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Non-commutative probability limit theorems

by

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Abstract. A non-commutative analogue of limit probability distributions of sums of independent random variables forming a uniformly infinitesimal array is considered. We give a complete description of all possible limit probability operators being a quantum analogue of infinitely divisible and self-decomposable probability distributions, respectively.

1. Preliminaries and notation. In the quantum probability theory the σ-field of random events is replaced by the lattice Π of orthogonal projectors in a separable infinite-dimensional Hilbert space H. A countably additive function from this lattice to the unit interval constitutes a state, the non-commutative analogue of a probability measure. The Theorem of Gleason [4] asserts that every state is of the form

$$P \to \operatorname{tr} PT \quad (P \in \Pi), \tag{1.1}$$

where T is a *probability operator* on H, i.e. a positive operator of unit trace. Conversely, every probability operator determines a state by (1.1). From now on let $\mathscr{P}$ stand for the set of all probability operators on H. We shall denote by $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively, the set of all nuclear and Hilbert-Schmidt operators on H. With the norm $\|T\|_1 = \operatorname{tr}\sqrt{T \cdot T^*}$ $\mathscr{L}_1$ becomes a Banach space. Furthermore, with the norm $\|T\|_2 = \sqrt{\operatorname{tr} T \cdot T^*}$ $\mathscr{L}_2$ becomes a Hilbert space. Obviously, $P \subset \mathscr{L}_1 \subset \mathscr{L}_2$ and $\|T\|_2 \leqslant \|T\|_1$ for $T \in \mathscr{L}_1$.

In the classical case the physical object is determined by its symmetry properties, i.e. by a group G. In this paper we restrict ourselves to the canonical non-relativistic case when G is the vector group R^{2s} $(s = 1, 2, \ldots)$. The quantum analogue of this object is defined by a representation of R^{2s} in the group of automorphisms of the lattice Π, i.e. according to the Theorem of Wigner [9] (p. 170) and the theorems on multiplier representations ([1], [10], Chapter X) by a projective unitary representation $V(z)$ $(z \in R^{2s})$ on H' satisfying the Weyl–Segal commutation relation

$$V(z)V(z') = e^{(i/2)\Delta(z,z')}V(z+z') \quad (z, z' \in R^{2s}),$$

where

$$\Delta(z, z') = \sum_{k=1}^{s} (x_k y'_k - x'_k y_k)$$

and

$$z = \langle x_1, y_1, x_2, y_2, \ldots, x_s, y_s \rangle,$$
$$z' = \langle x_1', y_1', x_2', y_2', \ldots, x_s', y_s' \rangle.$$

By D we shall denote the operator on R^{2s} corresponding to the skew form Δ, i.e. $(z, Dz') = \Delta(z, z')$ for all $z, z' \in R^{2s}$, where $(\cdot, \cdot)$ is the Euclidean inner product in R^{2s}. It is known ([5], p. 240) that the map $T \to \hat{T}$ $(T \in \mathcal{L}_1)$, where $\hat{T}(z) = \operatorname{tr} TV(z)$, extends uniquely to a linear isometric transformation from $\mathcal{L}_2$ onto the space $L_2(R^{2s})$ of all complex-valued square integrable (with respect to the Lebesgue measure) functions f on R^{2s} with the norm

$$\|f\|_2 = \Big((2\pi)^{-s} \int_{R^{2s}} |f(z)|^2\, dz\Big)^{1/2}.$$

A complex-valued function f on R^{2s} is said to be *Δ-positive-definite* if for arbitrary complex numbers $c_1, c_2, \ldots, c_n$ and vectors $z_1, z_2, \ldots, z_n \in R^{2s}$ the inequality

$$\sum_{j,k=1}^{n} c_j \bar{c}_k f(z_j - z_k)\, e^{(i/2)\Delta(z_j, z_k)} \geqslant 0 \tag{1.2}$$

holds. An analogue of Bochner's theorem asserts that $f = \hat{T}$ for a certain probability operator T if and only if f is Δ-positive-definite, continuous at the origin and $f(0) = 1$ ([5], p. 243). The function $\hat{T}$ is called the *characteristic function* of the probability operator T. It will be one of the main tools in the analysis of probability operators.

A probability operator T is said to be *Gaussian* if $\hat{T}(z) = e^{-1/2(qz,z)+i(z,z_0)}$, where $z_0 \in R^{2s}$ and q is non-negative self-adjoint operator on R^{2s}. The operator q is called the *covariance operator* for the Gaussian operator T. A necessary and sufficient condition for q to be the covariance operator for a certain Gaussian probability operator is given by the inequality

$$(qz, z) + (qz', z') \geqslant \Delta(z, z') \tag{1.3}$$

for all $z, z' \in R^{2s}$ ([5], p. 252).

It is evident that for $T \in \mathcal{P}$ both function $\hat{T}(z)$ and $\hat{T}(-z)$ fulfil condition (1.2). Thus there exists a probability operator $\tilde{T}$ such that $\hat{\tilde{T}}(z) = \hat{T}(-z)$. We say that a probability operator T is *symmetric* whenever $T = \tilde{T}$, i.e. the characteristic function of T is real. A projector is said to be a *ground state* if it is a symmetric Gaussian probability operator.

Let $M(R^{2s})$ denote the set of all Borel probability measures on R^{2s}. By δ_a $(a \in R^{2s})$ we shall denote the probability measure on R^{2s} concentrated at the point a. For any $T \in \mathcal{P}$ and $\mu \in M(R^{2s})$ we put

$$T \circ \mu = \int_{R^{2s}} V(Dz)\, TV^*(Dz)\, \mu(dz),$$

where the integral is taken in the weak sense. It is clear that $T \circ \mu \in \mathcal{P}$ and $\widehat{T \circ \mu} = \hat{T}\hat{\mu}$, where $\hat{\mu}$ denotes the classical characteristic function of μ, i.e. the Fourier transform of μ. Hence it follows that $(T \circ \mu) \circ \nu = T \circ (\mu * \nu)$, where the asterisk denotes the convolution in $M(R^{2s})$. Moreover, we have the following proposition.

PROPOSITION 1.1. *Let $T \in \mathcal{P}$ and $\mu \in M(R^{2s})$. Then $T \circ \mu$ is a projector if and only if T is a projector and $\mu = \delta_a$ for a certain vector $a \in R^{2s}$.*

Proof. The sufficiency is evident. To prove the necessity we note that always $|\hat{\mu}(z)| \leqslant 1$, $|\hat{T}(z)| \leqslant 1$ $(z \in R^{2s})$ and $\|T\|_2 \leqslant 1$. If $T \circ \mu$ is a projector, then

$$1 = \|T \circ \mu\|_2^2 = (2\pi)^{-s} \int_{R^{2s}} |\hat{T}(z)\, \hat{\mu}(z)|^2\, dz$$
$$\leqslant (2\pi)^{-s} \int_{R^{2s}} |\hat{T}(z)|^2\, dz = \|T\|_2^2,$$

which yields $\|T\|_2 = 1$ and $|\hat{\mu}(z)| = 1$ $(z \in R^{2s})$. Consequently, by Corollary 3.1 ([5], p. 241), T is a projector and μ is concentrated at a single point.

The projectors belonging to $\mathcal{P}$ will be called *pure states*. The probability operators of the form $Q \circ \mu$, where Q is a ground state and $\mu \in M(R^{2s})$, will be called *quasi-classical* probability operators. From Proposition 1.1 we get the following corollary.

COROLLARY 1.1. *A pure state is quasi-classical if and only if it is of the form $Q \circ \delta_a$, where Q is a ground state and $a \in R^{2s}$.*

Moreover, from the description of eigen-values of Gaussian probability operators ([5], p. 255) we get the following corollary.

COROLLARY 1.2. *A probability operator is Gaussian if and only if it is of the form $Q \circ \gamma$ where Q is a ground state and γ is a Gaussian measure on R^{2s}.*

2. A convolution algebra. Let B be the subset of $L^2(R^{2s})$ consisting of all continuous functions vanishing at ∞. The set B with pointwise addition, scalar multiplication, multiplication and the norm

$$\|f\| = \max_{z \in R^{2s}} |f(z)| + \|f\|_2$$

becomes a Banach algebra without the unit. It is very easy to check the inclusion

$$\hat{\mathcal{P}} \subset B. \tag{2.1}$$

In fact if $T \in \mathcal{P}$, then $\hat{T}^2$ is continuous and positive-definite ([3], p. 464). Consequently, $\hat{T}^2 = \hat{\mu}$ for a certain $\mu \in M(R^{2s})$. Since $\hat{T} \in L^2(R^{2s})$, we have $\hat{\mu} \in L(R^{2s})$ which yields that μ is absolutely continuous with respect to the Lebesgue measure on R^{2s}. By the Riemann–Lebesgue Theorem $\hat{\mu}$ vanishes at ∞. Thus $\hat{T}$ vanishes at ∞, too, which completes the proof.

From (2.1) we get the inclusion

$$\hat{\mathscr{L}}_1 \subset B.$$

Furthermore, we have the proposition.

PROPOSITION 2.1. *The set B is the closure of $\hat{\mathscr{L}}_1$ in the norm $\| \ \|$.*

Proof. Let Q be a ground state, $\hat{Q}(z) = e^{-\frac{1}{2}(qz,z)}$. Then for any $t \geq 1$ and $z_0 \in R^{2s}$ the functions

$$e^{-\frac{1}{2}t^2(qz,z)+i(z,z_0)} \tag{2.2}$$

being the characteristic functions of Gaussian operators belong to $\mathscr{P}$. It is easy to check that functions (2.2) separate points of the one-point compactification of R^{2s} and form the set invariant under complex conjugation. The algebra generated by these functions over the complex field is contained in $\hat{\mathscr{L}}_1$. Consequently, by the Stone-Weierstrass Theorem ([7], I. 4), the uniform closure of this algebra contains all functions from B. Hence it follows that for every positive number ε and every function $f \in B$ the functions

$$f(z)e^{-\varepsilon|z|^2}, \tag{2.3}$$

where $|z|^2 = (z, z)$, belong to the closure in the norm $\| \ \|$ of the algebra generated by functions (2.2). Since functions (2.3) form a dense subset of B in the norm $\| \ \|_2$, we conclude that the closure of $\hat{\mathscr{L}}_1$ in the norm $\| \ \|$ contains B which completes the proof.

Let $\mathscr{A}$ be the set of all Hilbert–Schmidt operators T for which $\hat{T} \in B$. We define the convolution $*$ in $\mathscr{A}$ by setting

$$\widehat{T_1 * T_2} = \hat{T}_1 \hat{T}_2.$$

Moreover, we put $\|T\| = \|\hat{T}\|$. Then

$$\|T_1 * T_2\| \leq \|T_1\| \, \|T_2\|$$

and, consequently, the *convolution algebra* $\mathscr{A}$ is a Banach algebra without the unit. A sequence $\{T_n\}$ of elements of $\mathscr{A}$ is said to be an *approximate unit* if for every $S \in \mathscr{A}$, $T_n * S \to S$ in the norm $\| \ \|$ when $n \to \infty$. It is easy to verify that $\{T_n\}$ is an approximate unit if and only if $\hat{T}_n$ tends to 1 uniformly on every compact subset of R^{2s} and the functions $\hat{T}_n$ are bounded in common on R^{2s}.

From (2.1) we get the inclusion

$$\mathscr{P} \subset \mathscr{L}_1 \subset \mathscr{A} \subset \mathscr{L}_2.$$

Moreover, by Proposition 2.1, $\mathscr{A}$ is the closure of $\mathscr{L}_1$ in the norm $\| \ \|$. It was proved in [3], p. 462, that if $T_n \in \mathscr{P}$ and $\hat{T}_n$ converges pointwise to a limit function f on R^{2s} which is continuous at the origin, then there exists $T \in \mathscr{P}$

such that $\hat{T} = f$. Moreover, on the set $\mathscr{P}$ the following convergences are equivalent:

(i) $T_n \to T$ in the norm $\| \ \|$.

(ii) $\hat{T}_n \to \hat{T}$ uniformly on R^{2s},

(iii) $\hat{T}_n \to \hat{T}$ pointwise on R^{2s},

(iv) $T_n \to T$ in the norm $\| \ \|_1$,

(v) $T_n \to T$ in the norm $\| \ \|_2$.

In fact, the implication (i) $\Rightarrow$ (ii) is a consequence of the inequality $\max\limits_{z \in R^{2s}} |\hat{T}_n(z) - \hat{T}(z)| \leq \|T_n - T\|$. The implication (ii) $\Rightarrow$ (iii) is obvious. The convergence (iii) by Theorem 2 ([3], p. 462) yields the weak convergence of T_n to T in $\mathscr{L}_1$ which, by the Theorem of Wehrl ([2], p. 287), implies the convergence (iv). Further, by the inequality $\|S\|_2 \leq \|S\|_1$ $(S \in \mathscr{L}_1)$ we get the implication (iv) $\Rightarrow$ (v). Conversely, the convergence (v) implies the weak convergence T_n to T in $\mathscr{L}_2$ which by Proposition 7 in [3], p. 465, yields the weak convergence T_n to T in $\mathscr{L}_1$. Applying the theorem of Wehrl ([2], p. 287) we get the convergence (iv). Finally, by the inequality $\|S\| \leq 2\|S\|_1$ $(S \in \mathscr{L}_1)$ we get the implication (iv) $\Rightarrow$ (i) which completes the proof.

By virtue of the isomorphism $T \to \hat{T}$ between $\mathscr{L}_2$ and $L^2(R^{2s})$ for each positive number a we define a linear transformation U_a of $\mathscr{L}_2$ by setting

$$\widehat{(U_a T)}(z) = \hat{T}(az) \quad (z \in R^{2s}). \tag{2.4}$$

It is clear that $U_{ab} = U_a U_b$,

$$\|U_a T\|_2 = a^{-s}\|T\|_2, \tag{2.5}$$

the algebra $\mathscr{A}$ is invariant under all transformations U_a,

$$U_a(T_1 * T_2) = U_a(T_1) * U_a(T_2),$$

and

$$\|U_a T\| \leq \max(1, a^{-s})\|T\|.$$

An n-tuple $(a_1, a_2, \ldots, a_n)$ of positive numbers is said to be *admissible* if $\mathop{*}\limits_{j=1}^{n} U_{a_j} T_j \in \mathscr{P}$ for every choice of probability operators $T_1, T_2, \ldots, T_n$. Let A_n denote the set of all admissible n-tuples $(n = 1, 2, \ldots)$.

PROPOSITION 2.2. *We have*

$$A_n \subset \{(a_1, a_2, \ldots, a_n): \sum_{j=1}^{n} a_j^2 \geq 1\}.$$

Proof. Let Q be a ground state. Then $\mathop{*}\limits_{j=1}^{n} U_{a_j} Q = U_b Q$, where

$b^2 = \sum_{j=1}^{n} a_j^2$. Suppose that $U_b Q \in \mathscr{P}$. Then $\|U_b Q\|_2 \leqslant 1$. Since $\|Q\|_2 = 1$, we have (by (2.5)) $b \geqslant 1$ which completes the proof.

PROPOSITION 2.3. *We have*

$$A_1 = \{1\}.$$

Proof. Let T be an arbitrary probability operator for which $\hat{T}$ is not positive-definite on R^{2s}. As an example of such operator we can take the operator T with the characteristic function

$$\hat{T}(z) = \left(1-\tfrac{1}{2}(x_1^2+y_1^2)\right) e^{-\frac{1}{4}|z|^2}, \tag{2.6}$$

where $z = \langle x_1, y_1, x_2, y_2, \ldots, x_s, y_s\rangle$. Since

$$\hat{T}(z) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{iy_1 t}\left(t+\frac{x_1}{2}\right)\left(t-\frac{x_1}{2}\right) e^{-t^2-\frac{x_1^2}{4}}\, dt \prod_{j=2}^{s} e^{-\frac{1}{4}(x_j^2+y_j^2)},$$

we conclude, by a simple calculation, that $\hat{T}$ is Δ-positive-definite. In other words $T \in \mathscr{P}$. Suppose that $\hat{T}$ is positive-definite. Then its Fourier transform f is continuous and non-negative. But, by a simple calculation, we have $f(0) = -(4\pi)^s$ which shows that $\hat{T}$ is not positive-definite. It is clear that the set A of all positive numbers a for which $U_a T \in \mathscr{P}$ is closed and non-empty because $1 \in A$. It is also bounded. Indeed, if there exists a sequence $a_r \in A$ tending to ∞, then $\hat{T}(a_r z)$ is Δ-positive-definite and setting $f(z) = \hat{T}(a_r z)$, $z_k = w_k/a_r$ into (1.2) we get the inequality

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \hat{T}(w_j - w_k)\, e^{(i/2a_r^2)\Delta(w_j, w_k)} \geqslant 0$$

which yields, when $a_r \to \infty$, the positive definiteness of $\hat{T}$. This contradiction shows that the set A is bounded. Let c be the greatest element of A. Put $S = U_c T$. Of course, $S \in \mathscr{P}$ and $U_a S \notin \mathscr{P}$ for all $a > 1$. Thus $A_1 \subset (0, 1]$ which together with Proposition 2.2 completes the proof.

PROPOSITION 2.4. *We have*

$$(1, 1) \notin A_2.$$

Proof. Let T be the probability operator with the characteristic function (2.6). Suppose that $(1, 1) \in A_2$. Then, in particular, $T * T \in \mathscr{P}$. Consequently, by Proposition 5 ([3], p. 464), $\widehat{T * T} \cdot \hat{T}$, or equivalently, $\hat{T}^3$ is positive definite. Thus its Fourier transform g is continuous and non-negative. But, by a simple calculation, $g(0) = -(4\pi/9)^s$ which yields the contradiction. Our statement is thus proved. Proposition 2.4 shows that the set $\mathscr{P}$ is not closed under the convolution.

PROPOSITION 2.5. *Given an arbitrary n-tuple* $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, *where* $\varepsilon_j = -1$ *or* 1 $(j = 1, 2, \ldots, n)$ *we have the inclusion*

$$\{(a_1, a_2, \ldots, a_n)\colon \sum_{j=1}^{n} \varepsilon_j a_j^2 = 1\} \subset A_n.$$

Proof. From the fact that f is Δ-positive-definite it follows that $\bar{f}$ is also Δ-positive definite. Consequently, we may conclude that for arbitrary $T_1, T_2, \ldots, T_n \in \mathscr{P}$, $z_1, z_2, \ldots, z_m \in R^{2s}$, positive numbers $a_1, a_2, \ldots, a_n$ and $\varepsilon_r = -1$ or 1 $(r = 1, 2, \ldots, n)$ the matrices

$$\left(\hat{T}_r\big(a_r(z_j - z_k)\big)\, e^{(i/2)\varepsilon_r a_r^2 \Delta(z_j, z_k)}\right)_{j,k=1,2,\ldots,m}$$

are positive-definite. Since the entry-by-entry product of such matrices is again positive-definite, we have that

$$\left(\prod_{r=1}^{n} \hat{T}_r\big(a_r(z_j - z_k)\big)\, e^{(i/2)\sum_{r=1}^{n}\varepsilon_r a_r^2 \Delta(z_j, z_k)}\right)_{j,k=1,2,\ldots,m}$$

is positive-definite. If $\sum_{r=1}^{n} \varepsilon_r a_r^2 = 1$, then $\prod_{r=1}^{n} \hat{T}_r(a_r z)$ is Δ-positive-definite or in other words $\mathop{*}_{r=1}^{n} U_{a_r} T_r \in \mathscr{P}$ which completes the proof.

Taking $\varepsilon_j = 1$ if $1 \leqslant j \leqslant n-r$ and $\varepsilon_j = -1$ if $n-r < j \leqslant n$ for every integer r satisfying the inequality $0 \leqslant r < n/2$ we have, by Proposition 2.5,

$$\left(\frac{1}{\sqrt{n-2r}}, \frac{1}{\sqrt{n-2r}}, \ldots, \frac{1}{\sqrt{n-2r}}\right) \in A_n.$$

PROPOSITION 2.6. *The set of all quasi-classical probability operators is invariant under transformations* $U_a (a \geqslant 1)$.

Proof. Let Q be a ground state and $\mu \in M(R^{2s})$. Then (by (2.4)) $U_a(Q \circ \mu) = U_a(Q) \circ \mu_a$, where $\hat{\mu}_a(z) = \hat{\mu}(az)$. By virtue of (1.3) the operator $U_a(Q)$ is Gaussian for $a \geqslant 1$. Consequently, by Corollary 1.2, it is of the form $Q_a \circ \nu_a$, where Q_a is a ground state and $\nu_a \in M(R^{2s})$. Thus $U_a(Q \circ \mu) = Q_a \circ (\mu_a * \nu_a)$ which completes the proof.

3. Statement of the problem. Let $\{T_{kn}\}$, $\{a_{kn}\}$ $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ be triangular arrays of probability operators and positive numbers with the property

$$(a_{1n}, a_{2n}, \ldots, a_{k_n n}) \in A_{k_n} \quad (n = 1, 2, \ldots), \tag{3.1}$$

respectively. Then of course

$$\mathop{*}_{k=1}^{k_n} U_{a_{kn}} T_{kn} \in \mathscr{P} \quad (n = 1, 2, \ldots).$$

The triangular array $\{U_{a_{kn}} T_{kn}\}$ $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ of operators from A is said to be *uniformly infinitesimal* if for every choice of r_n, $1 \leqslant r_n \leqslant k_n$ the sequence $\{U_{a_{r_n n}} T_{r_n n}\}$ $(n = 1, 2, \ldots)$ forms an approximate unit in the convolution algebra $\mathscr{A}$ or, equivalently, in terms of the characteristic function

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} |1 - \hat{T}_{kn}(a_{kn} z)| = 0 \tag{3.2}$$

uniformly on every compact subset of R^{2s}. Then $\{a_{kn}\}$ $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ will be called a *norming array*. Suppose that for a uniformly infinitesimal triangular array there exists a sequence $\{c_n\}$ of vectors from R^{2s} such that the sequence of probability operators

$$(\mathop{*}_{k=1}^{k_n} U_{a_{kn}} T_{kn}) \circ \delta_{c_n} \tag{3.3}$$

converges to an operator T in $\mathscr{A}$. Of course $T \in \mathscr{P}$. What can be said more about the limit probability operator T being a quantum analogue of a classical infinitely divisible probability distribution? Let $\mathscr{D}$ denote the set of all such limit probability operators T. By $\mathscr{D}_1$ we shall denote the subset of $\mathscr{D}$ corresponding to the quantum analogue of classical self-decomposable probability distributions, i.e. to the case $k_n = n$, $a_{1n} = a_{2n} = \ldots = a_{nn}$, $T_{kn} = T_k$ $(k = 1, 2, \ldots;\ n = 1, 2, \ldots)$. Further, $\mathscr{D}_2$ will denote the subset of $\mathscr{D}_1$ corresponding to the case $T_1 = T_2 = \ldots$ From the Cushen–Hudson Quantum Central Limit Theorem [3] it follows that $\mathscr{D}_2$ contains all Gaussian probability operators. Our aim is to characterize the sets $\mathscr{D}$, $\mathscr{D}_1$ and $\mathscr{D}_2$ of limit probability operators.

We recall that μ from $M(R^{2s})$ is *infinitely divisible* if for every positive integer n there exists a measure $\mu_n \in M(R^{2s})$ such that $\mu = \mu_n^{*n}$. A measure μ from $M(R^{2s})$ is *self-decomposable* if for every real number a from the interval $[0, 1]$ there exists a measure $\nu_a \in M(R^{2s})$ such that $\mu = \mu_a * \nu_a$, where $\hat{\mu}_a(z) = \hat{\mu}(az)$ $(z \in R^{2s})$.

THEOREM 3.1. *The set $\mathscr{D}$ consists of all quasi-classical probability operators $Q \circ \mu$, where Q is a ground state and μ is an infinitely divisible probability measure from $M(R^{2s})$.*

From this theorem and Corollary 1.1 we get the following statement.

COROLLARY 3.1. *The probability operators of the form $Q \circ \delta_a$, where Q is a ground state and $a \in R^{2s}$ are the only pure states in $\mathscr{D}$.*

THEOREM 3.2. *The set $\mathscr{D}_1$ consists of all quasi-classical probability operators $Q \circ \mu$, where Q is a ground state and μ is a self-decomposable measure from $M(R^{2s})$.*

THEOREM 3.3. *The set $\mathscr{D}_2$ consists of all Gaussian probability operators.*

Before proceeding to prove the theorems we shall establish auxiliary propositions.

LEMMA 3.1. *For each norming array of a uniformly infinitesimal triangular array $\{U_{a_{kn}} T_{kn}\}$ $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ the formula*

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} a_{kn} = 0$$

holds.

Proof. Suppose the contrary. Passing to a subsequence if necessary we may assume without loss of generality that for a sequence $\{r_n\}$ $(1 \leqslant r_n \leqslant k_n)$,

$$\lim_{n\to\infty} a_{r_n n} > 0.$$

By (3.2) we have then the convergence $\hat{T}_{r_n n}(z) \to 1$ uniformly on every compact subset of R^{2s}. The limit function 1 being Δ-positive-definite is the characteristic function of a probability operator belongs to $L^2(R^{2s})$ which yields the contradiction. The lemma is thus proved.

LEMMA 3.2. *For each norming array corresponding to a convergent sequence (3.3) the inequality*

$$\overline{\lim_{n\to\infty}} \sum_{k=1}^{k_n} a_{kn}^2 < \infty$$

holds.

Proof. Suppose that the sequence (3.3) is convergent to a probability operator T. Put $b_n^2 = \sum_{k=1}^{k_n} a_{kn}^2$, $b_n > 0$, $d_n = b_n^{-1} c_n$, $b_{kn} = b_n^{-1} a_{kn}$ $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$. Then, by Proposition 2.5

$$S_n = \mathop{*}_{k=1}^{k_n} U_{b_{kn}} T_{kn} \in \mathscr{P}$$

and $U_{b_n} S_n \to T$ in $\mathscr{A}$. Suppose that for a subsequence $\{m_n\}$ $b_{m_n} \to \infty$. Then the relation $\hat{S}_n(b_n z) \to \hat{T}(z)$ uniformly on every compact subset of R^{2s} yields $\hat{S}_n(z) \to 1$. The limit function 1 being Δ-positive-definite is the characteristic function of a probability operator belongs to $L^2(R^{2s})$ which gives the contradiction. The lemma is thus proved.

By Lemma 3.2 (passing to a subsequence if necessary) we may restrict ourselves to norming arrays for which the limit

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} a_{kn}^2 = a < \infty \tag{3.4}$$

exists. Moreover, by Proposition 2.2 and (3.1), $a \geqslant 1$. Let us denote by $\mathscr{D}_0$ the subset of $\mathscr{D}$ consisting of limits of sequences (3.3) with the norming arrays satisfying (3.4), where $a = 1$. From (3.1) and Proposition 2.5 we get the following corollary.

COROLLARY 3.2. *For every $T \in \mathscr{D}$ there exists $T_0 \in \mathscr{D}_0$ and $a \geqslant 1$ such that $T = U_a T_0$.*

By a simple calculation we get the following lemma.

LEMMA 3.3. *Let $\{c_{kn}\}$ $(k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$ be a triangular array of positive numbers with the properties*

$$\sum_{k=1}^{k_n} c_{kn} = 1 \qquad (n = 1, 2, \ldots)$$

and

$$\lim_{n\to\infty}\max_{1\leqslant k\leqslant k_n} c_{kn} = 0.$$

Then for every positive integer m there exist indices $1 = s_{0n} < s_{1n} < \ldots < s_{mn} = k_n$ such that

$$\lim_{n\to\infty}\sum_{k=s_{r-1n}+1}^{s_{rn}} c_{kn} = 1/m \qquad (r = 1, 2, \ldots, m).$$

LEMMA 3.4. *Let $T_n, S_n \in \mathscr{P}$. Suppose that for some positive numbers a_n and b_n with the property*

$$\varliminf_{n\to\infty} a_n > 0 \tag{3.5}$$

and for some vectors $z_n \in R^{2s}$, the sequence

$$(U_{a_n} T_n * U_{b_n} S_n)\circ\delta_{z_n} \qquad (n = 1, 2, \ldots)$$

is convergent in $\mathscr{A}$. Then the sequence $U_{1/\sqrt{2}}(T_n * \tilde{T}_n)$ $(n = 1, 2, \ldots)$ is conditionally compact in $\mathscr{P}$.

Proof. Let us denote by G_n and H_n the operators $U_{1/\sqrt{2}}(T_n * \tilde{T}_n)$ and $U_{1/\sqrt{2}}(S_n * \tilde{S}_n)$, respectively. By Proposition 2.5, $G_n, H_n \in \mathscr{P}$. Moreover, by the assumption, the sequence $U_{a_n} G_n * U_{b_n} H_n$ is convergent in $\mathscr{A}$. Thus the sequence

$$\hat{G}_n(a_n z)\hat{H}_n(b_n z) \qquad (n = 1, 2, \ldots)$$

is convergent uniformly on every compact subset of R^{2s}. Since $\hat{G}_n(z) = \hat{T}_n(z/\sqrt{2})\hat{\tilde{T}}_n(z/\sqrt{2})$, $\hat{H}_n(z) = \hat{S}_n(z/\sqrt{2})\hat{\tilde{S}}_n(z/\sqrt{2})$, we infer (by Proposition 5 in [3]) that both functions $\hat{G}_n$, $\hat{H}_n$ are positive definite. Consequently, there exist symmetric measures μ_n and ν_n in $M(R^{2s})$ such that $\hat{G}_n = \hat{\mu}_n$ and $\hat{H}_n = \hat{\nu}_n$. Thus the sequence $\hat{\mu}_n(a_n z)\hat{\nu}_n(b_n z)$ $(n = 1, 2, \ldots)$ is uniformly convergent on every compact subset of R^{2s}. Hence by the symmetry of μ_n, ν_n and inequality (3.5) we infer that the sequence $\{\mu_n\}$ is conditionally compact in $M(R^{2s})$ ([8], Theorem 2.2). Consequently, the sequence $\hat{\mu}_n$ or, in other words, $\hat{G}_n$ is conditionally compact in the topology of the uniform convergence on every compact subset of R^{2s}. Hence by the equivalence of the convergences (i)–(v) on $\mathscr{P}$ we get the assertion of the lemma.

LEMMA 3.5. *Let $T \in \mathscr{D}_0$. Then for every positive integer m there exist symmetric probability operators T_j $(j = 1, 2, \ldots, m)$ in $\mathscr{D}_0$ such that*

$$U_{1/\sqrt{2}}(T * \tilde{T}) = \mathop{*}_{j=1}^{m} U_{1/\sqrt{m}} T_j.$$

Proof. Suppose that T is the limit of the sequence (3.3) where the norming array fulfils condition (3.4) with $a = 1$. By Lemmas 3.1 and 3.3 for every positive integer m there exist indices $1 = s_{0n} < s_{1n} < \ldots < s_{mn} = k_n$ $(n = 1, 2, \ldots)$ such that setting

$$c_{rn}^2 = \sum_{k=s_{r-1n}+1}^{s_{rn}} a_{kn}^2, \quad c_{rn} > 0 \quad (r = 1, 2, \ldots, m)$$

we have

$$\lim_{n\to\infty} c_{rn}^2 = 1/m \qquad (r = 1, 2, \ldots, m) \tag{3.6}$$

Moreover, setting $b_{krn} = a_{kn}/c_{rn}$ $(s_{r-1n} < k \leqslant s_{rn})$ we infer that the triangulat arrays $\{U_{b_{krn}} T_{kn}\}$ $(s_{r-1n} < k \leqslant s_{rn};\ n = 1, 2, \ldots)$ are uniformly infinitesimal. Further, setting

$$S_{rn} = \mathop{*}_{k=s_{r-1n}+1}^{s_{rn}} U_{b_{krn}} T_{kn}$$

we have

$$\big(\mathop{*}_{r=1}^{m} U_{c_{rn}} S_{rn}\big)\circ\delta_{c_n} \to T \tag{3.7}$$

in $\mathscr{A}$. Taking into account (3.1) we conclude, in view of Lemma 3.4, that for every r the sequence $U_{1/\sqrt{2}}(S_{rn} * \tilde{S}_{rn})$ is conditionally compact in $\mathscr{P}$. Passing to a subsequence if necessary we may assume without loss of generality that

$$U_{1/\sqrt{2}}(S_{rn} * \tilde{S}_{rn}) \to T_r$$

when $n \to \infty$. Of course T_r is a symmetric probability operator from $\mathcal{S}_0$ and, by (3.6) and (3.7),

$$U_{1/\sqrt{2}}(T * \tilde{T}) = \mathop{*}_{r=1}^{m} U_{1/\sqrt{m}} T_r$$

which completes the proof.

It is well known that the characteristic function $\hat{\mu}$ of an infinitely divisible measure μ from $M(R^{2s})$ can be written in the Lévy–Khinchine canonical form

$$\hat{\mu}(z) = \exp\left\{ i(z, z_0) - \tfrac{1}{2}(qz, z) + \int_{R^{2s}} \left(e^{i(z,u)} - 1 - \frac{i(z, u)}{1+|u|^2} \right) \frac{1+|u|^2}{|u|^2} G(du) \right\},$$

where $z_0 \in R^{2s}$, q is a covariance operator on R^{2s} and G is a finite non-negative Borel measure on R^{2s} vanishing at the origin ([8], p. 181). The correspondence between $\hat{\mu}$ and the triple $[z_0, q, G]$ is one-to-one which enables us to use the notation $\hat{\mu} = [z_0, q, G]$. It is clear that

$$[z_1, q_1, G_1][z_2, q_2, G_2] = [z_1 + z_2, q_1 + q_2, G_1 + G_2]. \tag{3.8}$$

LEMMA 3.6. *For every $T \in \mathcal{S}_0$ there exists an infinitely divisible measure ν from $M(R^{2s})$ such that $\hat{T} = \hat{\nu}$. Moreover, if $\hat{\nu} = [z_0, q, G]$, then q is the covariance operator for a Gaussian probability operator.*

Proof. Let T be the limit of sequence (3.3) with the norming array satisfying (3.4), where $a = 1$. Then

$$\prod_{k=1}^{k_n} \hat{T}_{kn}^2(a_{kn} z) e^{2i(c_n, z)} \to \hat{T}^2(z) \tag{3.9}$$

uniformly on R^{2s}. Since by Proposition 5 in [3] the functions $\hat{T}_{kn}^2(z)$ are positive-definite, there exist $\mu_{kn} \in M(R^{2s})$ such that

$$\hat{\mu}_{kn}(z) = \hat{T}_{kn}^2(a_{kn} z) \quad (k = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots).$$

Of course, $\{\mu_{kn}\}$ form a uniformly infinitesimal array of probability measures and, by (3.9),

$$(\mathop{*}_{k=1}^{k_n} \mu_{kn}) * \delta_{2c_n} \to \mu,$$

where

$$\hat{\mu}(z) = \hat{T}^2(z) \quad (z \in R^{2s}). \tag{3.10}$$

Consequently, by the classical limit theorem ([8], p. 199) μ is infinitely divisible which yields the Lévy–Khinchine representation $\hat{\mu} = [2z_0, 2q, 2G]$.

Put $\hat{\nu} = [z_0, q, G]$. Since the function $\hat{T}$ is continuous, $\hat{T}(0) = 1$ and, by (3.10), does not vanish, we get the formula $\hat{T}(z) = \hat{\nu}(z)$ on R^{2s}. It remains to prove that q is the covariance operator for a Gaussian probability operator. Put $S = U_{1/\sqrt{2}}(T * \tilde{T})$. Then

$$\hat{S}(z) = \hat{\lambda}(z), \tag{3.11}$$

where $\hat{\lambda} = [0, q, H]$ and the measure H is symmetric on R^{2s}. Further, by Lemma 3.5, for every positive integer m, there exist symmetric probability operators T_{rm} $(r = 1, 2, \ldots, m)$ belonging to $\mathcal{S}_0$ such that

$$S = \mathop{*}_{r=1}^{m} U_{1/\sqrt{m}} T_{rm}. \tag{3.12}$$

We already know that

$$\hat{T}_{rm}(z) = \hat{\lambda}_{rm}(z),$$

where $\hat{\lambda}_{rm} = [0, q_{rm}, H_{rm}]$ and the measures H_{rm} are symmetric on R^{2s}. Taking into account (3.11) and (3.12) we get the equations

$$q = \frac{1}{m} \sum_{r=1}^{m} q_{rm}, \tag{3.13}$$

$$H(E) = \int_{\sqrt{m}E} \frac{m(1+|u|^2)}{m+|u|^2} H_m(du), \tag{3.14}$$

where

$$H_m = \frac{1}{m} \sum_{r=1}^{m} H_{rm}. \tag{3.15}$$

Put for any positive number t, $E_t = \{z\colon |z| \leqslant \sqrt[4]{t}\}$. Then, by (3.14),

$$H_m(E_m) \leqslant H(E_{1/m})$$

and

$$H_m(E_m^c) \leqslant \frac{m+\sqrt{m}}{m(1+\sqrt{m})} H(R^{2s}),$$

where E^c denotes the complement of E in R^{2s}. Since $H(\{0\}) = 0$, the last inequalities yield

$$\lim_{m\to\infty} H_m(R^{2s}) = 0. \tag{3.16}$$

Let $z_1, z_2, \ldots, z_{2s}$ be an arbitrary system of linearly independent vectors in R^{2s}. Put

$$h = \sum_{j=1}^{2s} (qz_j, z_j), \quad g = (qz_1, z_1)+(qz_2, z_2),$$

$$h_{rm} = \sum_{j=1}^{2s} (q_{rm} z_j, z_j), \quad g_{rm} = (q_{rm} z_1, z_1)+(q_{rm} z_2, z_2).$$

By (3.13) we have the equations

$$(3.17) \qquad h = \frac{1}{m}\sum_{r=1}^{m} h_{rm}, \quad g = \frac{1}{m}\sum_{r=1}^{m} g_{rm}.$$

Moreover, since q and q_{rm} are non-negative operators, all numbers h, g, h_{rm} and g_{rm} are non-negative. Consequently, for arbitrary positive number ε we can find a positive number η such that

$$(3.18) \qquad h\eta < \frac{\varepsilon}{g+\varepsilon}.$$

Put

$$A_m = \{r\colon 1 \leqslant r \leqslant m,\ h_{rm}\eta < 1\},$$
$$B_{mn} = \{r\colon 1 \leqslant r \leqslant m,\ H_{rm}(R^{2s}) < 1/n\},$$
$$C_m = \{r\colon 1 \leqslant r \leqslant m,\ g_{rm} < g+\varepsilon\}.$$

Then, denoting by $\operatorname{card} A$ the number of elements of A, we have, by (3.17),

$$\frac{1}{m}\operatorname{card} A_m \geqslant 1-h\eta \quad (m = 1, 2, \ldots).$$

Further, by (3.15) and (3.16),

$$\lim_{m\to\infty} \frac{1}{m}\operatorname{card} B_{mn} = 1 \quad (n = 1, 2, \ldots)$$

and consequently, by (3.18),

$$\varliminf_{m\to\infty} \frac{1}{m}\operatorname{card} A_m \cap B_{mn} \geqslant 1-h\eta > 1-\frac{\varepsilon}{g+\varepsilon} \quad (n = 1, 2, \ldots).$$

Finally, by (3.17),

$$\frac{1}{m}\operatorname{card} C_m \geqslant \frac{\varepsilon}{g+\varepsilon}.$$

From the last two inequalities it follows that for every positive integer n the set $A_m \cap B_{mn} \cap C_m$ is non-void for sufficiently large indices m. In other words we can find integers m_n, r_n with the properties $m_1 < m_2 < \ldots$, $1 \leqslant r_n \leqslant m_n$ and $r_n \in A_{m_n} \cap B_{m_n n} \cap C_{m_n}$. Then

$$(3.19) \qquad h_{r_m m_n} < \eta^{-1} \quad (n = 1, 2, \ldots),$$

$$(3.20) \qquad H_{r_n m_n}(R^{2s}) < 1/n \quad (n = 1, 2, \ldots)$$

and

$$(3.21) \qquad g_{r_n m_n} < g+\varepsilon \quad (n = 1, 2, \ldots).$$

From inequality (3.19), by the linear independence of the vectors $z_1, z_2, \ldots, z_{2s}$ we get the boundedness and, consequently, the conditional compactness of the sequence of covariance operators $q_{r_n m_n}$ on R^{2s}. Passing to a subsequence if necessary, we may assume without loss of generality that the sequence $q_{r_n m_n}$ converges to a covariance operator q_0. By (3.21) we then have the inequality

$$(3.22) \qquad (q_0 z_1, z_1)+(q_0 z_2, z_2) \leqslant (qz_1, z_1)+(qz_2, z_2)+\varepsilon.$$

Finally, inequality (3.20) shows that the sequence of the characteristic functions $\hat{T}_{r_n m_n}$ tends to the function $[0, q_0, 0]$ uniformly on every compact subset of R^{2s}. Hence it follows that the function $e^{-\frac{1}{2}(q_0 z, z)}$ is Δ-positive-definite and, consequently, is the characteristic function of a Gaussian probability operator. Thus q_0 fulfils condition (1.3) which, by (3.22), yields

$$\Delta(z_1, z_2) \leqslant (qz_1, z_1)+(qz_2, z_2)+\varepsilon$$

for every positive number ε. Consequently,

$$\Delta(z_1, z_2) \leqslant (qz_1, z_1)+(qz_2, z_2),$$

where z_1, z_2 are arbitrary linearly independent vectors from R^{2s}. For linearly dependent vectors z_1, z_2 the last inequality is evident because in this case $\Delta(z_1, z_2) = 0$. Thus q is the covariance operator for a Gaussian probability operator which completes the proof.

Now we are ready to prove the theorems.

Proof of Theorem 3.1. *Sufficiency*. Suppose that Q is a ground state and μ is an infinitely divisible measure from $M(R^{2s})$. Let $\mu_n^{*n} = \mu$. Put $\hat{\nu}_n(z) = \hat{\mu}_n(\sqrt{n}z)$ and $T_{kn} = Q \circ \nu_n$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$. The triangular array $\{U_{1/\sqrt{n}} T_{kn}\}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ is uniformly infinitesimal because $\hat{T}_{kn}(z/\sqrt{n}) = \hat{Q}(z/\sqrt{n})\sqrt[n]{\hat{\mu}(z)} \to 1$ uniformly on every compact subset of R^{2s}. Moreover, $(U_{1/\sqrt{n}} T_{kn})^{*n} = Q \circ \mu$ $(n = 1, 2, \ldots)$ which shows that $Q \circ \mu \in \mathscr{D}$.

Necessity. First suppose that $T \in \mathscr{D}_0$. By Lemma 3.6 there exist an infinitely divisible measure ν from $M(R^{2s})$ such that $\hat{T} = \hat{\nu}$, $\hat{\nu} = [z_0, q, G]$

and q is the covariance operator for a Gaussian probability operator. Let S be a Gaussian operator with the characteristic function $\hat{S}(z) = e^{-\frac{1}{2}(qz,z)}$. Put $\hat{\lambda} = [z_0, 0, G]$. Then (by (3.8)) $T = S \circ \lambda$. Further, by Corollary 1.2 $S = Q \circ \gamma$, where Q is a ground state and γ is a Gaussian measure on R^{2s}. Consequently, $T = Q \circ (\gamma * \lambda)$. The measure $\gamma * \lambda$ is infinitely divisible which completes the proof in the case $T \in \mathscr{P}_0$. For arbitrary probability operator from $\mathscr{P}$ our assertion is a consequence of Proposition 2.6 and Corollary 3.2.

Proof of Theorems 3.2 and 3.3. *Sufficiency*. We already know, by Cushen–Hudson Quantum Central Limit Theorem, that $\mathscr{P}_2$ contains all Gaussian probability operators. Given a ground state Q and a self-decomposable measure μ from $M(R^{2s})$, we have to prove that $Q \circ \mu \in \mathscr{P}_1$. By the self-decomposability of μ for every positive integer k there exists a measure ν_k in $M(R^{2s})$ such that

$$\hat{\mu}(z) = \hat{\mu}(\sqrt{(k-1)/k}\, z)\, \hat{\nu}_k(z) \qquad (z \in R^{2s}).$$

Let μ_k be the measure from $M(R^{2s})$ defined by the condition

$$\hat{\mu}_k(z) = \hat{\nu}_k(\sqrt{k}\, z).$$

Put $T_k = Q \circ \mu_k$ $(k = 1, 2, \ldots)$. Then $\prod_{k=1}^{n} \hat{T}_k(z/\sqrt{n}) = \hat{Q}(z)\hat{\mu}(z)$ $(n = 1, 2, \ldots)$ or, in other words, $Q \circ \mu = \mathop{*}_{k=1}^{n} U_{1/\sqrt{n}} T_k$ $(n = 1, 2, \ldots)$. Moreover,

$$\hat{T}_k(z/\sqrt{n}) = \hat{Q}(z/\sqrt{n}) \frac{\hat{\mu}(\sqrt{k/n}\, z)}{\hat{\mu}(\sqrt{(k-1)/n}\, z)}$$

which shows that the triangular array $\{U_{1/\sqrt{n}} T_k\}$ $(k = 1, 2, \ldots, n = 1, 2, \ldots)$ is uniformly infinitesimal. Thus $Q \circ \mu \in \mathscr{P}_1$.

Necessity. Suppose that $T \in \mathscr{P}_1$ and T is the limit of a sequence

$$(\mathop{*}_{k=1}^{n} U_{a_n} T_k) \circ \delta_{c_n}.$$

Put

$$S_k = U_{1/\sqrt{2}}(T_{2k-1} * T_{2k}) \qquad (k = 1, 2, \ldots).$$

Then, by Proposition 2.5, $S_k \in \mathscr{P}$ and, by Proposition 5 in [3], $\hat{S}_k$ is positive-definite. Thus there exist probability measures $\mu_k \in M(R^{2s})$ such that $\hat{\mu}_k = \hat{S}_k$. Moreover, the triangular array of probability measures $\{\mu_{kn}\}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$, where $\hat{\mu}_{kn}(z) = \hat{\mu}_k(z/\sqrt{n})$, is uniformly infinitesimal and

$$(\mathop{*}_{k=1}^{n} \mu_{kn}) * \delta_{2c_n} \to \nu,$$

where

$$\hat{\nu} = \hat{T}^2. \tag{3.23}$$

It is well known that the limit measure ν is self-decomposable ([6], p. 323). On the other hand (by Theorem 3.1) $T = Q \circ \mu$, where Q is a ground state and μ an infinitely divisible measure from $M(R^{2s})$. Put $\hat{\mu} = [z_0, q_0, G_0]$ and $\hat{\nu} = [z_1, q_1, G_1]$. Then, by (3.23), $2G_0 = G_1$ which shows that the measure μ is also self-decomposable which completes the proof of Theorem 3.2.

In order to prove that $\mathscr{P}_2$ consists of Gaussian probability operators only, we note that in the case $T_1 = T_2 = \ldots$ we have $\mu_1 = \mu_2 = \ldots$ and, consequently, the limit ν is a stable probability measure having (by Theorem 3.1 and (3.23)) a non-trivial Gaussian component. Thus ν itself is Gaussian which (by (3.23) and Theorem 3.1) shows that T is a Gaussian probability operator which completes the proof of Theorem 3.3.

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