If we formally let $H(-1,R)$ and $H(-1,\varphi)$ be the sets of weights that satisfy the corresponding inequality for all f, then (2.1), (2.2) and the above remark on the choice of φ in S show that $H(-1)=H(-1,R)=H(-1,\varphi)=A_p$.

Remark (4.2). By Remark (3.9) $H(N,R)$ is invariant under translation, dilation and rotation with comparable constants for the transformed weights. It is easy to see that the same holds for $H(N,\varphi)$ if φ is radial. Hence it holds for $H(N)$.

Lemma (4.3). *If u is in $H(N,R)$ or in $H(N,\varphi)$ for some φ in S with $\hat\varphi(0)\neq 0$, then either u is infinite a.e. or it is locally integrable and satisfies the doubling condition.*

Proof. Fix x_0 in $\boldsymbol{R}^n$. Either u is infinite a.e. or we can find a closed set E of positive measure with $\int_E u<\infty$ and $x_0\notin E$.

If u is in $H(N,R)$, we can show directly that u is locally integrable at x_0.

Let $f\in L_c^\infty(E)\cap L_N$, $R_1f(x_0)=2$. The existence of such a function follows from Lemma (2.9).

Since $x_0\notin\operatorname{supp}f$, R_1f is continuous at x_0 and hence $|R_1f|>1$ in some neighborhood U of x_0. Thus

$$\int_U u\leqslant\int_U|R_1f|^p u\leqslant C\int|f|^p u\leqslant C\|f\|_\infty^p\int_E u<\infty.$$

Since x_0 was arbitrary, u is locally integrable everywhere.

If u is in $H(N,\varphi)$ with φ in S, let f be as above. Since $f*\varphi_t$ is continuous and does not vanish identically for all $t>0$, we see from (4.1) that there exists an open ball B such that $\int_B u$ is finite. After translating and dilating we can assume that $B=\{x\colon |x|<1\}$.

If we can show that for an arbitrary x_1 in $\boldsymbol{R}^n$ there exists g in $L^\infty(B)\cap L_N$ with $g*\varphi_t(x_1)\neq 0$ for some $t>0$, then we get from (4.1) with $f=g$, that u is locally integrable in a neighborhood of x_1. The existence of such a function follows from Lemma (2.9), if the functions x^β, $|\beta| \leqslant N$ and φ_t are linearly independent over B for some $t > 0$.

Assume, to derive a contradiction, that for every positive t we had numbers $C(\beta, t)$ with

$$\sum_{|\beta| \leqslant N} C(\beta, t)\, x^\beta = \varphi(x/t), \qquad |x| < 1.$$

This implies

$$\sum_{|\beta| \leqslant N} C'(\beta, t)\, x^\beta = \varphi(x), \qquad |x| < 1/t.$$

It follows immediately that the $C'(\beta, t)$ have to be independent of t and that hence φ is a polynomial, which is false.

Thus g exists and since x, was arbitrary, u is locally integrable everywhere.

It remains to show that u satisfies the doubling condition. Let u be in $H(N)$ satisfying (1.1) or (4.1) with constant C_u. Let Q_0 denote the cube with sidelength 1 centered at the origin and x be in $2Q_0$. By the same argument as before for $\varphi(x) = x_k/(x)^{n+1}$ or φ in S, $\hat{\varphi}(0) \neq 0$, there exists a function f_x in $L^\infty(Q_0) \cap L_N$ with $\sup_{t>0} |f_x * \varphi_t| > 1$ in a neighborhood $U(x)$ of x. This gives

$$\int_{U(x)} u \leqslant C_u \|f_x\|_\infty^p \int_{Q_0} U.$$

We can select finitely many x_k in $2Q_0$ with corresponding functions f_k and neighborhoods U_k, which cover $2Q_0$. Then

$$\int_{2Q_0} u \leqslant \sum_k \int_{U_k} u \leqslant C_u \sum_k \|f_k\|_\infty^p \int_{Q_0} u \leqslant C C_u \int_{Q_0} u \tag{4.4}$$

with C independent of u.

For an arbitrary cube with sidelength r and center x_0, $Q = x_0 + rQ_0$. Thus

$$\int_{2Q} u(x)\,dx = r^n \int_{2Q_0} u(x_0 + rx)\,dx.$$

By (4.4) and Remark (4.2) the last expression is less than

$$r^n C C_u \int_{Q_0} u(x_0 + rx)\,dx = C C_u \int_{Q} u(x)\,dx$$

with C independent of r, x_0 and u. This completes the proof of Lemma (4.3).

The proof of the fact that every u in $H(N)$ if of the form as stated in the theorem, involves showing that some function related to u is in $H(N)$ or $H(N-1)$, i.e. satisfies (1.1) or (4.1) for all f in L_N or L_{N-1}. The following lemma shows that we only have to consider functions f with compact support.

LEMMA (4.5). *If u is such that* (3.1) *holds with $\varphi(x) = x_k/|x|^{n+1}$ or φ in S for all f in $L_c^1 \cap L_N$, then* (3.1) *holds for all f in L_N with the same constant.*

The proof is omitted. It consists of approximating f in L_N by a sequence of functions in $L_c^1 \cap L_N$ and using Fatou's lemma.

We note next that for $\varphi(x) = x_k/|x|^{n+1}$ or φ in S the following estimates hold which we will use frequently:

$$|D^\beta \varphi_t(x)| = t^{-n-|\beta|} |D^\beta \varphi(x/t)| \leqslant C|x|^{-n-|\beta|}, \qquad x \neq 0, \tag{4.6}$$

$$\big|\varphi_t(x-y) - T_m(\varphi_t(x-\cdot), y)\big| \leqslant C|y|^{m+1}|x|^{-n-m-1}, \qquad |y| < \tfrac{1}{2}|x|. \tag{4.7}$$

The first one is easily verified and the second one follows from the first and Taylor's formula. In both inequalities C depends only on φ, β and m.

We will also need the following fact. If

$$\alpha_\gamma \in C^\infty\{|x| < 1\}, \quad \int x^\beta \alpha_\gamma(x)\,dx = \delta_{\beta\gamma}, \qquad |\beta|, |\gamma| \leqslant N, \tag{4.8}$$

φ is in S *or* $\varphi(x) = x_k/|x|^{n+1}$, then there exists a constant C independent of x with

$$\sup_{t>0} |\alpha_\gamma * \varphi_t(x)| \leqslant C(1+|x|)^{-n-|\gamma|}. \tag{4.9}$$

To see this let $m = |\gamma|$. Using (4.8) we have

$$|\alpha_\gamma * \varphi_t(x)| = \Big|\int \alpha_\gamma(y)\big[\varphi_t(x-y) - T_{m-1}(\varphi_t(x-\cdot), y)\big]\,dy\Big|.$$

By (4.7) this is less than

$$C|x|^{-m-n} \int |\alpha_\gamma(y)|\,|y|^m\,dy \qquad \text{for} \qquad |x| > 2.$$

This together with $\|\alpha_\gamma * \varphi_t\|_\infty \leqslant \|\alpha_\gamma\|_\infty \|\varphi\|_1$ for φ in S or $R_k \alpha_\gamma \in L^\infty$ for $\varphi(x) = x_k/|x|^{n+1}$ proves (4.9).

To show the necessity part of the theorem we will consider three cases arranged in order of complexity:

(i) $u^{1-p'}$ is locally integrable but not in L^1,
(ii) $u^{1-p'}$ is in L^1,
(iii) $u^{1-p'}$ is not locally integrable at some c in $\boldsymbol{R}^n$.

We will show that for u in $H(N)$
(i) implies that u is in A_p,
(ii) implies that $u(x)(1+|x|)^{-p}$ is in $H(N-1)$ and that $u(x)(1+|x|)^{-np}$ is in A_p if $N = 0$,
(iii) implies that $u(x)|x-c|^{-p}(1+|x|)^p$ is also in $H(N)$.

This will reduce case (iii) to case (ii) and a sequence of simple arguments will complete the proof.

5. Case (i). We will proceed as follows. If $u \in H(N)$ and $u^{1-p'} \in L^1_{\mathrm{loc}} \setminus L^1$, then we show that $\int u(x)(1+|x|)^{-p(N+n)}\,dx < \infty$. This is the heart of case (i). A density argument gives then that u is in $H(N-1)$, so that after repeating this argument N times we have that u is in $H(-1) = A_p$.

LEMMA (5.1). *If u is in $H(N)$ and for all $\lambda > 0$, large $M > 0$ there exists a positive number h such that*

$$\int_T u(x)^{1-p'}|x|^{p'N}\,dx = 1, \tag{5.2}$$

where

$$T = \{x:\ M < |x| < M+h,\ |x| \leqslant (1+\lambda^2)x_1\}, \tag{5.3}$$

then

$$\int u(x)(1+|x|)^{-p(N+n)}\,dx < \infty. \tag{5.4}$$

Note. The set T is contained in the cone around the x_1-axis with opening $\arccos(1+\lambda^2)^{-1}$. If $u^{1-p'} \in L^1_{\mathrm{loc}}\backslash L^1$, hence $u(x)^{1-p'}|x|^{p'N} \in L^1_{\mathrm{loc}}\backslash L^1$, we can always find a cone Γ with that opening such that

$$\int_\Gamma u(x)^{1-p'}|x|^{p'N}\,dx = \infty.$$

Applying a rotation we can assume that Γ is around the x_1-axis. This shows that the assumptions of the lemma hold.

LEMMA (5.5). *If u is in $H(N)$ and (5.4) holds, then u is in $H(N-1)$.*

Proof of Lemma (5.1). We will choose $\lambda \in (0, 1]$ later, depending only on φ, n, and N. Let $\alpha_\gamma \in C^\infty\{|x| < 1\}$ with

$$\int x^\beta \alpha_\gamma(x)\,dx = \delta_{\beta\gamma}, \qquad |\beta|,\ |\gamma| \leqslant N. \tag{5.6}$$

We can assume that for all $|\gamma| \leqslant N$

$$\int |x|^{|\gamma|+1}|\alpha_\gamma(x)|\,dx \leqslant \lambda. \tag{5.7}$$

Otherwise replace α_γ by $\mu^{-|\gamma|-n}\alpha_\gamma(x/\mu)$, $\mu = \lambda\big(\int |x|^{|\gamma|+1}|\alpha_\gamma(x)|\,dx\big)^{-1}$.

For a given $M > 4$ let T be as in (5.3) with $h > 0$ such that (5.2) holds. Note that for x in T

$$x_1 \leqslant |x| \leqslant 2x_1 \quad\text{and}\quad |x_i| \leqslant 2\lambda x_1, \quad i = 2, \ldots, n. \tag{5.8}$$

Let

$$f(x) = u(x)^{1-p'}|x|^{N(p'-1)}\chi_T(x) - \sum_{|\gamma|\leqslant n} \alpha_\gamma(x) I_\gamma,$$

$$I_\gamma = \int_T u(z)^{1-p'}|z|^{N(p'-1)} z^\gamma\,dz.$$

Then f is in L_N and

$$\int \sup_{t>0} |f*\varphi_t(x)|^p u(x)\,dx \leqslant C\int |f(x)|^p u(x)\,dx. \tag{5.9}$$

The expression on the right is less than

$$C\int_T u(x)^{1-p'}|x|^{p'N}\,dx + \sum_{|\gamma|\leqslant N} \int |\alpha_\gamma|^p u\,\Big|\int_T u(z)^{1-p'}|z|^{|\gamma|+N(p'-1)}\,dz\Big|^p.$$

This is less than some C_0 independent of M, because of (5.2) and since for z in T, $|z|^{|\gamma|} \leqslant |z|^N$.

To estimate $\sup_{t>0}|f*\varphi_t(x)|$ from below we look at $t = |x|$, $x \in X$, $X = \{x:\ 2 < |x| < M/2\} \cap \Gamma$, where Γ is a cone centered at the origin so that for x in Γ

$$|x|^{n+N}|D_1^N \varphi_t(x)| = |D_1^N \varphi(x/|x|)| > \tfrac{1}{2}\max_{|y|=1}|D_1^N \varphi(y)|. \tag{5.10}$$

Let in abuse of notation $\alpha_N = \alpha_\gamma$ for $\gamma = (N, 0, \ldots, 0)$. Then using (5.6) we can estimate $|f*\varphi_t(t)|$ from below by the difference of

$$\Big|\int \alpha_N(y)\big[\varphi_t(x-y) - T_N(\varphi_t(x-\cdot), y) + (-y_1)^N D_1^N \varphi_t(x)/N!\big]\,dy\, I_{(N,0,\ldots,0)}\Big|, \tag{5.11}$$

and the sum of

$$\sum_{|\gamma|=N,\gamma_1\neq N} |\alpha_\gamma * \varphi_t(x)|\,|I_\gamma|, \tag{5.12}$$

$$\sum_{|\gamma|\leqslant N-1} |\alpha_\gamma * \varphi_t(t)|\,|I_\gamma|, \tag{5.13}$$

and

$$\Big|\int_T u(y)^{1-p'}|y|^{N(p'-1)}\varphi_t(x-y)\,dy\Big|. \tag{5.14}$$

In (5.11) use (5.8) and (4.7) to get the lower bound $C|D_1^N \varphi_t(x)| - C\int |y|^{N+1}|\alpha_N(y)|dy\,|x|^{-n-N-1}$, since $2^{-N} \leqslant |I_{(N,0,\ldots,0)}|$ by (5.8) and (5.2). The last difference is greater than $|x|^{-n-N}(C-\lambda C_1)$ for x in Γ by (5.10), (5.7) and since $|x| > 1$.

In (5.12) we use (5.8) and (5.2) to see that $|I_\gamma|$ is less than 2λ. Together with (4.9) we get that (5.12) is bounded above by $C\lambda|x|^{-n-N}$.

For (5.13) we get the upper bound C/M, since $\|\alpha_\gamma * \varphi_t\|_\infty \leqslant C$ uniformly in $t > 0$ and since for z in T, $|\gamma| < N$, $|z|^{|\gamma|} \leqslant |z|^N M^{|\gamma|-N} \leqslant |z|^N/M$, so that $|I_\gamma| \leqslant 1/M$.

We get the upper bound C/M for (5.14), too, since for $|y| > M > 2|x|$, $|\varphi_t(x-y)| \leqslant C|y|^{-n} \leqslant C|y|^N/M$.

Altogether we have for x in X

$$\sup_{t>0}|f*\varphi_t(x)| \geqslant |x|^{-n-N}(C-\lambda C_2) - C_3/M.$$

After choosing $\lambda = \min(1, C/2C_2)$ we see that $\sup_{t>0}|f*\varphi_t(x)| \geqslant |x|^{-n-N}C/4$, if x is in X and $|x| \leqslant M' = (MC/4C_3)^{1/(n+N)}$. Using this in (5.9) gives

$$\int_{x\in X, |x|\leqslant M'} u(x)|x|^{-p(n+N)}\,dx \leqslant CC_0.$$

The constants on the right are independent of M, so that letting $M \to \infty$, hence $M' \to \infty$, and observing that u satisfies the doubling condition completes the proof.

Proof of Lemma (5.5). Let α_γ in $C^\infty\{1<|x|<2\}$ satisfy (5.6). By Lemma (4.5) it is enough to consider functions in L_{N-1} with compact support. Let f be such a function and for $m>1$ let

$$f_m(x)=f(x)-\sum_{|\gamma|=N} m^{-n-N}\alpha_\gamma(x/m)\int y^\gamma f(y)\,dy.$$

Then f_m is in L_N and hence

$$\int\sup_{t>0}|f_m*\varphi_t|^p u\leqslant C\int|f_m|^p u. \tag{5.15}$$

We claim that $\sup_{t>0}|f_m*\varphi_t|\to\sup_{t>0}|f*\varphi_t|$ a.e. as $m\to\infty$. If $\varphi(x)=x_k/|x|^{n+1}$, this follows from the fact that

$$m^{-n-N}\|R_k\alpha_\gamma\|_\infty\int|y|^N|f(y)|\,dy\to 0\quad\text{ as } m\to\infty.$$

If φ is in S, then

$$\sup_{t>0}|(f-f_m)*\varphi_t(x)|\leqslant\|f-f_m\|_\infty\|\varphi\|_1\leqslant C\sum_{|\gamma|=N}m^{-n-N}\|\alpha_\gamma\|_\infty\to 0,$$

implies that the claim holds.

Thus we can apply Fatou's lemma to get

$$\int\sup_{t>0}|f*\varphi_t|^p u\leqslant\liminf_{m\to\infty}\int\sup_{t>0}|f_m*\varphi_t|^p u.$$

By (5.15) the last expression is bounded by

$$C\liminf_{m\to\infty}\int|f_m|^p u$$

$$\leqslant C\int|f|^p u+C\sum_{|\gamma|=N}\liminf_{m\to\infty}\int_{m<|x|<2m}\frac{|\alpha_\gamma(x/m)|}{m^{p(n+N)}}u(x)\,dx\left|\int y^\gamma f(y)\,dy\right|.$$

This completes the proof, since the last term is zero by (5.4).

6. Case (ii). Let u be in $H(N)$ and $u^{1-p'}$ in L^1. Let N_0 be the largest integer not exceeding N such that

$$\int u(x)^{1-p'}|x|^{p'N_0}\,dx<\infty. \tag{6.1}$$

If $N_0<N$, repeated application of Lemmas (5.1) and (5.5) shows that u is in $H(N_0)$. Hence we can assume that $N=N_0$.

With the help of several lemmas we will show that then (6.1) implies $u(x)(1+|x|)^{-p}\in H(N-1)$ and that also $u(x)(1+|x|)^{-pn}\in H(-1)=A_p$, if $N=0$.

For further use we state the following lemmas in a slightly more general form than needed here.

LEMMA (6.2). *If u is in $H(N)$ and if for some C_0, $r_0>2$*

$$\int_{|x|>r_0}u(x)^{1-p'}|x|^{p'N}\,dx\leqslant C_0, \tag{6.3}$$

then there exists C such that for all $r>r_0$

$$\int_{r_0<|x|<r}u(x)|x|^{-p(n+N)}\,dx\Big(\int_{|x|>r}u(x)^{1-p'}|x|^{p'N}\,dx\Big)^{p-1}\leqslant C. \tag{6.4}$$

LEMMA (6.5). *Under the assumptions of Lemma (6.2) there exists C such that for all $r>r_0$*

$$\int_{|x|>r}u(x)|x|^{-p(n+N+1)}\,dx\Big(\int_{r_0<|x|<r}u(x)^{1-p'}|x|^{p'(N+1)}\,dx\Big)^{p-1}\leqslant C. \tag{6.6}$$

Remark. It follows easily that if $u^{1-p'}$ is in L^1, then the conclusions of the two lemmas are equivalent to

$$\int_{|x|<r}u(x)(1+|x|)^{-p(n+N)}\,dx\Big(\int_{|x|>r}u(x)^{1-p'}(1+|x|)^{p'N}\,dx\Big)^{p-1}\leqslant C \tag{6.4$'$}$$

and

$$\int_{|x|>r}u(x)(1+|x|)^{-p(n+N+1)}\,dx\Big(\int_{|x|<r}u(x)^{1-p'}(1+|x|)^{p'(N+1)}\,dx\Big)^{p-1}\leqslant C \tag{6.6$'$}$$

with C independent of $r>0$. In this form they will allow us to apply Hardy's inequalities under appropriate circumstances.

We also have the following lemma which is similar to Lemma (3.3).

LEMMA (6.7). *If u is in $H(N)$ and $u(x)^{1-p'}(1+|x|)^{p'N}$ is in L^1, then there exists a constant C such that*

$$\int\sup_{t>0}\Big|\int_{\frac{|x|}{2}<|y|<2|x|}f(y)\varphi_t(x-y)\,dy\Big|^p|x|^{pb}u(x)\,dx\leqslant C\int|f(x)|^p|x|^{pb}u(x)\,dx \tag{6.8}$$

for all f in L^1 with compact support away from the origin, say, $\operatorname{supp}f\subset\{x\colon |x|\geqslant 2\}$ *and for $b=-1$, if $N>0$, and $b\leqslant -1$ if $N=0$.*

With the help of these three lemmas we can do the reduction.

LEMMA (6.9). *If u is in $H(N)$ and $u(x)^{1-p'}(1+|x|)^{p'N}$ is in L^1, then $u(x)(1+|x|)^{-pa}$ is in $H(N-1)$ for $a=1$, if $N>0$, and $1\leqslant a\leqslant n$, if $N=0$.*

COROLLARY (6.10). *If u is in $H(N)$ and $u^{1-p'}$ is in L^1, then $u(x)=(1+|x|)^{p(N_0+1)}w(x)$, where w and $w(x)(1+|x|)^{p(1-n)}$ are in A_p, and N_0 is the largest integer such that $u(x)^{1-p'}(1+|x|)^{p'N_0}$ is in L^1.*

Proof of Lemma (6.2). The proof is similar to that of Lemma (5.1) and we will refer to that one for some parts.

We will choose $\lambda\in(0,\frac{1}{2}]$ later depending only on φ, N and n. It suffices to show (6.4) for large r, say $r>\max(r_0, 3/\lambda)$. All constants appearing below will be independent of r and λ until it is fixed.

Let α_γ, $|\gamma| \leqslant N$, and f be as in the proof of Lemma (5.1), but with $T = \{x\colon\, r < |x|,\ |x| \leqslant (1+\lambda^2)x_1\}$.

Then (5.9) holds again and $\int |f|^p u$ can be estimated by $C\int_T u(x)^{1-p'}|x|^{p'N}dx$ using (6.3).

To estimate $\sup_{t>0}|f*\varphi_t(x)|$ from below let $t = |x|$ and $x \in X \cap \Gamma$ with Γ a cone such that (5.10) holds and $X = \{x\colon\, 2 < |x| < \lambda r\}$. Then a lower bound for $|f*\varphi_t(x)|$ is given by the difference of

$$(6.11)\qquad \Big|\int_T u(y)^{1-p'}|y|^{N(p'-1)}\big[\varphi_t(x-y) - T_N\big(\varphi_t(x-\cdot),\, y\big)\big]\,dy\Big|$$

and

$$(6.12)\qquad \Big|\sum_{|\gamma|\leqslant N} \alpha_\gamma(y)\big[\varphi_t(x-y) - T_N\big(\varphi_t(x-\cdot),\, y\big)\big]\,dy\, I_\gamma\Big|.$$

Splitting up (6.11), it is seen to be greater than the difference of

$$(6.13)\qquad \Big|\int_T u(y)^{1-p'}|y|^{N(p'-1)}\, y_1^N D_1^N \varphi_t(x)\,dy\Big|/N!$$

and the sum of

$$(6.14)\qquad C\int_T u(y)^{1-p'}|y|^{N(p'-1)} \sum_{|\eta|=N,\eta_1\neq N} |y^\eta D^\eta \varphi_t(x)|\,dy,$$

$$(6.15)\qquad C\int_T u(y)^{1-p'}|y|^{N(p'-1)} \sum_{|\eta|<N} |y^\eta D^\eta \varphi_t(x)|\,dy,$$

and

$$(6.16)\qquad C\int_T u(y)^{1-p'}|y|^{N(p'-1)}|x-y|^{-n}\,dy.$$

By (5.8) and (5.10), (6.13) has the lower bound

$$(6.17)\qquad C\int_T u(y)^{1-p'}|y|^{Np'}\,dy\,|x|^{-n-N}.$$

Using (5.8) we see that an upper bound for (6.14) is given by λ times (6.17).

We get the same upper bound for (6.15), if we use (5.8) and if we observe that $|y|^{|\eta|} \leqslant |y|^N|x|^{|\eta|-N}\lambda$, since $|x| < \lambda r < \lambda|y|$, $|\eta| < N$, $\lambda < 1$.

This bound holds for (6.16), too, since $|x| \leqslant \frac{1}{2}|y|$. and hence $|x-y| \geqslant \frac{1}{2}|y|$.

Thus (6.11) is greater than

$$C\int_T u(y)^{1-p'}|y|^{p'N}\,dy\,|x|^{-n-N}(1-\lambda C_1).$$

Since $|x| > 2|y|$ in (6.12) we can estimate it from above by

$$C\sum_{|\gamma|\leqslant N}\int |y|^{N+1}|\alpha_\gamma(y)|\,dy\,|x|^{-n-N-1}\int_T u(z)^{1-p'}|z|^{p'N}\,dz.$$

This is less than λ times (6.17) because of (5.7) and $|x| > 1$.

Thus for x in X $\sup_{t>0}|f*\varphi_t(x)|$ is greater than $1-\lambda C_2$ times (6.17). Letting $\lambda = \min(\frac{1}{2}, 1/2C_2)$ and using this estimate in (5.9) gives

$$\int_X u(x)|x|^{-p(n+N)}dx\Big(\int_T u(x)^{1-p'}|x|^{p'N}dx\Big)^{p-1} \leqslant C$$

after we divide by the integral over T on both sides.

Use the fact that u satisfies the doubling condition twice to see that for $a = -p(n+N)$

$$\int_{2<|x|<\lambda r} u(x)|x|^a\,dx \leqslant C\int_X u(x)|x|^a\,dx,$$

and that

$$\int_{r_0<|x|<r} u(x)|x|^a\,dx \leqslant C\int_{2<|x|<r} u(x)|x|^a\,dx \leqslant C\int_{2<|x|<\lambda r} u(x)|x|^a\,dx.$$

The last inequality uses $\lambda r > 3$.

We have now

$$\int_{r_0<|x|<r} u(x)|x|^{-p(n+N)}dx\Big(\int_T u(x)^{1-p'}|x|^{p'N}dx\Big)^{p-1} \leqslant C.$$

By Remark (4.2) the proof is complete since finitely many rotations of T cover the set $\{x\colon\, r < |x|\}$.

Proof of Lemma (6.5). We will again fix $\lambda \in (0, \frac{1}{2}]$ later depending only on φ, N and n. Let α_γ be as in the preceding proof.

The constants in the following will be independent of λ until chosen and of $r > r_0$, which we now fix.

Let $T = \{x\colon\, r_0 < |x| < r,\ \ |x| \leqslant (1+\lambda^2)x_1\}$, so that (5.8) holds. Let $f(x) = u(x)^{1-p'}|x|^{(N+1)(p'-1)}\chi_T(x) - \sum_{|\gamma|\leqslant N}\alpha_\gamma(x) I_\gamma$, where I_γ here denotes $\int_T u(z)^{1-p'}|z|^{(N+1)(p'-1)}z^\gamma\,dz$.

Then f is in L_N and hence

$$(6.18)\qquad \int \sup_{t>0}|f*\varphi_t|^p u \leqslant C\int |f|^p u.$$

The expression on the right is less than

$$(6.19)\quad C\int_T u(x)^{1-p'}|x|^{p'(N+1)}dx + C\sum_{|\gamma|\leqslant N}\int |\alpha_\gamma|^p u\,\Big|\int_T u(z)^{1-p'}|z|^{p'(N+1)-1}dx\Big|^p,$$

since $|z^\gamma| \leqslant |z|^N$ for z in T.

Writing the last integral as

$$\int_T \big[u(z)^{(1-p')/p}|z|^{(N+1)(p'-1)}\big]\big[u(z)^{(1-p')/p'}|z|^N\big]\,dz$$

and applying Hölder's inequality with exponents p and p' shows that (6.19) is less than $C\int_T u(x)^{1-p'}|x|^{p'(N+1)}dx$ using (6.3).

To estimate the left side of (6.18), we look at $f*\varphi_t(x)$ for $t=|x|$, x in $X=\{x\colon r<\lambda|x|\}\cap\Gamma$, where Γ is a cone centered at the origin such that for x in Γ

$$|x|^{n+N+1}|D_1^{N+1}\varphi_t(x)|=|D_1^{N+1}\varphi(x/|x|)|>\tfrac{1}{2}\max_{|y|=1}|D_1^{N+1}\varphi(y)|. \tag{6.20}$$

Then $|f*\varphi_t(x)|$ has as a lower bound the difference of

$$\Big|\int_T u(y)^{1-p'}|y|^{(N+1)(p'-1)}\big[\varphi_t(x-y)-T_{N+1}(\varphi_t(x-\cdot),y)+ \\ +(-y\cdot\nabla)^{N+1}\varphi_t(x)/(N+1)!\big]dy\Big| \tag{6.21}$$

and

$$\Big|\sum_{|\gamma|\leqslant N}\int\alpha_\gamma(y)\big[\varphi_t(x-y)-T_N(\varphi_t(x-\cdot),y)\big]dy\,I_\gamma\Big|. \tag{6.22}$$

Using that $|y|<r<\lambda|x|\leqslant\frac{1}{2}|x|$, (6.21) is seen to be greater than the difference of

$$\Big|\int_T u(y)^{1-p'}|y|^{(N+1)(p'-1)}y_1^{N+1}D_1^{N+1}\varphi_t(x)\,dy\Big|/(N+1)! \tag{6.23}$$

and the sum of

$$C\int_T u(y)^{1-p'}|y|^{(N+1)(p'-1)}\sum_{|\eta|=N+1,\eta_1\neq N+1}|y^\eta D^\eta\varphi_t(x)|\,dy \tag{6.24}$$

and

$$C\int_T u(y)^{1-p'}|y|^{(N+1)(p'-1)}|y|^{N+2}|x|^{-n-N-2}dy. \tag{6.25}$$

By (5.8) and (6.20), (6.23) has the lower bound

$$C\int_T u(y)^{1-p'}|y|^{(N+1)p'}dy\,|x|^{-n-N-1}. \tag{6.26}$$

Using (5.8) again shows that (6.24) is bounded above by λ times (6.26). The same is true for (6.25) since $|y|<\lambda|x|$.

Hence (6.21) is bounded below by $C_1-\lambda C_2$ times (6.26).

Since $|y|<1<|x|/2$ in (6.22), it is bounded above by

$$C\sum_{|\gamma|\leqslant N}\int|\alpha_\gamma(y)|\,|y|^{N+1}dy\,|x|^{-n-N-1}|I_\gamma|.$$

This is less than λ times (6.26) by (5.7) and since $|z^\gamma|\leqslant|z|^{N+1}$ for z in T.

Altogether for x in X $\sup_{t>0}|f*\varphi_t(x)|$ is greater than

$$C\int_T u(y)^{1-p'}|y|^{p'(N+1)}dy\,|x|^{-n-N-1}(1-\lambda C_3).$$

Taking $\lambda=\min(\frac{1}{2},1/2C_3)$ and using this estimate in (5.9) gives

$$\int_X u(x)|x|^{-p(n+N+1)}dx\Big(\int_T u(x)^{1-p'}|x|^{p'(N+1)}dx\Big)^{p-1}\leqslant C.$$

That u satisfies the doubling condition shows that the set X can be replaced by $\{x\colon |x|>r\}$ and the inequality still holds with a C independent of r. The same argument as in the proof of Lemma (6.2) then completes the proof.

Proof of Lemma (6.7). Since $u^{1-p'}\in L^1$, we can use the conclusions of the two previous lemmas as in (6.4)$'$ and (6.6)$'$.

Let $I_k=\{x\colon 2^k\leqslant|x|<2^{k+1}\}$, $k\in\mathbf{Z}$. We can decompose an arbitrary function f in L_c^1 into a sum of functions f_m, which are in L_N. Namely

$$f_m(x)=f(x)\chi_{I_m}(x)+ \\ +\sum_{|\gamma|<N}\Big[\frac{\alpha_\gamma(x/2^{m+1})}{(2^{m+1})^{n+|\gamma|}}\int_{|y|>2^{m+1}}y^\gamma f(y)\,dy-\frac{\alpha_\gamma(x/2^m)}{2^{m(n+|\gamma|)}}\int_{|y|>2^m}y^\gamma f(y)\,dy\Big] \\ +\sum_{|\gamma|=N}\Big[\frac{\alpha_\gamma(x/2^m)}{2^{m(n+N)}}\int_{|y|<2^m}y^\gamma f(y)\,dy-\frac{\alpha_\gamma(x/2^{m+1})}{(2^{m+1})^{n+N}}\int_{|y|<2^{m+1}}y^\gamma f(y)\,dy\Big],$$

where

$$\alpha_\gamma\in C^\infty(I_{-1}),\ |\gamma|\leqslant N-1,\qquad \alpha_\gamma\in C^\infty(I_0),\ |\gamma|=N,$$
$$\int x^\beta\alpha_\gamma(x)\,dx=\delta_{\beta\gamma},\ |\beta|,\ |\gamma|\leqslant N.$$

Note that $\operatorname{supp}f_m\subset I_{m-1}\cup I_m\cup I_{m+1}$ and $f_m\in L_N$.

If f vanishes on $\{x\colon |x|<2\}$ the left side of (6.8) can be written as

$$\sum_{k=0}^\infty\int_{I_k}\sup_{t>0}\Big|\int_{\frac{|x|}{2}<|y|<2|x|}f(y)\varphi_t(x-y)\,dy\Big|^p|x|^{pb}u(x)\,dx \\ =\sum_{k=0}^\infty\int_{I_k}\sup_{t>0}\Big|\int_{\frac{|x|}{2}<|y|<2|x|}\sum_{m=k-2}^{k+2}f_m(y)\varphi_t(x-y)\,dy\Big|^p|x|^{pb}u(x)\,dx.$$

This is less than a constant times

$$\sum_{k=0}^\infty\sum_{m=k-2}^{k+2}\int_{I_k}\sup_{t>0}\Big|f_m*\varphi_t(x)-\int_{G(x)}f_m(y)\varphi_t(x-y)\,dy\Big|^p|x|^{pb}u(x)\,dx,$$

where $G(x)=\{y\colon |y|<\frac{1}{2}|x|$ or $|y|>2|x|\}$. This in turn can be estimated from above by a multiple of the sum of

$$\sum_{k=0}^\infty\sum_{m=k-2}^{k+2}2^{kpb}\int\sup_{t>0}|f_m*\varphi_t|^p u, \tag{6.27}$$

$$\sum_{k=0}^\infty\sum_{m=k-2}^{k+2}2^{kp(b+N+1)}\int\Big(\int_{|y|<|x|}|f_m(y)|\,dy\Big)^p u(x)(1+|x|)^{-p(n+N+1)}dx, \tag{6.28}$$

and

$$\sum_{k=0}^{\infty}\sum_{m=k-2}^{k+2} 2^{kp(b+N)}\int\Big(\int_{|y|>|x|}|f_m(y)|\,dy\Big)^p u(x)(1+|x|)^{-p(n+N)}\,dx. \tag{6.29}$$

Since u is in $H(N)$ and $f_m \in L_N$, (6.27) is bounded by

$$C\sum_{k=0}^{\infty}\sum_{m=k-2}^{k+2} 2^{kpb}\int|f_m(x)|^p u(x)\,dx. \tag{6.30}$$

For (6.28) and (6.29) we observe that u satisfies (6.4)′ and (6.6)′, so that Hardy's inequalities give the bounds

$$C\sum_{k=0}^{\infty}\sum_{m=k-2}^{k+2} 2^{kp(b+N+1)}\int|f_m(x)|^p u(x)(1+|x|)^{-p(N+1)}\,dx$$

and

$$C\sum_{k=0}^{\infty}\sum_{m=k-2}^{k+2} 2^{kp(b+N)}\int|f_m(x)|^p u(x)(1+|x|)^{-pN}\,dx,$$

respectively.

Since $2^k \leqslant 4\cdot 2^m$ for $m \geqslant k-2$, the last two expressions are both less than a constant times (6.30), which in turn is less than $C\sum_{m=-2}^{\infty} 2^{mpb}\int|f_m|^p u$, after we interchange the order of summation. By the definition of f_m this is seen to be bounded by the sum of

$$C\sum_{m=-2}^{\infty}\int_{I_m}|f(x)|^p|x|^{pb}u(x)\,dx, \tag{6.31}$$

$$C\sum_{m=-2}^{\infty}\sum_{|\gamma|<N} 2^{mpb}\int_{I_{m-1}}\frac{|\alpha_\gamma(x/2^m)|^p}{2^{mp(n+|\gamma|)}}\Big|\int_{|y|>2^m} y^\gamma f(y)\,dy\Big|^p u(x)\,dx, \tag{6.32}$$

and

$$C\sum_{m=-2}^{\infty}\sum_{|\gamma|=N}\int_{I_m}\frac{|\alpha_\gamma(x/2^m)|^p}{2^{mp(n+N)}}\Big|\int_{|y|<2^m} y^\gamma f(y)\,dy\Big|^p u(x)\,dx. \tag{6.33}$$

We will estimate the last two expressions by a multiple of (6.31) which will complete the proof.

Since (6.32) only appears, if $N>0$, b equals -1, hence $N-|\gamma|+b \geqslant 0$, so that we have the following upper bounds:

$$C\sum_{m=-2}^{\infty}\sum_{|\gamma|<N}\int_{I_{m-1}} 2^{-mp(n+N)}\Big|\int_{|y|>2^m} 2^{m(N-|\gamma|+b)} y^\gamma f(y)\,dy\Big|^p u(x)\,dx$$

$$\leqslant C\sum_{m=-2}^{\infty}\int_{I_{m-1}}\Big(\int_{|y|>2^m}|y|^{N+b}|f(y)|\,dy\Big)^p u(x)|x|^{-p(n+N)}\,dx$$

$$\leqslant C\int\Big(\int_{|y|>|x|}|y|^{N+b}|f(y)|\,dy\Big)^p u(x)(1+|x|)^{-p(n+N)}\,dx.$$

Together with (6.4)′, Hardy's inequality shows that this is less than a multiple of (6.31), if we observe that $\operatorname{supp} f \subset \{x\colon |x| \geqslant 2\}$.

For (6.33) we only use $b \leqslant -1$ to get similar estimates:

$$C\sum_{m=-2}^{\infty}\sum_{|\gamma|=N}\int_{I_m} 2^{-pm(n+N+1)}\Big|\int_{|y|<2^m} 2^{m(b+1)} y^\gamma f(y)\,dy\Big|^p u(x)\,dx$$

$$\leqslant C\sum_{m=-2}^{\infty}\int_{I_m}\Big(\int_{|y|<2^m}|y|^{b+N+1}|f(y)|\,dy\Big)^p u(x)|x|^{-p(n+N+1)}\,dx$$

$$\leqslant C\int\Big(\int_{|y|<|x|}|y|^{b+N+1}|f(y)|\,dy\Big)^p u(x)(1+|x|)^{-p(n+N+1)}\,dx.$$

Using (6.6)′ and Hardy's inequality shows as above that this is less than a constant times (6.31).

This completes the proof.

Proof of Lemma (6.9). As in the preceding proof we will use (6.4)′ and (6.6)′.

Let f in L_{N-1} have compact support and write $f = f_0 + f_1$, where

$$f_0 = f\cdot\chi_{\{|x|<4\}} - \sum_{|\gamma|<N}\alpha_\gamma\int_{|y|<4} y^\gamma f(y)\,dy, \quad \alpha_\gamma \in C^\infty\{2<|x|<4\},$$

$$\int x^\beta\alpha_\gamma(x)\,dx = \delta_{\beta\gamma}, \quad |\beta|,\ |\gamma| \leqslant N.$$

Then f_0 and f_1 are in $L_{N-1}\cap L_c^1$ and it suffices to estimate

$$\int\sup_{t>0}|f_0 * \varphi_t(x)|^p u(x)(1+|x|)^{-pa}\,dx \tag{6.34}$$

and

$$\int\sup_{t>0}|f_1 * \varphi_t(x)|^p u(x)(1+|x|)^{-pa}\,dx \tag{6.35}$$

by

$$C\int|f(x)|^p u(x)(1+|x|)^{-pa}\,dx. \tag{6.36}$$

To estimate (6.34) let $g = f_0 - \sum_{|\gamma|=N}\alpha_\gamma\int y^\gamma f_0(y)\,dy$. Then g is in L_N and hence

$$\int\sup_{t>0}|g*\varphi_t|^p u \leqslant C\int|g|^p u. \tag{6.37}$$

First, (6.34) is less than

$$C\int\sup_{t>0}|g*\varphi_t|^p u + C\sum_{|\gamma|=N}\int\sup_{t>0}|\alpha_\gamma*\varphi_t(x)|^p u(x)(1+|x|)^{-pa}\,dx\Big|\int y^\gamma f_0(y)\,dy\Big|^p.$$

Using (6.37) this is seen to be less than

$$C\int|g|^p u + {} \tag{6.38}$$

$$+C\sum_{|\gamma|=N}\|(1+|x|)^{n+N}\alpha_\gamma*\varphi_t(x)\|_\infty^p\int u(x)(1+|x|)^{-p(a+n+N)}\,dx\,|\cdot|\int y^\gamma f_0(y)\,dy|^p.$$

Since $(1+|x|)^{n+N}\alpha_\gamma * \varphi_t(x)$ is in L^∞ by (4.9) and $u(x)(1+|x|)^{-p(a+n+N)}$ is in L^1, which follows from (6.6)′, the definition of g gives the following upper bound for (6.38):

$$C\int|f_0|^p u+C\Big(\sum_{|\gamma|=N}\int|\alpha_\gamma|^p u+1\Big)\Big(\int|y|^N|f_0(y)|\,dy\Big)^p$$
$$\leqslant C\int|f_0|^p u\Big(1+\Big(\int u(y)^{1-p'}|y|^{p'N}dy\Big)^{p-1}\Big)$$

by Hölder's inequality. By assumption this is bounded by

$$C\int|f_0|^p u\leqslant C\int|f_0(x)|^p u(x)(1+|x|)^{-pa}dx,$$

since f_0 is supported in $\{x\colon |x|\leqslant 4\}$.

The definition of f_0 gives the upper bound

$$C\int|f(x)|^p u(x)(1+|x|)^{-pa}dx+$$
$$+C\sum_{|\gamma|\leqslant N-1}\int|\alpha_\gamma(x)|^p u(x)(1+|x|)^{-pa}dx\Big|\int_{|y|<4}y^\gamma f(y)\,dy\Big|^p.$$

Since the second summand is bounded by

$$C\int_{|y|<4}|f(y)|^p u(y)(1+|y|)^{-pa}dy\Big(\int_{|y|<4}u(y)^{1-p'}(1+|y|)^{ap'}dy\Big)^{p-1}$$

by Hölder's inequality, (6.34) is less than a multiple of (6.36).

Since $|f_1|\leqslant|f|+|f_0|$, it will complete the proof to show that (6.35) is bounded by $C\int|f_1(x)|^p u(x)(1+|x|)^{-pa}dx$. This can be done by subtracting $T_{N-1}(\varphi_t(x-\cdot),y)$ from $\varphi_t(x-y)$ in the convolution integral, dividing it into the usual three parts, and applying the appropriate Hardy's inequalities as was shown in previous proofs.

Proof of Corollary (6.10). Since $u(x)^{1-p'}(1+|x|)^{p'(N_0+1)}$ is not in L^1 but locally integrable, Lemmas (5.1) and (5.5) show that u is in $H(N_0)$. Lemma (6.9) implies that $u(x)(1+|x|)^{-p}$ is in $H(N_0-1)$.

Since $[u(x)(1+|x|)^{-p}]^{1-p'}(1+|x|)^{p'(N_0-1)}$ is in L^1, Lemma (6.9) can be applied again and eventually yields the desired result.

7. Case (iii). Here we assume that u in $H(N)$ is such that $u^{1-p'}$ has a singularity at $x=0$, i.e. that for all positive numbers τ $\int_{|x|<\tau}u(x)^{1-p'}dx=\infty$. It is shown in the next section that then $u(x)|x|^{-p}(1+|x|)^p$ is also in $H(N)$.

The following lemmas will be needed.

LEMMA (7.1). *If u is in $H(N)$ for some N and if for some k*

$$\int_{|x|<\tau}u(x)^{1-p'}|x|^{p'k}dx=\infty \quad \textit{for all } \tau>0, \tag{7.2}$$

then

$$\int_{|x|<1}u(x)|x|^{-p(n+k)}dx<\infty. \tag{7.3}$$

LEMMA (7.4). *Under the assumptions of Lemma (7.1) there exist positive numbers τ_0 and C such that for $0<r<\tau_0$*

$$\int_{|x|<r}u(x)|x|^{-p(n+k)}dx\Big(\int_{r<|x|<\tau_0}u(x)^{1-p'}|x|^{p'k}dx\Big)^{p-1}\leqslant C. \tag{7.5}$$

The proof of Lemma (7.4) will show that in case $k=N$, any $\tau_0>0$ can be taken and C does not depend on τ_0. This gives the following corollary.

COROLLARY (7.6). *If u is in $H(N)$ and $\int_{|x|<\tau}u(x)^{1-p'}|x|^{p'N}dx=\infty$ for all $\tau>0$, then $u^{1-p'}$ has only a singularity at $x=0$, i.e. is locally integrable away from $x=0$.*

LEMMA (7.7). *If u is in $H(N)$, then $u^{1-p'}$ has only isolated singularities. Moreover, for every singularity z there exists a number $\tau>0$ such that*

$$\int_{|x-z|<\tau}u(x)^{1-p'}|x-z|^{p'(N+1)}dx<\infty.$$

LEMMA (7.8). *If u is in $H(N)$, then $u^{1-p'}$ has only a finite number of singularities.*

LEMMA (7.9). *If u is in $H(N)$ for some N, $u^{1-p'}$ has a singularity at $x=0$ and k is the largest integer such that (7.2) holds, then there exists τ_0, $C>0$ such that $0<r<\tau_0$*

$$\int_{r<|x|<\tau_0}u(x)|x|^{-p(n+k+1)}dx\Big(\int_{|x|<r}u(x)^{1-p'}|x|^{p'(k+1)}dx\Big)^{p-1}\leqslant C. \tag{7.10}$$

Note. By Lemma (7.7) this k exists and does not exceed N.

Lemmas (7.4) and (7.9) correspond to Lemmas (6.2) and (6.5). As in case (ii) they are the key lemmas which allow the use of Hardy's inequalities.

In the next lemma we assume that

$$\int_{|x|>r_0}u(x)^{1-p'}|x|^{p'N}dx\leqslant C \quad \text{for some } r_0>2. \tag{7.11}$$

This is not a serious restriction, since if this would not hold, Lemmas (5.1) and (5.5) would show that $u\in H(N-1)$.

By Lemma (7.8) $u^{1-p'}$ has only finitely many singularities $p_1=0$, $p_2,\ldots,p_J$. After a dilation we can assume that $|p_j-p_m|>3$, $j\neq m$.

LEMMA (7.12). *Under the assumptions of Lemma (7.9), if the singularities of u satisfy $|p_j-p_m|>3$, $j\neq m$, and if (7.11) holds, then there exists C such that*

(7.13)

$$\int\sup_{t>0}\Big|\int_{\frac{|x|}{2}<|y|<2|x|}f(y)\varphi_t(x-y)\,dy\Big|^p|x|^{-p}u(x)\,dx\leqslant C\int|f(x)|^p|x|^{-p}u(x)\,dx$$

for f in L^1 with $\operatorname{supp}f\subset\{x\colon |x|\leqslant 2\}$.

Proof of Lemma (7.1). We will show that (7.2) implies (7.3) by induction on k. This is possible since (7.2) for a given k implies (7.2) for all smaller k. The proof of the implication for $k=0$ is contained in the proof below, since for that case the induction step will not be used.

Hence we assume that

(7.14) $$\int_{|x|<1} u(x)|x|^{-p(n+m)}\,dx \leqslant C, \qquad m=0,\ldots,k-1.$$

Since $u\in H(N)$ implies $u\in H(\max(k,N))$, we can assume that $k\leqslant N$.

We will choose $\lambda\in(0,\frac{1}{2}]$ later depending only on φ, n and N. It follows from (7.2) that there exists a cone Γ centered at $x=0$ with opening $\arccos(1+\lambda^2)$ such that

$$\int_{X\in\Gamma,|x|<\tau} u(x)^{1-p'}|x|^{p'k}\,dx=\infty \quad \text{for all } \tau>0.$$

After a rotation we can assume that $\Gamma=\{x:\ |x|\leqslant(1+\lambda^2)x_1\}$.

For $\delta>0$ let $u_\delta(x)=\max(\delta,u(x))$. It follows now that for every $\varepsilon\in(0,\lambda/2)$ there exists $\delta\in(0,\varepsilon)$ such that

(7.15) $$\int_T u_\delta(x)^{1-p'}|x|^{p'k}\,dx=1,$$

where $T=\{x:\ \delta<|x|<\varepsilon,\ |x|\leqslant(1+\lambda^2)x_1\}$. Then (5.8) holds again for x in T.

Finally, let $\mu\in(0,\lambda\delta)$, $\alpha_\gamma\in C^\infty\{|x|<1\}$, $\int x^\beta\alpha_\gamma(x)\,dx=\delta_{\beta\gamma}$, $|\beta|,|\gamma|\leqslant N$ and

$$f(x)=u_\delta(x)^{1-p'}|x|^{k(p'-1)}\chi_T(x)-\sum_{|\gamma|\leqslant k-1}\alpha_\gamma(x/\mu)\mu^{-|\gamma|-n}I_\gamma-\sum_{k\leqslant|\gamma|\leqslant N}\alpha_\gamma(x)I_\gamma,$$

where $I_\gamma=\int_T u_\delta(z)^{1-p'}|z|^{k(p'-1)}z^\gamma\,dz$. Then f is in L_N and hence

(7.16) $$\int\sup_{t>0}|f*\varphi_t|^p u\leqslant C\int|f|^p u.$$

All constants appearing below will be independent of ε, δ, μ and also of λ, until we fix λ.

The expression on the right of (7.16) is less than the sum of

(7.17) $$C\int_T u_\delta(x)^{1-p'}|x|^{p'k}\,dx,$$

(7.18) $$C\sum_{|\gamma|\leqslant k-1}\int|\alpha_\gamma(x/\mu)|^p\mu^{-p(|\gamma|+n)}u(x)\,dx\,|I_\gamma|^p,$$

and

(7.19) $$C\sum_{k\leqslant|\gamma|\leqslant N}\int|\alpha_\gamma|^p u\,|I_\gamma|^p.$$

By (7.15) (7.17) equals a constant.

Since $\mu<1$ and $T\subset\{x:\ |x|<1\}$, (7.18) is less than

(7.20) $$C\int_{|x|<\mu} u(x)|x|^{-p(k-1+n)}\,dx\Big(\int_T u_\delta^{1-p'}\Big)^p.$$

For $|\gamma|\geqslant k$, $|I_\gamma|\leqslant\int_T u_\delta(z)^{1-p'}|z|^{p'k}\,dz=1$, so that (7.19) is also bounded by a constant. Thus an upper bound for $\int|f|^p u$ is given by the sum of (7.20) and a constant.

To estimate the left side of (7.16) we look at $f*\varphi_t(x)$ for $t=|x|$, $x\in X=\{x:\ 2\varepsilon/\lambda<|x|<1\}\cap\Gamma_0$, where Γ_0 is a cone centered at the origin such that for x in Γ_0

(7.21) $$|x|^{n+k}|D_1^k\varphi_t(x)|=|D_1^k\varphi(x/|x|)|>\tfrac{1}{2}\max_{|y|=1}|D_1^k\varphi(y)|.$$

Since f is in L_N, $f*\varphi_t(x)$ is equal to the sum of

$$\int_T u_\delta(y)^{1-p'}|y|^{k(p'-1)}[\varphi_t(x-y)-T_k(\varphi_t(x-\cdot),y)+(-y\cdot)^k\varphi_t(x)/k!]\,dy,$$

$$-\sum_{|\gamma|\leqslant k-1}\int\alpha_\gamma(y/\mu)\mu^{-|\gamma|-n}[\varphi_t(x-y)-T_{k-1}(\varphi_t(x-\cdot),y)]\,dy\,I_\gamma,$$

and

$$-\sum_{k\leqslant|\gamma|\leqslant N}\alpha_\gamma*\varphi_t(x)I_\gamma.$$

Hence $|f*\varphi_t(x)|$ is greater than the difference of

(7.22) $$\Big|\int_T u_\delta(y)^{1-p'}|y|^{k(p'-1)}y_1^k D_1^k\varphi_t(x)\,dy\Big|/k!$$

and the sum of

(7.23) $$C\sum_{|\eta|=k,\eta_1\neq k}\int_T u_\delta(y)^{1-p'}|y|^{k(p'-1)}|y^\eta D^\eta\varphi_t(x)|\,dy,$$

(7.24) $$\int_T u_\delta(y)^{1-p'}|y|^{k(p'-1)}|\varphi_t(x-y)-T_k(\varphi_t(x-\cdot),y)|\,dy,$$

(7.25) $$\sum_{|\gamma|\leqslant k-1}\int|\alpha_\gamma(y/\mu)|\mu^{-|\gamma|-n}|\varphi_t(x-y)-T_{k-1}(\varphi_t(x-\cdot),y)|\,dy\,|I_\gamma|,$$

and

(7.26) $$\sum_{k\leqslant|\gamma|\leqslant N}|\alpha_\gamma*\varphi_t(x)|\,|I_\gamma|.$$

By (5.8), (7.15) and (7.21), (7.22) is greater than $C|x|^{-n-k}$.

Again from (5.8) and (7.15) we get for (7.23) the upper bound

(7.27) $$C\lambda|x|^{-n-k}.$$

Since in (7.24) $|y|<\varepsilon<\lambda|x|/2\leqslant|x|/4$, it is less than

$$C\int_T u_\delta(y)^{1-p'}|y|^{k(p'-1)}|y|^{k+1}|x|^{-n-k-1}\,dy.$$

By $|y|<\lambda|x|$ and (7.15) this is less than a multiple of (7.27).

In (7.25) we have $|y| \leqslant \mu < \lambda\delta < \varepsilon < |x|/2$. Also note that for $|\gamma| \leqslant k-1$, $|I_\gamma| \leqslant \delta^{|\gamma|-k} \int_T u_\delta(z)^{1-p'} |z|^{kp'} dz = \delta^{|\gamma|-k}$. Hence (7.25) is bounded above by

$$C \sum_{|\gamma| \leqslant k-1} \int |\alpha_\gamma(y/\mu)| \mu^{-|\gamma|-n} |y|^k |x|^{-n-k} dy\, \delta^{|\gamma|-k}.$$

Changing the variable of integration gives the bound $C(\mu/\delta)^{k-|\gamma|} |x|^{-n-k}$, which is less than a multiple of (7.27), since $\mu < \lambda\delta$ and $|\gamma| < k$.

For (7.26) note that $\alpha_\gamma * \varphi_t$ is in L^∞ and that $|I_\gamma| \leqslant 1$ for $|\gamma| \geqslant k$. Thus (7.26) is bounded by a constant.

Altogether for x in X $\sup_{t>0} |f*\varphi_t(x)|$ is greater than $C|x|^{-n-k}(1-\lambda C_1) - C_2$. This exceeds $C/4 \cdot |x|^{-n-k}$, if we let $\lambda = \min(\frac{1}{2}, 1/2C_1)$ and if $|x| < (C/4C_2)^{1/(k+n)} = C_0$.

Using this estimate in (7.16) we get

$$\int_{x \in X, |x| < C_0} u(x)|x|^{-p(k+n)} dx \leqslant C + C \int_{|x| < \mu} u(x)|x|^{-p(k-1+n)} dx \,\Big|\int_T u_\delta^{1-p'}\Big|^p.$$

Observe that the last expression only appears, if $k > 0$.

Use the induction assumption (7.16) and let $\mu \to 0$ to get

$$\int_{x \in X, |x| < C_0} u(x)|x|^{-p(k+n)} dx \leqslant C.$$

Let $\varepsilon \to 0$ in the definition of X to see that

$$\int_{x \in \Gamma_0, |x| < C_0} u(x)|x|^{-p(k+n)} dx \leqslant C.$$

Thus (7.3) holds, because u satisfies the doubling condition.

Proof of Lemma (7.4). Since $u \in H(N)$ implies $u \in H(\max(k, N))$, we can assume that $k \leqslant N$. Since (7.5) for a given k implies (7.5) for all smaller k, we may assume that either $k = N$ or that $k < N$ and that there exists $\tau_0 \in (0, 1)$, such that

$$\int_{|x| < \tau_0} u(x)^{1-p'} |x|^{p'(k+1)} dx \leqslant C. \tag{7.28}$$

If $k = N$, let τ_0 be arbitrary but fixed.

We will choose $\lambda \in (0, \frac{1}{2}]$ later depending only on φ, n and N. Let $0 < r < \tau_0$ be given and $\varepsilon \in (0, \lambda r/2)$, $\nu \in (0, \lambda\varepsilon/2)$, $\mu = 2\lambda r$.

Let $T = \{x\colon r < |x| < \tau_0,\ |x| \leqslant (1+\lambda^2) x_1\}$, so that (5.8) holds. Let $u_\varepsilon(x) = \max(\varepsilon, u(x))$ and

$$\alpha_\gamma \in C^\infty\{1 < |x| < 2\}, \quad \int x^\beta \alpha_\gamma(x)\, dx = \delta_{\beta\gamma}, \quad |\beta|,\ |\gamma| \leqslant N. \tag{7.29}$$

Finally, let

$$f(x) = u_\varepsilon(x)^{1-p'} |x|^{k(p'-1)} \chi_T(x) - \sum_{|\gamma| \leqslant k-1} \alpha_\gamma(x/\mu)\mu^{-|\gamma|-n} \cdot I_\gamma - \sum_{|\gamma| = k} \alpha_\gamma(x/\nu)\nu^{-|\gamma|-n} \cdot I_\gamma - \sum_{k+1 \leqslant N} \alpha_\gamma(x) I_\gamma,$$

where $I_\gamma = \int_T u_\varepsilon(z)^{1-p'} |z|^{k(p'-1)} z^\gamma dz$.

The constants appearing below will be independent of τ_0, r, ε, ν and also of λ, until λ is chosen.

Since f is in L_N we have

$$\int \sup_{t>0} |f*\varphi_t|^p u \leqslant C \int |f|^p u. \tag{7.30}$$

The right side is less than the sum of

$$C \int_T u_\varepsilon(x)^{1-p'} |x|^{p'k} dx. \tag{7.31}$$

$$C \sum_{|\gamma| \leqslant k-1} \int |\alpha_\gamma(x/\mu)|^p \mu^{-p(|\gamma|+n)} u(x)\, dx\, |I_\gamma|^p, \tag{7.32}$$

$$C \sum_{|\gamma| = k} \int |\alpha_\gamma(x/\nu)|^p \nu^{-p(k+n)} u(x)\, dx\, |I_\gamma|^p, \tag{7.33}$$

and

$$C \sum_{k+1 \leqslant |\gamma| \leqslant N} \int |\alpha_\gamma|^p u\, |I_\gamma|^p. \tag{7.34}$$

Observing that $|z| > r$ for z in T, and that $|\gamma| \leqslant k-1$ in (7.32), we see that it is less than

$$C \sum_{|\gamma| \leqslant k-1} \int_{\mu < |x| < 2\mu} u(x)|x|^{-p(k+n)} dx \Big(\int_T u_\varepsilon(z)^{1-p'} |x|^{p'k} dz\Big)^p (\mu/r)^{p(k-|\gamma|)}.$$

This is bounded by

$$C\lambda \int_{\mu < |x| < 2\mu} u(x)|x|^{-p(k+n)} dx \Big(\int_T u_\varepsilon(z)^{1-p'} |z|^{p'k} dz\Big)^p, \tag{7.35}$$

since $\mu = 2\lambda r$ and $p(k-|\gamma|) \geqslant 1$.

The following bound for (7.33) is immediate:

$$C \int_{\nu < |x| < 2\nu} u(x)|x|^{-p(k+n)} dx \Big(\int_T u_\varepsilon(z)^{1-p'} |z|^{p'k} dz\Big)^p. \tag{7.36}$$

Note that (7.34) only appears, if $k < N$. Hence Hölder's inequality applied to $|I_\gamma|^p$ together with (7.28) and $|z| < \tau_0 < 1$ shows that (7.34) is less than a multiple of (7.31).

Altogether, $\int |f|^p u$ is bounded by a constant times the sum of (7.31), (7.35) and (7.36).

To estimate $\sup_{t>0} |f*\varphi_t(x)|$, take $t = |x|$, $x \in X = \{x\colon \varepsilon < |x| < \lambda r\} \cap \Gamma_0$, where Γ_0 is as in (7.21). We can then estimate $|f*\varphi_t(x)|$ from below by the difference of

$$\Big|\sum_{|\gamma| = k} \nu^{-k-n} \alpha_\gamma(\cdot/\nu) * \varphi_t(x)\, I_\gamma\Big| \tag{7.37}$$

and the sum of

$$\Big|\int_T u_\varepsilon(y)^{1-p'} |y|^{k(p'-1)} \varphi_t(x-y)\, dy\Big|, \tag{7.38}$$

(7.39) $$\Big|\sum_{|\gamma|\leqslant k-1}\mu^{-|\gamma|-n}\alpha_\gamma(\cdot/\mu)*\varphi_t(x)I_\gamma\Big|$$

and

(7.40) $$\Big|\sum_{k+1\leqslant|\gamma|\leqslant N}\alpha_\gamma*\varphi_t(x)I_\gamma\Big|.$$

In abuse of notation let α_k denote α_γ for $\gamma=(k,0,\ldots,0)$. Then using (7.29), (7.37) is seen to be greater than the difference of

(7.41) $$\Big|\int\alpha_k(y/\nu)\,\nu^{-k-n}\big(\varphi_t(x-y)-\\ -T_k(\varphi_t(x-\cdot),y)\big)+(-y_1)^k D_1^k\varphi_t(x)/k!\;dy\,I_{(k,0,\ldots,0)}\Big|$$

and

(7.42) $$\Big|\sum_{|\gamma|=k,\gamma_1\neq k}\int\alpha_\gamma(y/\nu)\,\nu^{-k-n}\big(\varphi_t(x-y)-T_{k-1}(\varphi_t(x-\cdot),y)\big)\,dy\,I_\gamma\Big|.$$

Since $|y|<2\nu<\lambda\varepsilon\leqslant|x|/2$ in (7.41), and because of (7.29), (7.21) and (5.8), (7.41) is greater than the difference of

(7.43) $$C\int_T u_\varepsilon(z)^{1-p'}|z|^{p'k}\,dz\,|x|^{-k-n}$$

and

(7.44) $$C\int|\alpha_k(y/\nu)|\nu^{-k-n}|y|^{k+1}|x|^{-n-k-1}\,dy\int_T u_\varepsilon(z)^{1-p'}|z|^{p'k}\,dz.$$

Using that $|y|<2\nu<\lambda\varepsilon<\lambda|x|$ in (7.44) and changing the variable of integration show that (7.41) is greater than (7.43) times $1-\lambda C_1$.

Since $|y|<|x|/2$ in (7.42) and because of (5.8), (7.42) is bounded above by

$$C\sum_{|\gamma|=k,\gamma_1\neq k}\int|\alpha_\gamma(y/\nu)|\nu^{-k-n}|y|^k|x|^{-k-n}\,dy\,\lambda\int_T u_\varepsilon(z)^{1-p'}|z|^{p'k}\,dz,$$

which is less than a multiple of λ times (7.43).

Hence (7.37) is greater than $1-\lambda C_2$ times (7.43).

In (7.38) use $|x|<\lambda r<\lambda|x|$ to get the upper bound

$$C\int_T u_\varepsilon(y)^{1-p'}|y|^{p'k}(|x|/|y|)^{n+k}\,dy\,|x|^{-n-k},$$

which is less than λ times (7.43).

For (7.40) note that it only appears, if $k<N$, hence $\tau_0<1$. This implies that $|I_\gamma|\leqslant\int_T u_\varepsilon(z)^{1-p'}|z|^{p'k}\,dz$, since $|\gamma|>k$ and $|z|\leqslant\tau_0$.

Since also $\|\alpha_\gamma*\varphi_t\|_\infty\leqslant C$, uniformly in t, (7.40) is bounded above by $C\int_T u_\varepsilon(z)^{1-p'}|z|^{p'k}\,dz$. This is less than a multiple of λ times (7.34), since $|x|^{k+n}<(\lambda r)^{k+n}\leqslant\lambda^{k+n}\leqslant\lambda$.

Altogether we have for x in X

$$\sup_{t>0}|f*\varphi_t(x)|\geqslant C\int_T u_\varepsilon(y)^{1-p'}|y|^{p'k}\,dy\,|x|^{-k-n}(1-\lambda C_3).$$

For $\lambda\leqslant\min(\tfrac12,1/2C_3)$ we can use this estimate in (7.30), to get that

$$\int_X u(x)|x|^{-p(k+n)}\,dx\Big(\int_T u_\varepsilon(y)^{1-p'}|y|^{p'k}\,dy\Big)^p$$

is bounded by (7.31) times the sum of a constant and $\big(\int_T u_\varepsilon(x)^{1-p'}|x|^{p'k}\,dx\big)^p$ times the sum of

(7.45) $$C\lambda\int_{\mu<|x|<2\mu}u(x)|x|^{-p(k+n)}\,dx$$

and

(7.46) $$C\int_{\nu<|x|<2\nu}u(x)|x|^{-p(k+n)}\,dx.$$

Let $\nu\to0$ in (7.46), which by Lemma (7.1) tends to zero.

Since $\mu=2\lambda r$, $\varepsilon<\lambda r/2$ and u satisfies the doubling condition, (7.45) is bounded by

$$\lambda C_4\int_{\frac{\lambda r}{2}<|x|<\lambda r}u(x)|x|^{-p(k+n)}\,dx\leqslant\lambda C_5\int_X u(x)|x|^{-p(k+n)}\,dx.$$

Let $\lambda=\min(\tfrac12,1/2C_3,1/2C_5)$. Then after dividing by (7.31) and subtracting the nonconstant term on both sides we get that

(7.47) $$\int_X u(x)|x|^{-p(k+n)}\,dx\Big(\int_T u_\varepsilon(x)^{1-p'}|x|^{p'k}\,dx\Big)^{p-1}\leqslant C.$$

Let $\varepsilon\to0$ in the definition of X and u_ε. The fact that u satisfies the doubling condition shows then that we can extend the range of integration in the first integral first to $\{x\colon|x|<\lambda r\}$ and then to $\{x\colon|x|<r\}$, so that the new constant on the right is still independent of r.

Since finitely many rotations of T cover the set $\{x\colon r<|x|<\tau_0\}$ the proof is complete by Remark (4.2).

Proof of Lemma (7.7). The first assertion follows immediately from Lemma (7.4). To show the second one, let $z=0$ without loss of generality.

In order to derive a contradiction, assume that $\int_{|x|<\tau}u(x)^{1-p'}|x|^{p'(N+1)}\,dx=\infty$, $\tau>0$. Lemma (7.4) shows then that

(7.48) $$\int_{|x|<\tau_0}u(x)|x|^{-p(N+n+1)}\,dx<\infty\quad\text{for some }\tau_0>0.$$

Let $\alpha\in C^\infty\{|x|<1\}$, $\int x^\beta\alpha(x)\,dx=0$ for $|\beta|\leqslant N+1$, $\beta\neq\eta$, $\int x^\eta\alpha(x)\,dx=1$ for some fixed η, $|\eta|=N+1$.

For $m>1$ let $f(x)=m^{N+n+1}\alpha(mx)$. Then f is in L_N and

$$\int \sup_{t>0}|f*\varphi_t|^p u \leqslant C\int |f|^p u = C\int_{|x|<\frac{1}{m}} m^{p(N+n+1)}|\alpha(mx)|^p u(x)\,dx.$$

The last expression tends to zero as $m\to\infty$ by (7.48).

But for $2<t=|x|$, x in Γ, where Γ is a cone centered at the origin in which $|x|^{N+n+1}|D^\eta\varphi_t(x)|=|D^\eta\varphi(x/|x|)|>\frac{1}{2}\max_{|y|=1}|D^\eta\varphi(y)|$ holds, we have that

$$|f*\varphi_t(x)| = \Big|\int m^{N+n+1}\alpha(my)\big[\varphi_t(x-y)-T_{N+1}(\varphi_t(x-\cdot),y)+$$
$$+C_\eta y^\eta D^\eta\varphi_t(x)/(N+1)!\big]\,dy\Big|$$
$$\geqslant C|x|^{-N-n-1}-C\int m^{N+n+1}|\alpha(my)|\,|y|^{N+2}|x|^{-N-n-2}\,dy,$$

since then $|y|<1/m<1<|x|/2$.

Changing the variable of integration gives the lower bound $C|x|^{-N-n-1}(1-C_1/m|x|)\geqslant C|x|^{-N-n-1}/2$, if $m>C_1$.

Since all constants are independent of m, we arrive at a contradiction, if we let $m\to\infty$. This completes the proof.

Proof of Lemma (7.8). Lemma (7.7) shows that there can only be a finite number of singularities in any ball.

Assume now that $u^{1-p'}$ has infinitely many singularities. Then for every $\lambda>0$ there exists a cone Γ_λ centered at the origin with opening $\arccos(1+\lambda)^{-1}$, such that $\int_{x\in\Gamma_\lambda,|x|>M} u(x)^{1-p'}\,dx=\infty$ for all $M>0$. Otherwise we could for some $\lambda>0$ cover $\boldsymbol{R}^n$ with finitely many cones Γ_j with opening $\arccos(1+\lambda)^{-1}$ and for all j there would exist M_j with $\int_{x\in\Gamma_j,|x|>M_j} u(x)^{1-p'}\,dx<\infty$. This would imply that

$$\int_{|x|>\max_j M_j} u(x)^{1-p'}\,dx<\infty,$$

with contradicts the assumption that $u^{1-p'}$ has infinitely many singularities and the above remark.

Using a compactness argument we can assume that $\Gamma_\lambda\subset\Gamma_{\lambda'}$, for $\lambda<\lambda'$. Thus after a rotation the assumptions of Lemma (5.1) are satisfied, so that an application of Lemma (5.5) shows that u is in $H(N-1)$.

Repeating this argument we eventually arrive at $u\in H(-1)=A_p$. But then $u^{1-p'}$ is in $A_{p'}$ and has no singularities at all.

This contradiction completes the proof.

Proof of Lemma (7.9). After a dilation of u, if necessary, we have

$$\int_{|x|<1} u(x)^{1-p'}|x|^{p'(k+1)}\,dx\leqslant C. \tag{7.49}$$

We will show that then (7.10) holds with $\tau_0=1$.

We will choose $\lambda\in(0,\frac{1}{2}]$ later depending only on φ, n and N. It suffices to prove (7.10) for small r, so fix r in $(0,\lambda/2)$.

Let $\varepsilon\in(0,r)$, $T=\{x:\ \varepsilon<|x|<r,\ |x|\leqslant(1+\lambda^2)x_1\}$. As before (5.8) holds.

Let α_γ be as in the proof of Lemma (5.1) with (5.6) and (5.7) and set

$$f(x)=u(x)^{1-p'}|x|^{(k+1)(p'-1)}\chi_T(x)-\sum_{|\gamma|\leqslant k}\alpha_\gamma(x/\varepsilon)\varepsilon^{-|\gamma|-n}I_\gamma-\sum_{k+1\leqslant|\gamma|\leqslant N}\alpha_\gamma(x)I_\gamma,$$

$$\text{where}\quad I_\gamma=\int_T u(z)^{1-p'}|z|^{(k+1)(p'-1)}z^\gamma\,dz.$$

Since $u^{1-p'}$ is integrable over T by (7.49), f is in L_N and

$$\int\sup_{t>0}|f*\varphi_t|^p u\leqslant C\int|f|^p u. \tag{7.50}$$

All constants appearing below will be independent of ε, r and also of λ, until it is fixed.

The right side of (7.50) is less than the sum of

$$C\int_T u(x)^{1-p'}|x|^{p'(k+1)}\,dx, \tag{7.51}$$

$$C\sum_{|\gamma|\leqslant k}\|\alpha_\gamma\|_\infty^p\int_{|x|<\varepsilon}u(x)|x|^{-p(|\gamma|+n)}\,dx\,|I_\gamma|^p \tag{7.52}$$

and

$$C\sum_{k+1\leqslant|\gamma|\leqslant N}\int|\alpha_\gamma|^p u\,|I_\gamma|^p. \tag{7.53}$$

By Hölder's inequality $|I_\gamma|^p$ does not exceed

$$\int_T u(z)^{1-p'}|z|^{p'(k+1)}\,dz\Big(\int_{\varepsilon<|z|<1}u(z)^{1-p'}|z|^{p'|\gamma|}\,dz\Big)^{p-1}.$$

Next we note that (6.5) certainly holds with k replaced by $|\gamma|$, $|\gamma|\leqslant k$. The number τ_0 in (6.5) can be chosen to be 1 by (7.49). This shows that (7.52) is bounded by a multiple of (7.51).

This is also true for (7.53), since

$$|I_\gamma|^p\leqslant C\int_T u(z)^{1-p'}|z|^{p'(k+1)}\,dz\,r^{|\gamma|-k}\quad\text{for}\quad|\gamma|\geqslant k+1, \tag{7.54}$$

using the above estimate on I_γ, (7.49) and $r<1$.

To get an estimate for the expression on the left side of (7.50), we look at $t=|x|$, $x\in X=\{x:\ \frac{r}{\lambda}<|x|<1\}\cap\Gamma$, where Γ is a cone centered at the origin such that (7.21) holds with k replaced by $k+1$ and $x\in\Gamma$.

Using that f is in L_N and (5.6) we see that $f*\varphi_t(x)$ equals the sum of

$$\int_T u(y)^{1-p'}|y|^{(k+1)(p'-1)}(-y\cdot\nabla)^{k+1}\varphi_t(x)\,dy/(k+1)!, \tag{7.55}$$

(7.56) $$\int_T u(y)^{1-p'}|y|^{(k+1)(p'-1)}[\varphi_t(x-y)-T_{k+1}(\varphi_t(x-\cdot),y)],$$

(7.57) $$-\sum_{|\gamma|\leqslant k}\int \alpha_\gamma(y|\varepsilon)\varepsilon^{-|\gamma|-n}[\varphi_t(x-y)-T_k(\varphi_t(x-\cdot),y)]\,dy\cdot I_\gamma$$

and

(7.58) $$-\sum_{k+1\leqslant|\gamma|\leqslant N}\alpha_\gamma*\varphi_t(x)\cdot I_\gamma.$$

In (7.55) we split $(-y\cdot\nabla)^{k+1}$ into $(-y_1\cdot D_1)^{k+1}$ and $\sum_{|\eta|=k+1,\eta_1\neq k+1} C_\eta y^\eta D^\eta$ and use (5.7) and (7.21) to see that (7.55) in absolute value is greater than

$$C\int_T u(y)^{1-p'}|y|^{p'(k+1)}\,dy\,|x|^{-n-k-1}(1-\lambda C_1).$$

Since $|y|\leqslant r<\lambda|x|\leqslant|x|/2$ in (7.56), it is less in absolute value than

(7.59) $$\lambda C\int_T u(y)^{1-p'}|y|^{p'(k+1)}\,dy\,|x|^{n-k-1}.$$

In (7.57) $|y|\leqslant\varepsilon<r\leqslant|x|/2$ and $|x|>\varepsilon$ for z in T show that (7.57) is bounded in absolute value from above by

$$C\sum_{|\gamma|\leqslant k}\int|\alpha_\gamma(y/\varepsilon)|\varepsilon^{-|\gamma|-n}|y|^{k+1}|x|^{-n-k-1}\,dy$$

times $\varepsilon^{|\gamma|-k-1}\int_T u(z)^{1-p'}|z|^{p'(k+1)}\,dz$.

After a change of variable in the first integral and by (5.7) this is less than a multiple of (7.59).

The same is true for the absolute value of (7.58) by $\|\alpha_\gamma*\varphi_t\|_{\infty}\leqslant C$, uniformly in t, (7.54), and since $r^{|\gamma|-k}\leqslant r\leqslant\lambda|x|^{-n-k-1}$ for $|\gamma|\geqslant k+1$, $r<\lambda$, $|x|<1$.

Altogether for x in X $\sup_{t>0}|f*\varphi_t(x)|$ is greater than

$$C\int_T u(y)^{1-p'}|y|^{p'(k+1)}\,dy\,|x|^{-n-k-1}(1-\lambda C_2).$$

Letting $\lambda=\min(\tfrac12,1/2C_2)$ and using this estimate in (7.50) shows that

(7.60) $$\int_X u(x)|x|^{-p(n+1+k)}\,dx\Big(\int_T u(x)^{1-p}|x|^{p'(k+1)}\,dx\Big)^{p-1}\leqslant C.$$

The usual arguments used before then complete the proof.

Proof of Lemma (7.12). The proof is similar to the one of Lemma (6.7) and we will sometimes refer to that one.

We remarked before the statement of the lemma that we can assume that the singularities of $u^{1-p'}$ satisfy $|p_j-p_m|>3$, $j\neq m$. After a further dilation we can assume that there exists a cone Γ centered at the origin such that for x in Γ, $|x-p_j|>1$, $j=2,\ldots,J$.

Let $\alpha_\gamma\in C^\infty(\{x:\ 1<|x|<2\}\cap\Gamma)$ with $\int x^\beta\alpha_\gamma(x)\,dx=\delta_{\beta\gamma}$, $|\beta|$, $|\gamma|\leqslant N$. Hence for any $\lambda>0$

$$\operatorname{supp}\alpha(\lambda\cdot)\subset\Gamma\subset\mathbf{R}^n\setminus\bigcup_{j=2}^{J}\{x:\ |x-p_j|<1\}.$$

Let I_m denote the set $\{x:\ 2^m\leqslant|x|\leqslant 2^{m+1}\}$. We decompose f in L^1_c into a sum of functions in L_N, namely $f=\sum_{m=-\infty}^{\infty} f_m$, where $f_m(x)$ is the sum of

$$f(x)\chi_{I_m}(x),\quad \sum_{|\gamma|\leqslant k-1}\left[\frac{\alpha_\gamma(x/2^m)}{2^{m(|\gamma|+n)}}\int_{|y|>2^{m+1}} y^\gamma f(y)\,dy-\frac{\alpha_\gamma(x/2^{m-1})}{2^{(m-1)(|\gamma|+n)}}\int_{|y|>2^m} y^\gamma f(y)\,dy\right]$$

and

$$\sum_{k\leqslant|\gamma|\leqslant N}\left[\frac{\alpha_\gamma(x/2^m)}{2^{m(|\gamma|+n)}}\int_{|y|<2^m} y^\gamma f(y)\,dy-\frac{\alpha_\gamma(x/2^{m-1})}{2^{(m-1)(|\gamma|+n)}}\int_{|y|<2^{m+1}} y^\gamma f(y)\,dy\right].$$

Note that f_m is in L_N and that f_m is supported in

$$(I_{m-1}\cup I_m\cup I_{m+1})\cap\Big(\mathbf{R}^n\setminus\bigcup_{j=2}^{J}\{x:\ |x-p_j|<1\}\Big),$$

if f is supported in $\{x:\ |x|\leqslant 2\}$.

If χ_0 denotes the characteristic function of the set $\mathbf{R}^n\setminus\bigcup_{j=2}^{J}\{x:\ |x-p_j|<1\}$, then we show that the following inequalities hold:

(7.61) $$\int_{|x|<r} u(x)|x|^{-p(k+n)}\,dx\Big(\int_{|x|>r} u(x)^{1-p'}|x|^{p'k}\chi_0(x)\,dx\Big)^{p-1}\leqslant C,$$

(7.62) $$\int_{|x|>r} u(x)|x|^{-p(n+N+1)}\,dx\Big(\int_{|x|<r} u(x)^{1-p'}|x|^{p'(N+1)}\chi_0(x)\,dx\Big)^{p-1}\leqslant C,$$

C independent of $r>0$.

For $r>r_0$ (7.62) follows from Lemma (6.5) and the fact that $\int_{|x|<r_0} u(x)^{1-p'}|x|^{p'(N+1)}\chi_0(x)\,dx\leqslant C$, since χ_0 is supported away from p_j, $j=2,\ldots,J$ and $u(x)^{1-p'}|x|^{p'(N+1)}$ is integrable at $x=p_1=0$ by Lemma (7.7). Thus we only have to show (7.62) for small $r<\tau_0$, where τ_0 is given by (7.9).

Again by Lemma (6.5) the first integral is bounded by

$$C+\int_{r<|x|<\tau_0} u(x)|x|^{-p(n+k+1)}\,dx\,r^{p(k-N)}.$$

For $r<1$ the second integral is less than

$$\int_{|x|<r} u(x)^{1-p'}|x|^{p'(k+1)}\,dx\cdot r^{p'(N-k)}.$$

Since this is uniformly bounded in $r < \tau_0$ and by Lemma (7.9), (7.62) follows.

Similarly, using Lemmas (6.2) and (7.1), as well as (7.11), we see that (7,61) holds.

Analogous to the proof of Lemma (6.7) the left side of (7.13) can be estimated by a multiple of the sum of

$$(7.63)\qquad \sum_{i=-\infty}^{1}\sum_{m=i-2}^{i+1} 2^{-ip}\int \sup_{t>0}|f_m * \varphi_t|^p u,$$

$$(7.64)\qquad \sum_{i=-\infty}^{1}\sum_{m=i-2}^{i+2} 2^{-ip}\int_{I_i}\Big(\int_{|y|<|x|} |f_m(y)|\,|x|^{-n}\,dy\Big)^p u(x)\,dx$$

and

$$(7.65)\qquad \sum_{i=-\infty}^{1}\sum_{m=i-2}^{i+2} 2^{-ip}\int_{I_i}\Big(\int_{|y|>|x|} |f_m(y)|\,|y|^{-n}\,dy\Big)^p u(x)\,dx.$$

Using similar arguments as in the proof of Lemma (6.7) and (7.61), (7.62), the last three sums can be seen to be less than $C\sum_{m=-\infty}^{3} 2^{-mp}\int |f_m|^p u$.

The definition of f_m and Hardy's inequalities show that this is bounded by $C\int |f(x)|^p |x|^{-p} u(x)\,dx$.

8. Case (iii) (continued). This section contains the reduction of case (iii) to case (ii). We let p_j, $j = 1, \ldots, J$ be the singularities of $u^{1-p'}$, u in $H(N)$, i.e.

$$\int_{|x|<\tau} u(x+p_j)^{1-p'}\,dx = \infty \qquad \text{for all } \tau > 0.$$

After a dilation we can assume that $|p_j - p_i| > 5$, $j \neq i$.

If we denote by m_j the largest integer such that

$$(8.1)\qquad \int_{|x|<\tau} u(x+p_j)^{1-p'}|x|^{p'(m_j-1)}\,dx = \infty, \qquad \tau > 0,$$

then by Lemma (7.7), $1 \leqslant m_j \leqslant N+1$, and we have

$$(8.2)\qquad \int_{|x|<1} u(x+p_j)^{1-p'}|x|^{p'm_j}\,dx \leqslant C$$

and

$$(8.3)\qquad \int_{|x|<1} u(x+p_j)\,|x|^{-p(m_j-1+n)}\,dx \leqslant C$$

by Lemma (7.1).

We cannot a priori give an estimate on the number of singularities p_j or the sum of the orders m_j, but we have the following interesting result, which says in effect that $M = \sum_{j=1}^{J} m_j - 1$ cannot be too large. The notation is as in the introduction.

LEMMA (8.4). *If u in $H(N)$ satisfies (8.3) for $j = 1, \ldots, J$, then*

$$(8.5)\qquad \boldsymbol{P}_N \cup \boldsymbol{R}_M \quad \textit{spans} \quad \boldsymbol{P}_M.$$

Proof. If $M \leqslant N$, (8.5) holds trivially.

If $M > N$, assume (8.5) does not hold. Let L, $N+1 \leqslant L \leqslant M$, be the smallest integer such that $\boldsymbol{P}_L$ is not spanned by $\boldsymbol{P}_N \cup \boldsymbol{R}_M$.

Then by Lemma (2.9) there exists a function f in C_c^∞ satisfying

$$\int x^\beta f(x)\,dx = 0, \qquad |\beta| \leqslant N,$$

$$\int f(x) R(x)\,dx = 0, \qquad R \text{ in } \boldsymbol{R}_M,$$

and

$$\int x^\beta f(x)\,dx = \delta_{\beta\eta} \qquad \text{for all } \beta,\ |\beta| = L,$$

and a fixed η, $|\eta| = L$. Otherwise all x^γ, $|\gamma| = L$, and hence $\boldsymbol{P}_L$ would be in the span of $\boldsymbol{P}_N \cup \boldsymbol{R}_M$.

Let $P_{j,\gamma}$ be as in Section 2 and set for $r > 0$,

$$h_r(x) = \sum_{j=1}^{J}\sum_{|\gamma|<m_j} \gamma!\,\alpha_\gamma\Big(\frac{x-p_j}{r}\Big) r^{-|\gamma|-n}\int f P_{j,\gamma},$$

$$\alpha_\gamma \in C^\infty\{|x| < 1\}, \qquad \int x^\beta \alpha_\gamma(x)\,dx = \delta_{\beta\gamma}, \qquad |\beta|,\ |\gamma| \leqslant M.$$

Since $P_{j,\gamma}(x)$ can be written as

$$(x-p_j)^\gamma/\gamma! + \sum_{m_j \leqslant |\eta| \leqslant M} C_{j,\gamma,\eta}(x-p_j)^\eta,$$

$$\int \gamma!\,\alpha_\gamma\Big(\frac{x-p_j}{r}\Big) r^{-|\gamma|-n} P_{i,\beta}(x)\,dx = \delta_{ij}\,\delta_{\beta\gamma}, \qquad |\gamma| < m_j,\ |\beta| < m_i.$$

Hence $\int h_r P_{j,\gamma} = \int f P_{j,\gamma}$.

Also, $R(x) = \sum_{m_j \leqslant |\eta| \leqslant M} C_{j,\eta}(x-p_j)^\eta$ and $\int \alpha_\gamma\Big(\frac{x-p_j}{r}\Big) R(x)\,dx = 0$, thus $\int h_r R = 0 = \int f R$, R in $\boldsymbol{R}_M$.

The fact that $\{P_{j,\gamma}\}$ and $\boldsymbol{R}_M$ span $\boldsymbol{P}_M$ shows that $\int x^\gamma h(x)\,dx = \int x^\gamma f(x)\,dx = \delta_{\gamma\eta}$ for $|\gamma| = L$, since x^γ is in $\boldsymbol{P}_M$.

Also, h_r is in L_{L-1}, since by the choice of L every P in $\boldsymbol{P}_{L-1}$ can be written as $Q+R$, Q in $\boldsymbol{P}_N$, R in $\boldsymbol{R}_M$ and $\int h_r P = \int h_r(Q+R) = \int f(Q+R) = 0$.

Since L_{L-1} is contained in L_N,

(8.6) $$\int \sup_{t>0} |h_r * \varphi_t|^p u \leqslant C \int |h_r|^p u.$$

The right side tends to zero with r, because

$$\int_{|x-p_j|<r} \left|\alpha_\gamma\left(\frac{x-p_j}{r}\right)\right|^{pr-p(|\gamma|+n)} u(x)\,dx \to 0 \quad \text{as} \quad r \to 0$$

by (8.3) and since $|\gamma| < m_j$.

But as shown in the proof of Lemma (7.7), there exists a constant C independent of r and x such that $\sup_{t>0} |h_r * \varphi_t(x)| \geqslant C|x|^{-n-L}$ for large $|x|$ and x contained in a cone depending on φ.

Together, the last two facts contradict (8.6), and hence (8.5) holds.

Now we note that unless u is also in $H(N-1)$ Lemmas (5.1) and (5.5) show that

(8.7) $$\int_{|x|>r_0} u(x)^{1-p'} |x|^{p'N}\,dx \leqslant C, \quad r_0 = \max_j |p_j| + 2.$$

Then we have

LEMMA (8.8). *If u is in $H(N)$, (8.1), (8.2), hence (8.3), and also (8.7) hold, then $u(x)|x-p_j|^{-p}(1+|x|)^p$ is also in $H(N)$.*

Proof. Without loss of generality let $j = 1$, $p_1 = 0$.

We will distinguish between the cases $M \leqslant N$ and $M > N$. In either case we will write a function f in L_N as $f_0 + \sum_{j=1}^{J} f_j$, with f_j supported around p_j and f_0 away from all p_j's.

We will show that with $v(x) = u(x)|x|^{-p}(1+|x|)^p$

(8.9) $$\int \sup_{t>0} |f_0 * \varphi_t|^p v \leqslant C \int |f_0|^p v,$$

(8.10) $$\int \sup_{t>0} |f_1 * \varphi_t|^p v \leqslant C \int |f_1|^p v,$$

and

(8.11) $$\int \sup_{t>0} |f_j * \varphi_t|^p v \leqslant C \int |f_j|^p v, \quad j > 1.$$

Then we can appeal to arguments used before to complete the proof.

If $M \leqslant N$ let f_j be as in (3.17), otherwise as in (3.18). Then f_0 and f_j are in L_N.

To show (8.9), consider

(8.12) $$\int_{|x|<1/2} \sup_{t>0} \left|\int f_0(y)\,\varphi_t(x-y)\,dy\right|^p u(x)|x|^{-p}\,dx$$

and

(8.13) $$\int_{|x|>1/2} \sup_{t>0} |f_0 * \varphi_t)|^p v(x)\,dx.$$

Since $|y| > 2|x|$ in (8.12), $|\varphi_t(x-y)| \leqslant C|y|^{-n}$, and Hölder's inequality gives the upper bound

(8.14) $$C \int_{|x|<1} u(x)|x|^{-p}\,dx \int |f_0(y)|^p u(y)\,dy \int u(y)^{1-p'} |y|^{-p'n} \chi_{\operatorname{supp} f_0}(y)\,dy.$$

The fact that $u^{1-p'}$ is locally integrable on $\operatorname{supp} f_0$, (8.3) and (8.7) show that (8.14) is bounded by $C\int |f_0|^p u$, which is less than $C\int |f_0|^p v$.

For (8.13) we simply use $v(x) \sim u(x)$ on $\{|x| > \frac{1}{2}\}$, $u \in H(N)$ and $f_0 \in L_N$, to get the desired bound.

By the same argument it will be sufficient to estimate

$$\int_{|x|<1} \sup_{t>0} \left|\int f_1(y)\,\varphi_t(x-y)\,dy\right|^p u(x)|x|^{-p}\,dx \quad \text{by} \quad C\int |f_1(x)|^p u(x)|x|^{-p}\,dx.$$

This is done by subtracting $T_{m_1-2}(\varphi_t(x-\cdot), y)$ from $\varphi_t(x-y)$ in the inner integral and using a three parts proof as before.

Similarly we subtract $T_{m_j-1}(\varphi_t(x-p_j-\cdot), y-p_j)$ in the inner integral of (8.11) and use a three parts proof.

This completes the proof of (8.9), (8.10) and (8.11).

It remains to show that

$$\int |f_j|^p v \leqslant C \int |f|^p v, \quad j = 0, 1, \ldots, J.$$

But the proof is virtually the same as that of (3.20) with only obvious modifications.

This completes the proof of Lemma (8.8).

9. Completion of the proof of the theorem. We will show that any u in $H(N)$ satisfies (1.2) through (1.6).

By Lemma (7.8) $u^{1-p'}$ has at most a finite number of singularities p_j, $j = 1, \ldots, J$. Let $r_0 = \max_j |p_j| + 2$.

If $\int_{|x|>r_0} u(x)^{1-p'}\,dx = \infty$, then u is in $H(-1) = A_p$ by repeated application of Lemmas (5.1) and (5.5). Hence $J = 0$ and (1.2) through (1.6) hold with $N_0 = M = -1$.

If $\int_{|x|>r_0} u(x)^{1-p'}\,dx < \infty$, let N_0 be the largest integer with

(9.1) $$\int_{|x|>r_0} u(x)^{1-p'} |x|^{p'N_0}\,dx < \infty.$$

That N_0 exists and does not exceed N is seen as follows. Assume that $\int_{|x|>r_0} u(x)^{1-p'} |x|^{p'(N+1)}\,dx < \infty$. Then since $H(N)$ is contained in $H(N+1)$, Lemma (6.5) implies that also

$$\int_{|x|>r_0} u(x)|x|^{-p(n+N+1)}\,dx < \infty.$$

But Hölder's inequality gives

$$\infty = \int_{|x|>r_0} |x|^{-n}\,dx = \Big(\int_{|x|>r_0} u(x)|x|^{-p(n+N+1)}\,dx\Big)^{1/p}\Big(\int_{|x|>r_0} u(x)^{1-p'}|x|^{p'(N+1)}\,dx\Big)^{1/p'}.$$

This contradiction shows that $N_0 \leqslant N$.

Since $\int_{|x|>r_0} u(x)^{1-p'}|x|^{p'(N_0+1)}\,dx = \infty$, u is in $H(N_0)$ be repeated application of Lemmas (5.1) and (5.5). Thus it will suffice to show (1.2) through (1.6) with $N = N_0$.

If $J = 0$, $u(x)^{1-0'}(1+|x|)^{p'N}$ is in L^1 and Corollary (6.10) gives (1.2), (1.4) and (1.5) with $N_0 = N$, $M = -1$, and (1.3) is trivially true.

If $J > 0$, let m_j be as in the previous section, $M = \sum_{j=1}^{J} m_j - 1$. Applying Lemma (8.8) $M+1$ times shows that $v(x) = u(x)\prod_{j=1}^{J}|x-p_j|^{-pm_j}(1+|x|)^{p(M+1)}$ is also in $H(N)$.

Since (9.1) holds with u replaced by v and $N_0 = N$, Lemma (6.10) shows that $v(x) = (1+|x|)^{p(N+1)}w(x)$, with w satisfying (1.4) and (1.5).

Since (1.2) follows from that and (1.3) was shown in Lemma (8.4), it only remains to show (1.6).

Without loss of generality let $j = 1$, $p_1 = 0$. We first show that for some $C > 0$

$$\text{(9.2)}\qquad \int_{|x|<r} w(x)|x|^{p(1-n)}\,dx\Big(\int_{|x|>r} w(x)^{1-p'}|x|^{-p'}\,dx\Big)^{p-1} \leqslant C \quad \text{for} \quad r > 0.$$

Note that both integrals exist, since

$$w(x)|x|^{p(1-n)} = u(x)(1+|x|)^{p(M-N)}|x|^{p(1-n-m_1)}\prod_{j=2}^{J}|x-p_j|^{-pm_j}$$

is locally integrable by (8.3), and the second integral is finite, since $w(x)^{1-p'}(1+|x|)^{p'(n-1)}$ is in $B_{p'}$.

So we only have to check (9.2) for small r and for large r. The former case is taken care of by Lemma (7.4) with $k = m_1 - 1$. The latter one by $w(x)(1+|x|)^{p(1-n)} \in A_p$ and $w(x)^{1-p'}(1+|x|)^{p'(n-1)} \in B_{p'}$.

Hence (9.2) holds.

Now we can show that $w(x)|x|^{-ap}$ is in A_p for $0 \leqslant a \leqslant n-1$, which implies and is actually equivalent to (1.4) and (1.6).

We will show that for any function f

$$\text{(9.3)}\qquad \int \sup_{t>0}|f*\varphi_t(x)|^p w(x)|x|^{-ap}\,dx \leqslant C\int |f(x)|^p w(x)|x|^{-ap}\,dx,$$

where φ is a fixed positive function in S, radial and decreasing in $|x|$.

For such φ, $\sup_{t>0}|f*\varphi_t| \geqslant Cf^*$, if $f \geqslant 0$, so that (2.1) then implies that $w(x)|x|^{-ap}$ is in A_p.

It suffices to estimate the following integrals by a multiple of the right side of (9.3):

$$\text{(9.4)}\qquad \int\Big(\int_{|y|<|x|/2} |f(y)|\,|x|^{-n}\,dy\Big)^p w(x)|x|^{-ap}\,dx,$$

$$\text{(9.5)}\qquad \int \sup_{t>0}\Big|\int_{\frac{|x|}{2}<|y|<2|x|} f(y)f(y)\varphi_t(x-y)\,dy\Big|^p w(x)|x|^{-ap}\,dx$$

and

$$\text{(9.6)}\qquad \int\Big(\int_{|y|>2|x|} |f(y)|\,|y|^{-n}\,dy\Big)^p w(x)|x|^{-ap}\,dx.$$

Hardy's inequalities can be applied to both (9.4) and (9.6), respectively, to yield the desired bound, since

$$\int_{|x|>r} w(x)|x|^{-p(n+a)}\,dx\Big(\int_{|x|<r} w(x)^{1-p'}|x|^{ap'}\,dx\Big)^{p-1} \leqslant C,$$

since $a \geqslant 0$, $w \in A_p$, hence $w^{1-p'} \in B_{p'}$, and

$$\int_{|x|<r} w(x)|x|^{-ap}\,dx\Big(\int_{|x|>r} w(x)^{1-p'}|x|^{p'(a-n)}\,dx\Big)^{p-1} \leqslant C,$$

since $a \leqslant n-1$ and by (9.2).

Since φ is bounded, (9.5) is seen to be bounded by

$$C\int [|y|^{-a}f(y)]^{*p}(x)\,w(x)\,dx.$$

The fact that w is in A_p completes the proof of (9.3) and hence of the theorem.

10. Proof of Corollary (1.8). Let u satisfy (1.7) and (1.1) for all f in $S_{0,0}$. As in the necessity part of the proof of the theorem we will assume that

$$\text{(10.1)}\qquad \int \sup_{t>0}|f*\varphi_t|^p u \leqslant C\int |f|^p u$$

for all f in $S_{0,0}$ and either for all $\varphi(x) = x_k/|x|^{n+1}$, $k = 1,\ldots,n$, or for some φ in S with nonvanishing integral.

We will prove the corollary by showing that (10.1) actually holds for all f in $L_c^1 \cap L_N$.

We first note that (1.7) implies that every f in S_N can be approximated by a sequence of functions f_m in $S_{0,0}$ in the norm of L_u^p and L^2. This follows from Theorem (6.13) in [7]. The proof given there is in one dimension, but the generalization to higher dimensions is straightforward.

Now let f be in $L_N \cap L_C^1$, $\Phi \in C^\infty\{|x| < 1\}$, $\int \Phi = 1$. Then for $\varepsilon > 0$

$f_\varepsilon = f * \Phi_\varepsilon$ is in S_N. Hence there exists a sequence of functions $f_{\varepsilon,k}$ in $S_{0,0}$ converging to f_ε in L_u^p and L^2 as $k \to \infty$.

We claim that after selecting a suitable subsequence of $\{f_{\varepsilon,k}\}_{\varepsilon,k>0}$ for every $\delta > 0$ and a.e. x

$$\lim_{\varepsilon\to 0} \lim_{k\to\infty} \sup_{t>\delta} |(f - f_{\varepsilon,k}) * \varphi_t(x)| = 0. \tag{10.2}$$

This will follow from

$$\lim_{\varepsilon\to 0} \sup_{t>\delta} |(f - f_\varepsilon) * \varphi_t(x)| = 0, \tag{10.3}$$

and

$$\lim_{k\to\infty} \sup_{t>\delta} |(f_\varepsilon - f_{\varepsilon,k}) * \varphi_t(x)| = 0 \quad \text{for every } \varepsilon > 0. \tag{10.4}$$

We first consider the case that φ is in S. To show (10.3), observe that for fixed y in $\boldsymbol{R}^n$, $\delta > 0$,

$$\lim_{\varepsilon\to 0} \sup_{t>\delta} |\hat f(y/t)[1 - \hat\Phi(\varepsilon y/t)]\,\hat\varphi(y)|$$
$$\leqslant \lim_{\varepsilon\to 0} \sup_{S>\delta/\varepsilon} \|\hat f\|_\infty \|\hat\varphi\|_\infty |1 - \hat\Phi(\varepsilon y/s)| = 0, \qquad \text{since } \hat\Phi(0) = 1.$$

Since the left side of (10.3) is less than

$$C \lim_{\varepsilon\to 0} \sup_{t>\delta} \int |\hat f(y)[1 - \hat\Phi(\varepsilon y)]\,\hat\varphi(ty)|\,dy$$
$$\leqslant C \lim_{\varepsilon\to 0} \sup_{t>\delta} \int |\hat f(y/t)[1 - \hat\Phi(\varepsilon y/t)]\,\hat\varphi(y)|\,dy \cdot t^{-n},$$

and since the last integrand is bounded by $\|\hat f\|_\infty (1 + \|\hat\Phi\|_\infty) |\hat\varphi(y)|$, which is integrable, we can use the above observation and apply Lebesgue dominated convergence theorem to get (10.3).

For the left side of (10.4) we have the upper bound

$$\lim_{k\to\infty} \sup_{t>\delta} \|f_\varepsilon - f_{\varepsilon,k}\|_2 \|\varphi_t\|_2 \leqslant \lim_{k\to\infty} \delta^{-(n/2)} \|f_\varepsilon - f_{\varepsilon,k}\| \cdot \|\varphi\|_2 = 0.$$

To show (10.3) and (10.4) in case $\varphi(x) = x_k/|x|^{n+1}$, note that $f_\varepsilon \to f$ in L^1 and $f_{\varepsilon,k} \to f_\varepsilon$ in L^2. Using that the Riesz transforms are of weak-type (1.1) and of strong type (2.2), both equalities follow and hence (10.2) holds, if we select an appropriate subsequence first from $\{f_\varepsilon\}_{\varepsilon>0}$ and then from $\{f_{\varepsilon,k}\}_{k>0}$ for every ε.

From the above we get for every $\delta > 0$ and a.e.

$$\sup_{t>\delta} |f * \varphi_t(x)| \leqslant \liminf_{\varepsilon\to 0} \liminf_{k\to\infty} \sup_{t>0} |f_{\varepsilon,k} * \varphi_t(x)|.$$

Since $\delta > 0$ was arbitrary, Fatou's lemma shows that

$$\int \sup_{t>0} |f * \varphi_t|^p u \leqslant \liminf_{\varepsilon\to 0} \liminf_{k\to\infty} \int \sup_{t>0} |f_{\varepsilon,k} * \varphi_t|^p u.$$

By assumption the last expression is bounded by

$$C \liminf_{\varepsilon\to 0} \liminf_{k\to\infty} \int |f_{\varepsilon,k}|^p u = C \liminf_{\varepsilon\to 0} \int |f_\varepsilon|^p u.$$

It remains to show that this equals $C \int |f|^p u$. But this follows from $\|f - f_\varepsilon\|_{p,u} \leqslant \sup_{|z|<\varepsilon} \|(f(\cdot - z)) - f\|_{p,u} \|\Phi\|_1$, and the last expression tends to zero with ε by standard arguments, since u is locally integrable.

This completes the proof of the corollary.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF TEXAS AT AUSTIN
AUSTIN, TEXAS 78712

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